Homological Mirror Symmetry for Theta Divisors

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[ACLL]

Haniya Azam, Catherine Cannizzo, Heather Lee, Chiu-Chu Melissa Liu

Complex moduli of theta divisors in principally polarized abelian variety (ppav)

Siegel upper half space $\mathcal{H}_g := \{\tau = B + i\Omega \in S_g(\mathbb{C}) \mid \Omega \text{ positive definite}\}$

$g \times g$ symmetric matrices

Abelian variety V_τ of dimension g

$$V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g \ni (x_1, \dots, x_g) = (e^{2\pi i z_1}, \dots, e^{2\pi i z_g})$$

$$\updownarrow \qquad\qquad \uparrow \exp$$

$$V_\tau^+ = \mathbb{C}^g / \mathbb{Z}^g + \tau\mathbb{Z}^g \ni (z_1, \dots, z_g)$$

multiplicative action: $\tau n \cdot (x_1, \dots, x_g) = (e^{2\pi i (\tau n)_1} x_1, \dots, e^{2\pi i (\tau n)_g} x_g)$

Theta divisor Θ_τ

line bundle $\mathcal{L}_\tau = (\mathbb{C}^*)^g \times \mathbb{C} / \tau\mathbb{Z}^g \longrightarrow V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g$

$$\tau n . (x_1, \dots, x_g, v) = \left(e^{2\pi i (\tau n)_1} x_1, \dots, e^{2\pi i (\tau n)_g} x_g, \; e^{-\pi i n^T \tau n} x_1^{-n_1} \cdots x_g^{-n_g} v \right)$$

Riemann theta function $\vartheta(\tau, \cdot) : (\mathbb{C}^*)^g \longrightarrow \mathbb{C}$

$$\vartheta(\tau, x) = \sum_{n \in \mathbb{Z}^g} x_1^{n_1} \cdots x_g^{n_g} e^{\pi i n^T \tau n}$$

descends to a section $\vartheta \in H^0(V_\tau, \mathcal{L}_\tau) = \mathbb{C}\vartheta$

$$\Theta_\tau := \{ \vartheta(\tau, x) = 0 \} \subseteq V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g$$

(when $g = 2$
Θ_τ genus 2 curve)

Principal polarization

$$c_1(\mathcal{L}_\tau) = [\omega_{V_\tau}] = \sum_{j=1}^{g} a_j \wedge b_j \in H^{1,1}(V_\tau) \cap H^2(V_\tau, \mathbb{Z})$$

$\{a_j, b_j\}$ dual to $\{\alpha_j, \beta_j\}$ an integral symplectic basis of $H_1(V_\tau, \mathbb{Z})$

(More general polarization $[\omega] = \sum \delta_j a_j \wedge b_j$, $\delta_j \in \mathbb{Z}$, $\delta_j \mid \delta_{j+1}$)

Moduli $\mathcal{A}_g$ of g dimensional ppav's $(V_\tau, c_1(\mathcal{L}_\tau))$

Moduli of ppav

+ Torelli structure

choice of integral symplectic basis $\{\alpha_j, \beta_j\}$ of

$$\left(H_1(V_\tau, \mathbb{Z}),\ c_1(\mathcal{L}_\tau)\right) \cong \left(\mathbb{Z}^{2g},\ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\right)$$

$$\mathcal{H}_g \longrightarrow \mathcal{A}_g = \left[\mathcal{H}_g / Sp(2g, \mathbb{Z})\right]$$

$\circlearrowleft$ $Sp(2g, \mathbb{Z})$

$$\begin{bmatrix} A & C \\ E & D \end{bmatrix} \circ \tau = (A\tau + C)(E\tau + D)^{-1}$$

SYZ fibration

$$\begin{tikzcd}
 & \widetilde{\Theta}_\tau \arrow[r] \arrow[d, hook] & \Theta_\tau \arrow[d, hook] & \\
(x_1, \dots, x_g) \in \arrow[d, mapsto] & (\mathbb{C}^*)^g \arrow[r] \arrow[d] & V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g \arrow[d, "\pi_\tau^{SYZ}"] & T_F \text{ fiber} \arrow[l] \\
(\xi_1, \dots, \xi_g) \in & \mathbb{R}^g \arrow[r] & \mathbb{R}^g / \Omega\mathbb{Z}^g &
\end{tikzcd}$$

$$\xi_j = \frac{1}{2\pi} \log|x_j|$$

base of SYZ fibration

$$G_g = \left\{ \begin{bmatrix} A & C \\ 0 & D \end{bmatrix} \in Sp(2g,\mathbb{Z}) \right\} = \left\{ \begin{bmatrix} A & C \\ 0 & (A^T)^{-1} \end{bmatrix} : A \in GL_g(\mathbb{Z}),\ AC^T = CA^T \right\}$$

