Ribbon Concordances and Slice Obstructions: experiments and examples

Gauge Theory and Low Dimensional Topology Conference - University of Miami

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(on work-in-progress, joint with Nathan Dunfield)

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The goal: Finding concordances

- A concordance between knots $K_1, K_2 \subset S^3$ is a cylinder $F \subset S^3 \times [0,1]$ such that its boundary consists of $K_1 \subset S^3 \times 0$ and $K_2 \subset S^3 \times 1$.

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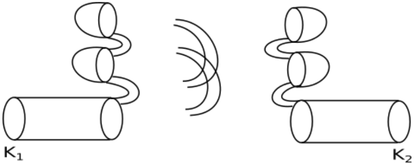
- Goal: find slice knots and concordances between various knots.

Ribbon concordances and ribbon knots

- Consider the concordance embedded in $S^3 \times [0,1]$ in such a way that the projection to $[0,1]$ is a Morse function, so that critical points are maxima, minima, or saddle points.



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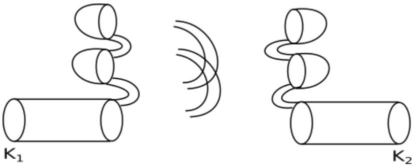
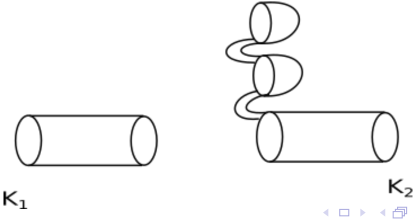
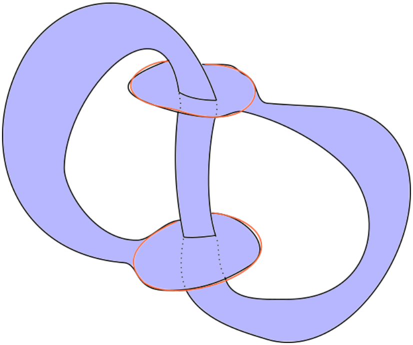


- We say that a concordance is a ribbon concordance from K_1 to K_2 if there are no local minima. We say that a knot is ribbon if it admits a ribbon concordance to the unknot.



A ribbon disk

Ribbon knots can be thought of as an unlink with some ribbons attached.



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  - Generalized version of this conjecture: each concordance class has a single sink
- Gordon's conjecture: If there is a ribbon concordance from K_1 to K_2, does this mean K_2 has smaller volume?
  - We generally expect that if there is a ribbon concordance from K_1 to K_2, then K_2 is simpler than K_1.
  - Zemke, 2019: If there is a ribbon concordance from K_1 to K_2, then the knot Floer homology of K_2 is a direct summand of that of K_1.
  - Agol, 2022: Ribbon concordance is a partial ordering

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- Sq^1 invariant for odd Khovanov homology: A refinement of the s invariant corresponding to the first Steenrod square on odd Khovanov.

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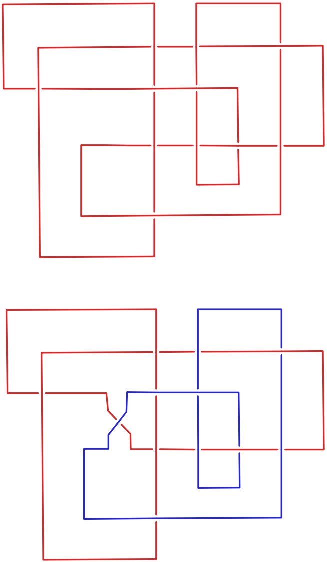
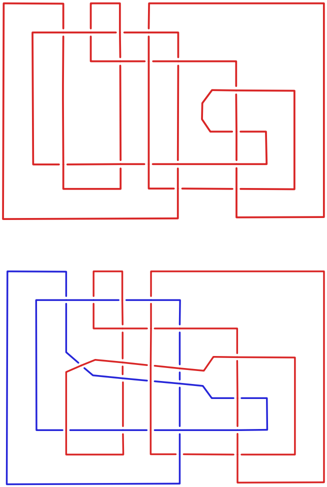
- Randomly add k bands that increase the number of components.
- Check if the result is a link composed of a knot K' along with k unknotted, unlinked components. If so then you have obtained a ribbon concordance from K to K'.

