Ribbon Concordances and Slice Obstructions: experiments and examples
Gauge Theory and Low Dimensional Topology Conference - University of Miami

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The goal: Finding concordances

- A concordance between knots $K_1, K_2 \subset S^3$ is a cylinder $F \subset S^3 \times [0, 1]$ such that its boundary consists of $K_1 \subset S^3 \times 0$ and $K_2 \subset S^3 \times 1$. 

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Ribbon concordances and ribbon knots

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- We say that a concordance is a ribbon concordance from $K_1$ to $K_2$ if there are no local minima. We say that a knot is ribbon if it admits a ribbon concordance to the unknot.
A ribbon disk

Ribbon knots can be thought of as an unlink with some ribbons attached.
Conjectures about slice and ribbon knots

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Conjectures about slice and ribbon knots

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- **Gordon’s conjecture**: If there is a ribbon concordance from $K_1$ to $K_2$, does this mean $K_2$ has smaller volume?
  - We generally expect that if there is a ribbon concordance from $K_1$ to $K_2$, then $K_2$ is simpler than $K_1$.
  - Zemke, 2019: If there is a ribbon concordance from $K_1$ to $K_2$, then the knot Floer homology of $K_2$ is a direct summand of that of $K_1$.
  - Agol, 2022: Ribbon concordance is a partial ordering
Obstructions to being slice

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- $Sq^1$ invariant for odd Khovanov homology: A refinement of the $s$ invariant corresponding to the first Steenrod square on odd Khovanov.
Method for finding ribbon concordances

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- Randomly add $k$ bands that increase the number of components.
- Check if the result is a link composed of a knot $K'$ along with $k$ unknotted, unlinked components. If so then you have obtained a ribbon concordance from $K$ to $K'$. 
Example of a ribbon disk we found
A longer band
An example where we needed two bands
A few optimizations

- After each band, we check some invariants of ribbon links, to make sure that what we have is still a ribbon link.
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  - signature is zero
  - multi-variate Fox-Milnor test
Summary of concordances found

- Of the 350 million knots of up to 19 crossings, 3.87 million have signature 0 and satisfy the Fox Milnor condition.
  - Of these:
    - 2,218,555 (57.3%) are not slice
    - 1,632,995 (42.2%) are ribbon
    - 17,991 (0.5%) unknown
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- Of the ribbon cobordisms:
  - 1,249,589 used 1 band
  - 381,703 used 2 bands
  - 1,644 used 3 bands
  - 59 used 4 bands
The ribbon concordance graph

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- As an undirected graph, it has 524 connected components (singleton removed)

Each component has a unique sink as a directed graph. The ribbon-slice conjecture states that the unknot is the unique sink of the component of slice knots.

The largest component is the unknot with 1,632,995 nodes. The second largest has $K_{11}^n_{34}$ as the sink and has 1673 nodes.
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Summary of obstructions

- 56.7% (2,194,701) Herald-Kirk-Livingston
- 5.0% (195,069) tau/epsilon/nu
- 5.0% (195,155) s-invariant (over $F_2$ or $F_3$)
- 6.5% (252,805) $Sq^1$ for odd Khovanov
- 0.0% (1) The Conway knot isn’t slice
- 1.2% (4,677) Ribbon concordances
- $Sq^1$ for even homology, and s with rational coefficients did not obstruct anything that others did not obstruct
Owens-Swenton computations for alternating knots

- Owens and Swenton have a method for generating ribbon disks for alternating knots
- Our sample has 203,488 alternating knots; we have ribbon disks for 81,577.
- They have ribbon disks for 82,015.
- They have 475 knots that we don’t. We have 37 knots they don’t.
Knots that share a zero surgery

Freedman, Gompf, Morrison and Walker’s potential method to find a counter-example to the smooth 4-dimensional Poincaré conjecture: Find $K$ that bounds a disk in $W \setminus B^4$ for a homotopy 4-sphere $W$, so that it doesn't bound a disk in the standard $B^4$.
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Manolescu and Piccirillo constructed some potential counter-examples thus: If two knots $K$ and $K'$ share a zero-surgery

$$S_0^3(K) = S_0^3(K')$$

and one of them is slice and the other is not, this can be used to construct the above. (They propose some pairs constructed using RBG links.)
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26,844 of these had $\leq 60$ crossings.
Knots that share a 0-surgery with a knot of $\leq 18$ crossings

For those where the larger knot had $\leq 60$ crossings:

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<th>Value</th>
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(Note there are 70 knots for which we know the status of the larger one and not the one in our sample.)
Comparison to previously expected behaviour

The decrease in complexity is generally confirmed:

- Ribbon concordances generally seem to drop volume and ranks of $\hat{\text{HFK}}$ and Khovanov homology quite significantly.
- We have not found any ribbon concordances from an alternating knot to a non-alternating knot.
- Ribbon concordances do not preserve rank of $\hat{\text{HFK}}$ of $\tilde{\text{Kh}}$ mod 8.
- This seemed true up to 17 crossings, but there are 18 crossing knots that violate this.
- Recently, Kyle Hayden also found ribbon knots that violate 1 mod 4.

Sinks in the directed graph of ribbon concordances:

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Thank you!

Thank you for the invitation and thank you for listening!