* G_g is the subgroup preserving $T_F = H_1(T_F,\mathbb{Z}) \subseteq H_1(V_\tau,\mathbb{Z})$

$$T_F = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j$$

* G_g is generated by the following two subgroups

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} : A \in GL_g(\mathbb{Z}) \right\} \cong GL_g(\mathbb{Z}) \qquad \tau \mapsto A\tau A^T$$

$$\left\{ \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} : C^T = C \right\} \cong S_g(\mathbb{Z}) \qquad \tau \mapsto \tau + C$$

* $Sp(2g,\mathbb{Z})$ is generated by G_g and $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ $\qquad \tau \mapsto -\tau^{-1}$

[Yingdi Qin 2020 thesis]

$\mathcal{A}_g^F := [\mathbb{H}_g / G_g]$ moduli of pairs (V_τ, T_F)

moduli of principally polarized and SYZ fibered abelian variety

Pure tropical Siegel space

$$\mathcal{H}_g^{trop,p} := \{\Omega \in S_g(\mathbb{R}) \mid \Omega \text{ positive definite}\}$$

[Brannetti - Melo - Viviani]
[Chan - Melo - Viviani]

$$\begin{tikzcd}
\mathcal{H}_g = \underbrace{S_g(\mathbb{R})}_{\mathbb{R}^{g(g+1)/2}} \times \mathcal{H}_g^{trop,p} \arrow[d] \arrow[drr] & & \\
{[\mathcal{H}_g / S_g(\mathbb{Z})] = [\underbrace{S_g(\mathbb{R})/S_g(\mathbb{Z})}_{(S^1)^{g(g+1)/2}} \times \mathcal{H}_g^{trop,p}]} \arrow[rr] \arrow[d] & & \mathcal{H}_g^{trop,p} \arrow[d] \\
A_g^F := [\mathcal{H}_g / G_g] \arrow[rr] \arrow[d] & & A_g^{trop,p} = [\mathcal{H}_g^{trop,p} / GL_g(\mathbb{Z})] \\
A_g = [\mathcal{H}_g / Sp(2g;\mathbb{Z})] & &
\end{tikzcd}$$

SYZ fibered ppav $\longmapsto$ its base

moduli of pure tropical ppav's

Kähler moduli of Y

mirror LG model
(Y, v_0)

Theorem [ACLL]

(1) $\mathcal{H}_g^{trop,P}$ is the kähler space $K(Y)$ of Y.

space of all kähler classes

$(S_g(\mathbb{R})/S_g(\mathbb{Z})) \times \mathcal{H}_g^{trop,P}$ is the complexified kähler space $K_{\mathbb{C}}(Y)$ of Y

$K_{\mathbb{C}}(Y) = \{ \omega^{\mathbb{C}} \in H^2(Y, \mathbb{C}) : \operatorname{Im} \omega^{\mathbb{C}} \in K(Y) \} / \operatorname{im} H^2(Y, \mathbb{Z})$

(2) $A_g^{trop,P}$ is the kähler moduli of Y

$K(Y)/\operatorname{Aut}(Y)$

A_g^F is the complexified kähler moduli of Y.

(3) when $g=2$, $\dim K(Y) = 3$

kähler cones $\longleftrightarrow$ 3 cones in Voronoi decomposition

Generalized SYZ mirror [Abouzaid - Auroux - Katzarkov 2020]

SYZ mirror to $Bl_{\Theta_\tau \times \{0\}} V_\tau \times \mathbb{C}$

$\boxed{\text{LG model } (\tilde{Y}_\tau, \tilde{v}_0) \text{ mirror to } \tilde{\Theta}_\tau}$ ($\tilde{Y}_\tau$ toric CY of infinite type)

Moment polyhedron for $\tilde{Y}_\tau$

$$\Delta_\Omega = \Big\{ (\xi, \eta) \in \mathbb{R}^{g+1} \;\Big|\; \eta \geq \varphi(\xi) = \max_{n \in \mathbb{Z}^g} \{ \langle \xi, n \rangle + \underbrace{k(n)} \} \Big\}$$

$\xi \in \mathbb{R}^g$ $\qquad$ $k(n) = -\frac{1}{2} n^T \Omega n$

$$= \bigcap_{n \in \mathbb{Z}^g} \Big\{ (\xi, \eta) \in \mathbb{R}^{g+1} \;\Big|\; L_n(\xi, \eta) = -\langle \xi, n \rangle + \eta - k(n) \geq 0 \Big\}$$

polyhedron with facets $\{ L_n(\xi, \eta) = 0 \}$ normal to $\nu_m = \begin{pmatrix} -n_1 \\ \vdots \\ -n_g \\ 1 \end{pmatrix}$

