The Toda lattice, billiards and symplectic geometry

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Question 2

Given $X_1, X_2 \subset \mathbb{R}^{2n}$, does there exist an embedding $\varphi : X_1 \hookrightarrow X_2$ such that $\varphi^*\omega = \omega$?

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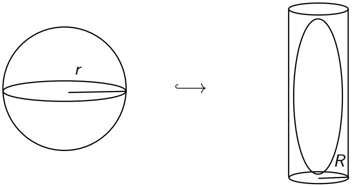

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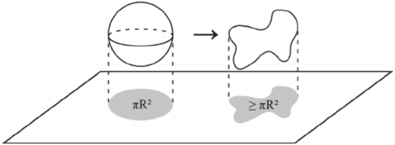

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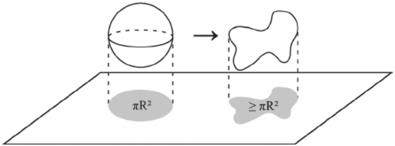

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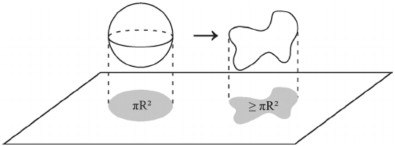

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The existence of a normalized symplectic capacity is equivalent to Gromov's nonsqueezing theorem.

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- First embedded contact homology capacity c_1^{ECH} (Hutchings 2011) - only in dimension 4.

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Moreover equality holds if, and only if, X is symplectomorphic to a ball.

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Strong Viterbo conjecture

All normalized capacities coincide on convex sets.

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In particular $M(K) = \text{vol}(K) \cdot \text{vol}(K^\circ)$ is preserved.

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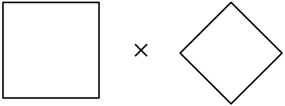

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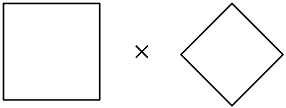

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For each n, $M(K)$ attains its minimum when K is a hypercube.

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Theorem (Iriyeh–Shibata 2020)

The Mahler conjecture holds for $n = 3$.

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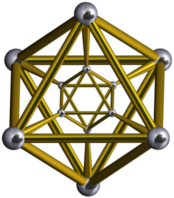

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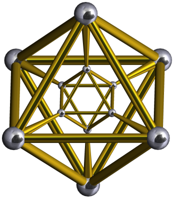

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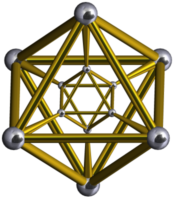

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Conjecture

$M(K)$ is minimized precisely by the Hanner polytopes.

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Strong Viterbo $\Rightarrow$ Viterbo $\Rightarrow$ Weak Viterbo $\Rightarrow$ Mahler

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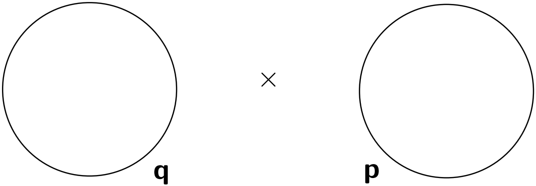

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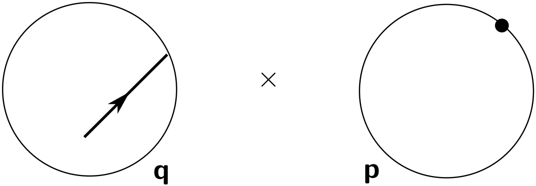

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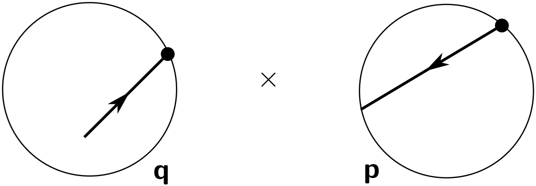

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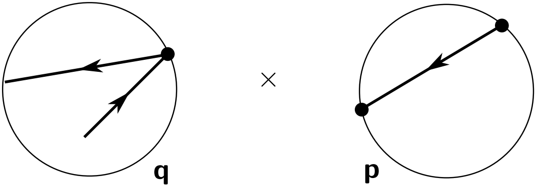

