Self-organized criticality and Tropical Geometry

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First Part: Self-Organized Criticality

Figure: *La trahison des images*, 1928, René Magritte

*Figure: La trahison des images, 1928, René Magritte*
The naturals under addition

Figure: *Ce n’est pas des mathématiques.* A super-computer
CV by Leonardo (30 years old)

Figure: “and in paiting I am as good as anyone”
Los cuadernos de Leonardo

Figure: “all branches of the tree, in each of their developments, together equal the thickness of the tree”
Las citas de Leonardo

Figure: Physical Review Letters, 2011
Las citas de Leonardo

Tree Branching: Leonardo da Vinci’s Rule versus Biomechanical Models

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Abstract
This study examined Leonardo da Vinci’s rule (i.e., the sum of the cross-sectional area of all tree branches above a branching point at any height is equal to the cross-sectional area of the trunk or the branch immediately below the branching point) using simulations based on two biomechanical models: the uniform stress and elastic similarity models. Model calculations of the daughter/mother ratio (i.e., the ratio of the total cross-sectional area of the daughter branches to the cross-sectional area of the mother branch at the branching point) showed that both biomechanical models agreed with da Vinci’s rule when the branching angles of daughter branches and the weights of lateral daughter branches were small; however, the models deviated from da Vinci’s rule as the weights and/or the branching angles of lateral daughter branches increased. The calculated values of the two models were largely similar but differed in some ways. Field measurements of Fagus crenata and Abies homolepis also fit this trend, wherein models deviated from da Vinci’s rule with increasing relative weights of lateral daughter branches. However, this deviation was small for a branching pattern in nature, where empirical measurements were taken under realistic measurement conditions; thus, da Vinci’s rule did not critically contradict the biomechanical models in the case of real branching patterns, though the model calculations described the contradiction between da Vinci’s rule and the biomechanical models. The field data for Fagus crenata fit the uniform stress model best, indicating that stress uniformity is the key constraint of branch morphology in Fagus crenata rather than elastic similarity or da Vinci’s rule. On the other hand, mechanical constraints are not necessarily significant in the morphology of Abies homolepis branches, depending on the number of daughter branches. Rather, these branches were often in agreement with da Vinci’s rule.


Figure: PLOS One, 2014
The Sandpile Cellular Automaton

Figure: Xiuhcoatl, the super-computer
Zipf’s Law (from G. West, Scale)
Body Weight

HEART RATES OF ANIMALS

Heart rate (beats/min) vs. Body weight (gm) for various animals:
- Mouse
- Rat
- Guinea pig
- Rabbit
- Small dog
- Hare
- Large dog
- Sheep
- Human
- Ox
- Horse

The graph shows a logarithmic relationship between heart rate and body weight.
Metabolism

Figure: Metabolic rate of animals versus body mass.
Earthquake frequency by size

![Graph showing earthquake frequency by size with a power law relationship.](image-url)
Earthquake frequency by size and region

Figure: Power Law
The Sandpile Cellular Automaton

Figure: Intermediate State
The Sandpile Cellular Automaton

Figure: Intermediate State
The Sandpile Cellular Automaton

Figure: The unique final state
The Sandpile Cellular Automaton

A billion grains of sand at the center: Start.

Figure: A very large table
The Sandpile Cellular Automaton
Self-Organized Criticality

PHYSICAL REVIEW LETTERS

Self-Organized Criticality: An Explanation of $1/f$ Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld

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(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

PACS numbers: 05.40.Đı, 05.40.-a

Figure: The most cited paper in Physics in the 90’s
Self-Organized Criticality

Figure: The original computer calculations.
Real Sand
SOC Timeline

- Bak, Tang, Wiesenfeld (1987)
  - SOC

- Kadanoff et al. (1989)
  - Scaling and universality in avalanches.

- Dhar (1990)
  - Group structure.

- Manna (1991)
  - Calculation of exponents.

- Caracciolo, Paoletti, Sportiello (2010)
  - Patterns of piecewise linear.