Complex structure on $\tilde{Y}_\tau$ (invariant under $(\mathbb{C}^*)^{g+1}$)

Each vertex of Δ_Ω is of the form $\sigma = \sigma_{n^1, \dots, n^{g+1}} = \bigcap_{j=1}^{g+1} \{ L_{n^j}(\xi, \eta) = 0 \}$

Complex toric coordinates $t \in (\mathbb{C}^*)^{g+1} \leftrightsquigarrow$ inhomogeneous coordinates $y^\sigma \in \mathbb{C}^{g+1}$

$$t_1 = \prod_{j=1}^{g+1} (y_j^\sigma)^{-n_1^j}, \; \dots, \; t_g = \prod_{j=1}^{g+1} (y_j^\sigma)^{-n_g^j}, \quad t_{g+1} = y_1^\sigma \cdots y_{g+1}^\sigma$$

Symplectic structure on $\tilde{Y}_\tau$ (invariant under $U(1)^3$) [Guillemin 1994, Kanazawa-Lau 2016]

Dual Kähler potential $\tilde{G} : \Delta_\Omega \to \mathbb{R}$

$$\tilde{G}(\xi,\eta) = \sum_{n\in\mathbb{Z}^g} \chi(\xi,\eta)\, L_n(\xi,\eta) \log L_n(\xi,\eta)$$

Kähler potential: $\tilde{F}(\rho) = \langle \rho, (\xi,\eta)\rangle - \tilde{G}$ $\qquad \rho = \left(\rho_j = \frac{\partial G}{\partial \xi_j}\right)_{j=1}^{g+1}, \quad \xi_{g+1} := \eta$

$$t_j = e^{2\pi(\rho_j + i\theta_j)}$$

Kähler form: $\omega = \sum_{k=1}^{g+1} d\xi_k \wedge d\theta_k = \sum_{j,k=1}^{g+1} \Psi_{jk}\, d\rho_j \wedge d\theta_k$ $\qquad (\Psi_{jk})^{-1} = \Psi^{jk} = \frac{\partial^2 G}{\partial \xi_j \partial \xi_k}$

Superpotential $\tilde{v}_0 : \tilde{Y}_\tau \to \mathbb{C}$ (invariant under $(\mathbb{C}^*)^g \subseteq (\mathbb{C}^*)^{g+1}$)

holomorphic function extending t_{g+1}, Symplectic fibration

Symplectic fibration with singular fiber $\tilde{v}^{-1}(0) = \bigcup_{n\in\mathbb{Z}^g} D_n$

B-field (invariant under $U(1)^3$)

$B = \mathrm{Re}(\tau)$ determines $[B_\tau] \in H^2(\tilde{Y}_\tau;\mathbb{R})$ via injective map $i^* : H^2(\tilde{Y}_\tau;\mathbb{R}) \to H^2(\tilde{v}_0^{-1}(\varepsilon);\mathbb{R})$

Choose $U(1)^3$ invariant $(1,1)$-form B_τ $\qquad i^* B_\tau = \sum_{j,k=1}^{g} B_{jk}\, dr_j \wedge d\theta_k$