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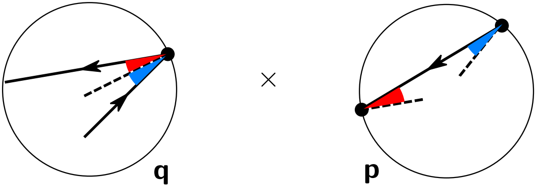

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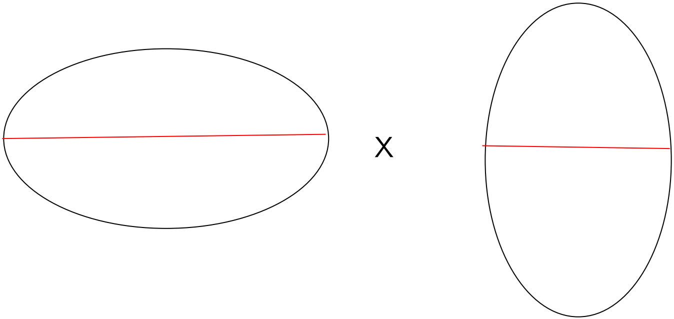

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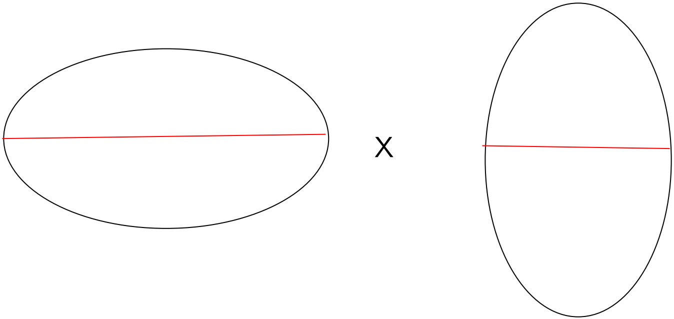

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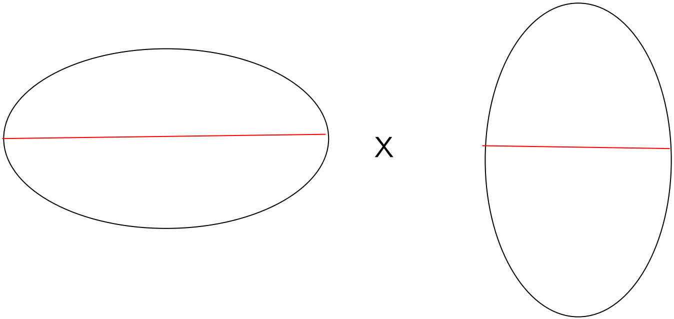

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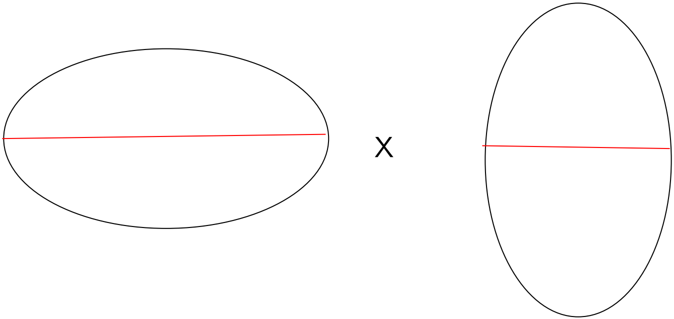

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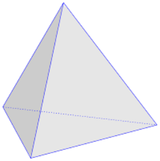

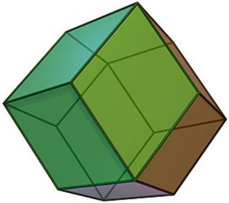

The Toda lattice

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \sum_{i=1}^{n+1} e^{q_i - q_{i+1}}.$$

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$$\begin{cases} \dot{b}_i = a_i^2 - a_{i-1}^2 \\ \dot{a}_i = \frac{1}{2} a_i (b_{i+1} - b_i). \end{cases}$$

Lax pair formulation

There exists a Lax pair (L, B) such that the Hamiltonian system above is equivalent to $\dot{L} = [L, B]$,

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The spectrum of L is invariant under the flow.