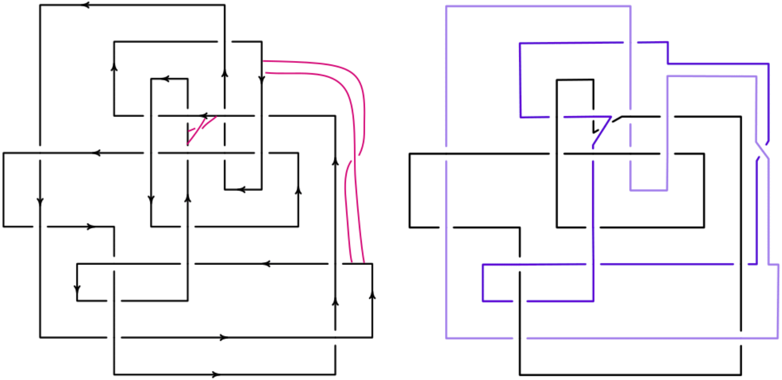
Example of a ribbon disk we found



A longer band



An example where we needed two bands



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  - linking number between components is zero
  - signature is zero
  - multi-variate Fox-Milnor test

Summary of concordances found

- Of the 350 million knots of up to 19 crossings, 3.87 million have signature 0 and satisfy the Fox Milnor condition
  Of these:
  - 2,218,555 (57.3%) are not slice
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- Of the ribbon cobordisms:
  - 1,249,589 used 1 band
  - 381,703 used 2 bands
  - 1,644 used 3 bands
  - 59 used 4 bands

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  - The largest is the unknot with 1,632,995 nodes. Second largest has K11n34 as the sink and has 1673 nodes.

Summary of obstructions

- 56.7% (2,194,701) Herald-Kirk-Livingston
- 5.0% (195,069) tau/epsilon/nu
- 5.0% (195,155) s-invariant (over F_2 or F_3)
- 6.5% (252,805) Sq^1 for odd Khovanov
- 0.0% (1) The Conway knot isn't slice
- 1.2% (4,677) Ribbon concordances
- Sq^1 for even homology, and s with rational coefficients did not obstruct anything that others did not obstruct

Owens-Swenton computations for alternating knots

- Owens and Swenton have a method for generating ribbon disks for alternating knots
- Our sample has 203,488 alternating knots; we have ribbon disks for 81,577.
- They have ribbon disks for $82,015$.
- They have 475 knots that we don't. We have 37 knots they don't.

Knots that share a zero surgery

Freedman, Gompf, Morrison and Walker's potential method to find a counter-example to the smooth 4-dimensional Poincaré conjecture: Find K that bounds a disk in $W \setminus B^4$ for a homotopy 4-sphere W, so that it doesn't bound a disk in the standard B^4

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Manolescu and Piccirillo constructed some potential counter-examples thus: If two knots K and K' share a zero-surgery

$$S_0^3(K) = S_0^3(K')$$

and one of them is slice and the other is not, this can be used to construct the above. (They propose some pairs constructed using RBG links.)

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$26,844$ of these had ≤ 60 crossings

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For those where the larger knot had ≤ 60 crossings:

Base slice	other slice	
-1	-1	1639
-1	0	3293
0	-1	11
0	0	180
0	1	59
1	0	2236
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(Note there are 70 knots for which we know the status of the *larger* one and not the one in our sample.)

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Sinks in the directed graph of ribbon concordances

- every component we found had a unique sink

Thank you!

Thank you for the invitation and thank you for listening!