- Kalinin, Shkolnikov (2016)
  - Tropical Geometry and Sandpiles

- Bhupatiraju, Hanson, Järal (2016)
  - Formal proofs of SOC.
The Sandpile Cellular Automaton

Brains May Teeter Near Their Tipping Point

In a renewed attempt at a grand unified theory of brain function, physicists now argue that brains optimize performance by staying near — though not exactly at — the critical point between two phases.

Figure: Is the brain in SOC?
First relation to geometry

• This can be generalized to any graph $G$ (finite, with a sink).
• The configuration space of this discrete dynamical system is meant to be though of as the space of divisors of a graph (or tropical curve).
• There is the subgroup of stable configurations,
• and the subgroup of recurrent configurations (a stable configuration is recurrent if it can be obtained from any other configuration by adding chips and stabilizing.) Think probability one in the Markov chain.
First relation to geometry

- The **sandpile group** is the set of recurrent configurations.
- This is the same as the ”tropical jacobian” of the ”tropical curve” $J(G)$.
- It has as many elements as spanning trees has $G$, that is to say, the determinant of the ”tropical laplacian” (matrix tree theorem).
- But this relation to geometry is NOT what we mean to discuss today.
Figure: Notice the thin graphs inside the triangles
Fig. 2. In (A), (B), (C) and (D), a very large number \( N \) of grains of sand is placed at the origin of the everywhere empty integral lattice, the final relaxed state shows fractal behavior. Here, as we advance from (A) to (D), we see successive sandpiles for \( N = 10^3 \) (A), \( 10^4 \) (B), \( 10^5 \) (C), and \( 10^6 \) (D), rescaled by factors of \( \sqrt{N} \). In (E), we zoom in on a small region of (D) to show its intricate fractal structure, and, finally, in (F), we further zoom in on a small portion of (E). We can see proportional growth occurring in the patterns as the fractal limit appears. The balanced graphs inside the roughly triangular regions of (F) are tropical curves.
The Laplacian

• The toppling function $H(i, j)$ defined as follows: Given an initial state $\phi$ and its relaxation $\phi^\circ$, the value of $H(i, j)$ equals the number of times that there was a toppling at the vertex $(i, j)$ in the process taking $\phi$ to $\phi^\circ$.

• The discrete Laplacian of $H$ is defined by the net flow of sand,

$$\Delta H(i, j) := H(i-1, j) + H(i+1, j) + H(i, j-1) + H(i, j+1) - 4H(i, j).$$
The Laplacian determines the evolution

The toppling function is clearly non-negative on $\Omega$ and vanishes at the boundary. The function $\Delta H$ completely determines the final state $\varphi^\circ$ by the formula:

$$\varphi^\circ(i,j) = \varphi(i,j) + \Delta H(i,j).$$  \hfill \text{(1)}
The Least Action Principle

It can be shown by induction that the toppling function $H$ satisfies the *Least Action Principle*: if $\varphi(i, j) + \Delta F(i, j) \leq 3$ is stable, then $F(i, j) \geq H(i, j)$. Ostojic noticed that $H(i, j)$ is a piecewise quadratic function in the usual sandpile.
Tropical Sandpiles

Consider a state $\varphi$ which consists of 3 grains of sand at every vertex, except at a finite family of points

$$P = \{p_1 = (i_1, j_1), \ldots, p_r = (i_r, j_r)\}$$

where we have 4 grains of sand:

$$\varphi := \langle 3 \rangle + \delta_{p_1} + \cdots + \delta_{p_r} = \langle 3 \rangle + \delta_P.$$  \hspace{1cm} (2)

The state $\varphi^\circ$ and the evolution of the relaxation can be described by means of tropical geometry. This was discovered by Caracciolo et al. while a rigorous mathematical theory to prove this fact has been given by Kalinin and Shkolnikov.