Toric domains

Definition

A toric domain $\mathbb{X}_\Omega \subset \mathbb{C}^n$ is a set of the form $\mathbb{X}_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n_{\geq 0}$ is an open set and

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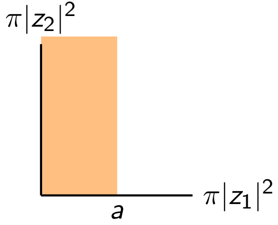

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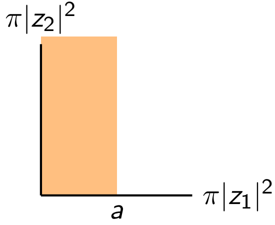

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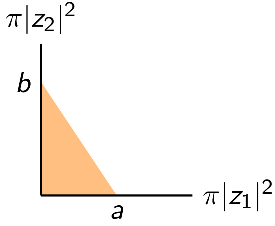

$E(a, b) := \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\right\}$

Monotone toric domains

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A toric domain $X_\Omega \subset \mathbb{R}^{2n}$ is called *monotone* if for each point $p \in \partial\Omega \setminus \{x_i = 0, \text{ for some i}\}$, the normal vector $\nu = (\nu_1, \ldots, \nu_n)$ satifies $\nu_i \geq 0$ for every i.

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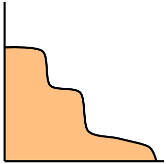

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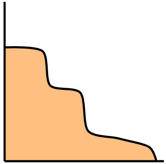

Theorem (Gutt–Hutchings–R. 2020)

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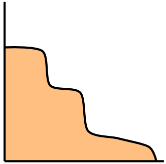

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For a monotone toric domain $X_\Omega \subset \mathbb{R}^{2n}$,

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The Arnold-Liouville theorem

Fix (M^{2n}, ω) and let $F = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ such that $\{H_i, H_j\} = 0$ for all i, j.

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- Let U be an open set such that $F(U)$ is simply-connected and does not contain critical values.

The Arnold-Liouville theorem

Fix (M^{2n}, ω) and let $F = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ such that $\{H_i, H_j\} = 0$ for all i, j.

- If $c \in \mathbb{R}^n$ is a regular value of F and $F^{-1}(c)$ is compact and connected, then $F^{-1}(c) \cong \mathbb{T}^n$.
- Let U be an open set such that $F(U)$ is simply-connected and does not contain critical values. Then there exists a diffeomorphism $\phi : F(U) \to \Omega$ and a symplectomorphism $\Phi : U \to \mathbb{X}_\Omega$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathbb{X}_\Omega \\ \downarrow F & & \downarrow \mu \\ F(U) & \xrightarrow{\phi} & \Omega \end{array}$$

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There is a difference equation related to L:

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and they induce a symplectomorphism

$$\Phi : \left\{ (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n+1} \mid \sum_i q_i = \sum_i p_i = 0 \right\} \longrightarrow \mathbb{R}^{2n}.$$

A deformation of the Toda lattice

For $c > 0$, let

$$H_c(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + e^{-c} \sum_{i=1}^{n+1} e^{c(q_i - q_{i+1})}.$$

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The flow of X_{H_c} converges to the billiard flow in

$$\{\mathbf{q} \in \mathbb{R}^n \mid q_i - q_{i+1} < 1, \text{ for all } i = 1, \dots, n\}.$$

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Theorem (Ostrover–R.–Sepe)

$\mathcal{S}^n \times \{\sum_i p_i = 0\}$ admits a toric action whose moment map is given by

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Corollary

The ball is symplectomorphic to $\mathcal{S}^n \times \mathcal{R}^n$.

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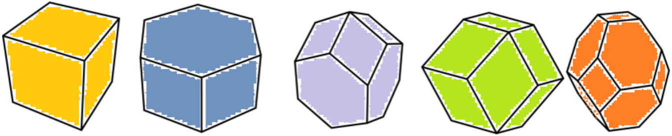

Figure: The Fedorov polyhedra

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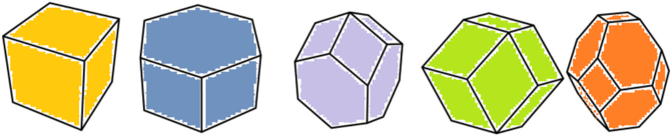

Figure: The Fedorov polyhedra

Question 2

Do other root systems B_n, C_n, D_n, G_2, etc, give rise to interesting symplectomorphisms?

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- The Lagrangian product of a simplex $\mathcal{S}^n$ and a symmetric region in $\mathbb{R}^n$ is symplectomorphic to a toric domain. (Ostrover– R.– Sepe, 2023)