It is a remarkable fact that, in this case, the toppling function $H(i, j)$ is piecewise linear (after passing to the scaling limit).
A Tropical Sandpile (Kalinin-Shkolnikov)

Fig. 4. The evolution of $\langle 3 \rangle + \delta_P$. Sand falling outside the border disappears. Time progresses in the sequence (A), (B), (C), and finally (D). Before (A), we add grains of sand to several points of the constant state $\langle 3 \rangle$ (we see their positions as blue disks given by $\delta_P$). Avalanches ensue. At time (A), the avalanches have barely started. At the end, at time (D), we get a tropical analytic curve on the square $\Omega$. White represents the region with 3 grains of sand while green represent 2, yellow represents 1, and red represents the zero region. We can think of the blue disks $\delta_P$ as the genotype of the system, of the state $\langle 3 \rangle$ as the nutrient environment, and of the thin graph given by the tropical function in (D) as the phenotype of the system.

Figure: Time advances from left to right
A Tropical Sandpile

A movie:
Idea of the Proof (1)

To prove this, one considers the family $\mathcal{F}_P$ of functions on $\Omega$ that are:

1. piecewise linear with integral slopes,
2. non-negative over $\Omega$ and zero at its boundary,
3. concave, and
4. not smooth at every point $p_i$ of $P$.

Let $F_P$ be the pointwise minimum of functions in $\mathcal{F}_P$. Then $F_P \geq H$ by the Least Action Principle.
Lemma

*In the scaling limit* $H = F_P$.

**A sketch of a proof.** K-S introduce the wave operators $W_p$ at the cellular automaton level and the corresponding tropical wave operators $G_p$. Given a fixed vertex $p = (i_0, j_0)$, we define the wave operator $W_p$ acting on states $\varphi$ of the sandpile as:

$$W_p(\varphi) := (T_p(\varphi + \delta_p) - \delta_p)^\circ,$$

where $T_p$ is the operator that topples once the state $\varphi + \delta_p$ at $p$ if at all possible. In a computer simulation, the application of this operator looks like a wave of topplings spreading from $p$, while each vertex topples at most once.
The wave operator (1)

Fig. 5. Top: The action of the wave operator $W_p$ on a tropical curve. The tropical curve steps closer to $p$ by an integral step. Thus $W_p$ shrinks the face that $p$ belongs to; the combinatorial morphology of the face that $p$ belongs to, can actually change. Bottom: The function $G_p \Omega$, where $p$ is the center of the circle, and its associated omega-tropical curve are shown.
The first important property of $W_p$ is that, for the initial state $\varphi := \langle 3 \rangle + \delta_P$, we can achieve the final state $\varphi^\circ$ by successive applications of the operator $W_{p_1} \circ \cdots \circ W_{p_r}$ a large but finite number of times (in spite of the notation):

$$\varphi^\circ = (W_{p_1} \cdots W_{p_r})^\infty \varphi + \delta_P.$$ 

This process decomposes the total relaxation $\varphi \mapsto \varphi^\circ$ into layers of controlled avalanching.
The wave operator (3)

The second important property of the wave operator $W_p$ is that its action on a state $\varphi = \langle 3 \rangle + \Delta f$ has an interpretation in terms of tropical geometry. To wit, whenever $f$ is a piecewise linear function with integral slopes that, in a neighborhood of $p$, is expressed as $a_{i_0j_0} + i_0x + j_0y$, we have that

$$W_p(\langle 3 \rangle + \Delta f) = \langle 3 \rangle + \Delta W(f),$$

where $W(f)$ has the same coefficients $a_{ij}$ as $f$ except one, namely $a'_{i_0j_0} = a_{i_0j_0} + 1$. This is to emulate the fact that the support of the wave is exactly the face where $a_{i_0j_0} + i_0x + j_0y$ is the leading part of $f$. 
The dynamical system

• We will write $G_p := W_p^\infty$ to denote the operator that applies $W_p$ to $\langle 3 \rangle + \Delta f$ until $p$ lies in the corner locus of $f$.

• It has an elegant interpretation in terms of tropical geometry: $G_p$ increases the coefficient $a_{i_0j_0}$ corresponding to a neighborhood of $p$ lifting the plane lying above $p$ in the graph of $f$ by integral steps until $p$ belongs to the corner locus of $G_pf$. Thus $G_p$ has the effect of pushing the tropical curve closer towards $p$ until it contains $p$. 
From the properties of the wave operators, it follows immediately that:

$$F_P = (G_{p_1} \cdots G_{p_r})^\infty 0,$$

where $0$ is the function which is identically zero on $\Omega$.

Each intermediate function $(G_{p_1} \cdots G_{p_r})^k 0$ is less than $H$ since they represent partial relaxations, but their limit belongs to $F_P$, and this, in turn, implies that $H = F_P$. 

End of the Proof
The Tropical Sandpile model (KGPSKL) (1)

Now, we define a new model, tropical sandpile (TS), reflecting structural changes when a sandpile evolves. The definition of this dynamical system is inspired by the mathematics of the previous section, and TS is not a cellular automaton but it exhibits SOC.
The Tropical Sandpile model (2)

The dynamical system lives on the convex set $\Omega = [0, N] \times [0, N]$; namely, we will consider $\Omega$ to be a very large square. The input data of the system is a large but finite collection of points $P = \{p_1, \ldots, p_r\}$ with integer coordinates on the square $\Omega$. Each state of the system is an $\Omega$-tropical series (and the associated $\Omega$-tropical curve).
Tropical Series

Definition
An $\Omega$-tropical series is a piecewise linear function in $\Omega$ given by:

$$F(x, y) = \min_{(i, j) \in A} (a_{ij} + ix + jy),$$

where the set $A$ is not necessarily finite and $F|_{\partial \Omega} = 0$. An $\Omega$-tropical curve is the set where $F$ is not smooth. Each $\Omega$-tropical curve is a locally finite graph satisfying the balancing condition.
The Tropical Sandpile model (3)

The initial state for the dynamical system is $F_0 = 0$, and its final state is the function $F_P$ defined previously. Notice that the definition of $F_P$, while inspired by sandpile theory, uses no sandpiles or cellular automaton whatsoever. Intermediate states $\{F_k\}_{k=1,\ldots,r}$ enjoy the property that $F_k$ is not smooth at $p_1, p_2, \ldots, p_k$, i.e. the corresponding tropical curve passes through these points.
In other words, the tropical curve is first attracted to the point $p_1$. Once it manages to pass through $p_1$ for the first time, it continues to try to pass through $\{p_1, p_2\}$. Once it manages to pass through $\{p_1, p_2\}$, it proceeds in the same manner towards $\{p_1, p_2, p_3\}$. The same process is repeated until the curve passes through all of $P = \{p_1, \ldots, p_r\}$. 
The Tropical Sandpile model (5)

\[ \Omega = [0, 100] \times [0, 100] \]
The Tropical Sandpile model (6)

We will call the modification $F_{k-1} \rightarrow F_k$ the $k$-th avalanche and it occurs as follows: To the state $F_{k-1}$, we apply the tropical operators $G_{p_1}, G_{p_2}, \ldots, G_{p_k}; G_{p_1}, \ldots$ in cyclic order until the function stops changing; the discreteness of the coordinates of the points in $P$ ensures that this process is finite\(^1\). Again, as before, while sandpile inspired, the operators $G_p$ are defined entirely in terms of tropical geometry without a mention to sandpiles.

There is a dichotomy: Each application of a $G_p$ either does something changing the shape of the current tropical curve (in this case $G_p$ is called an active operator), or does nothing, leaving the curve intact (if $p$ already belongs to the curve).

\(^1\)If the coordinates of the points in $P$ are not integers, the model is well-defined, but we need to take a limit which is not suitable for computer simulations.
Definition
The size of the $k$-th avalanche is the number of distinct active operators $G_{pi}$ (that actually do something) used to take the system from the self-critical state $F_{k-1}$ to the next self-critical state $F_k$, divided by $k$. In particular, the size $s_k$ of the $k$-th avalanche is a number between zero and one: $0 \leq s_k \leq 1$, and it estimates the proportional area of the picture which changed during the avalanche.
Spatial SOC

Fig. 6. The first two pictures show the comparison between the classical (A) and tropical (B) sandpiles for \(|P| = 100\) generic points on the square. In (C), the square \(\Omega\) has side \(N = 1000\); a large number (\(|P| = 40000\)) of grains has been added, showing the spatial SOC behavior on the tropical model compared to the identity (D) of the sandpile group on the square of side \(N = 1000\). In the central square region on (C) (corresponding to the solid block of the otherwise fractal unit), we have a random tropical curve with edges on the directions \((1, 0)\), \((0, 1)\), and \((\pm 1, 1)\), which is given by a small perturbation of the coefficients of the tropical polynomial defining the usual square grid.
The Tropical Sandpile model (8)

In the previous example, as the number of points in $P$ grows and becomes comparable to the number of lattice points in $\Omega$, the tropical sandpile exhibits a phase transition going into spatial SOC (fractality). This provides the first evidence in favor of SOC on the tropical sandpile model, but there is a more subtle spatio-temporal SOC behavior that we proceed to exhibit in the following slides.

While the ordering of the points from the first to the $r$-th is important for the specific details of the evolution of the system, its statistical behavior and the final state are insensitive to it. This we called the Abelian property.
SOC in tropical geometry

The tropical sandpile dynamics exhibits slow driving avalanching.

Once the tropical dynamical system stops after $r$ steps, we can ask ourselves what the statistical behavior of the number $N(s)$ of avalanches of size $s$ is like. We posit that the tropical dynamical system exhibits spatio-temporal SOC behavior, namely, we have a power law:

$$\log N(s) = \tau \log s + c.$$  

To confirm this, we have performed experiments in the supercomputing clusters ABACUS and Xiuhcoatl at Cinvestav (Mexico City); the code is available on GitHub. In the figure below, we see the graph of $\log N(s)$ vs $\log s$ for the tropical (piecewise linear, continuous) sandpile dynamical system, the resulting experimental $\tau$ in this case was $\tau \sim -0.9$. 
Figure 6: A) The power law for sandpiles. The logarithm of the frequency is linear on the logarithm of the avalanche size, except near the right where the avalanches have bigger size than the half of the system. $\Omega = [0, 100]^2$, initially filled with 3 grains everywhere, followed by $10^6$ dropped grains. B) The power-law for the Tropical (piece-wise linear, continuous) dynamical system. In this computer experiment $\Omega$ has a side of 1000 units and we throw at random a set $P$ of 10000 points (a random large genotype) using two super-computer clusters.
The dichotomy between continuous and discrete models of our paper (already appearing in the biological models of Turing) has an analogue in topological string theory.

Iqbar-Vafa-Nekrasov-Okunkov have argued that, when we "probe space-time beyond the scale $\alpha'$ and going below Planck's scale", the "resulting fluctuations of space time" can be computed with a classical cellular automaton (a melting crystal) representing quantum gravitational foam.

Their theory is a three-tier system whose levels are respectively classical geometry (Kähler gravity), tropical geometry (toric manifolds) and cellular automata (a discrete melting crystal).
Work in progress: SOC in "Quantum Gravity" (with R. López Vázquez)

The theory described above is also a three-tier system whose levels are classical complex algebraic geometry, tropical geometry (analytic tropical curves) and cellular automata (sandpiles). This seems to be not a coincidence and suggests connections between our model for SOC and their model for quantum gravitational foam.
We have progressed by proving so far that, at the level of partition functions:

\[ Z_{\text{Sandpile}} = Z_{\text{IVNO}}, \]

by using the Temperley bijection for the dual graph, and ONLY for the hexagonal tiling.

(DETAILS: My 2nd talk next week)