$$\Omega r = \xi$$

LG model (Y_τ, V_0) mirror to Θ_τ

$\tau\mathbb{Z}^g$ action on $\tilde{Y}_\tau$

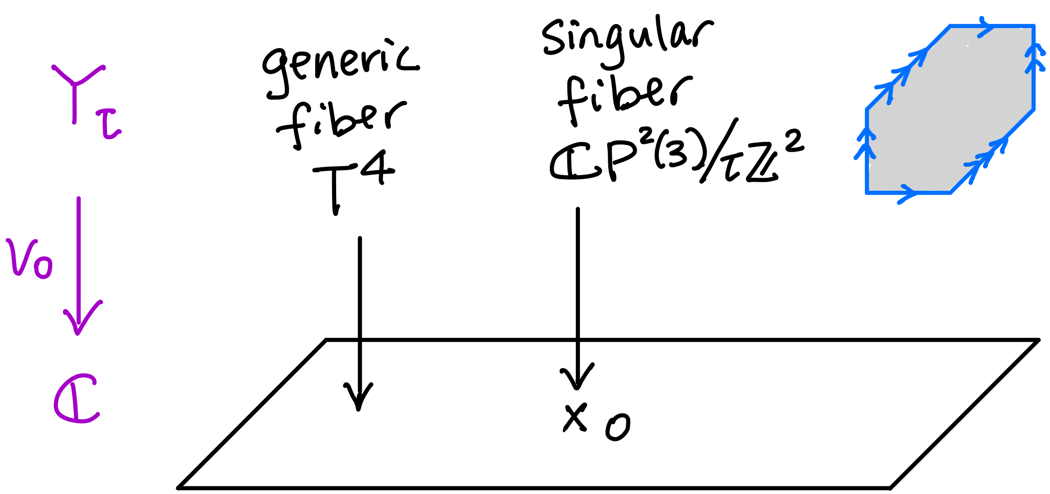
* $(\tau n)\cdot(t_1,\dots,t_g,t_{g+1}) = (t_{g+1}^{-n_1}t_1,\dots,t_{g+1}^{-n_g}t_g\,,\ t_{g+1})$
* preserves the complex structure, $\tilde{V}_0$, and $[\omega],[B]\in H^2(\tilde{Y}_\tau;\mathbb{R})$
* free on $\tilde{Y}'_\tau := \tilde{V}_0^{-1}(\mathbb{D})$ $\mathbb{D}$ = open unit disk.

$Y_\tau := \tilde{Y}_\tau^\epsilon/\tau\mathbb{Z}^g$, $\tilde{Y}_\tau^\epsilon := \tilde{V}_0^{-1}(\{|z|<\varepsilon\})$ quotient complex structure

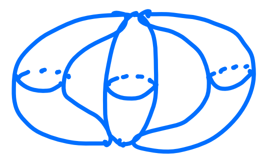
$V_0: Y_\tau \to \mathbb{C}$, generic fiber $\cong T^{2g}$ descends from $\tilde{V}_0|_{\tilde{Y}'_\tau}: \tilde{Y}'_\tau \to \mathbb{C}$

ω, B descends to Y

$\boxed{\text{When } g=2}$ Θ_τ = genus 2 curve



Critical locus



Chamber I

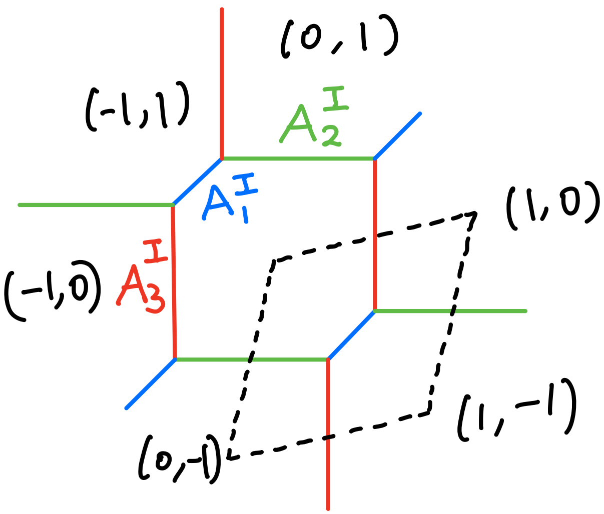
$$\begin{cases} \Omega_{12} = \Omega_{21} > 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

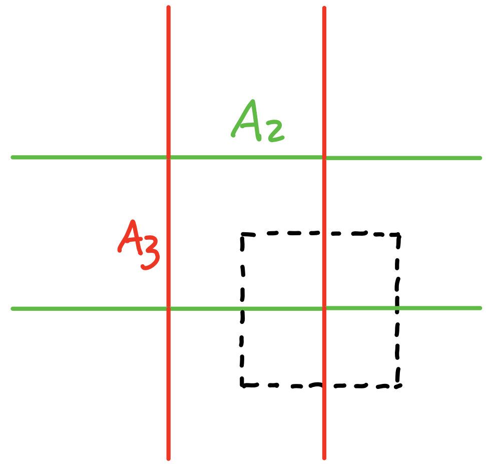
$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1^{I} + A_2^{I} & A_1^{I} \\ A_1^{I} & A_1^{I} + A_3^{I} \end{pmatrix}$$

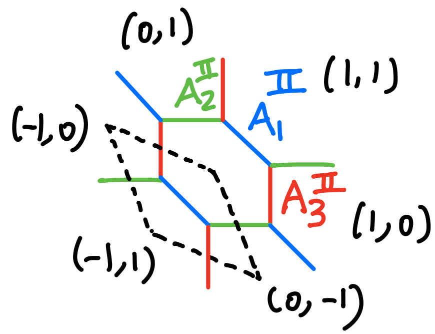
Chamber II

$$\begin{cases} \Omega_{12} = \Omega_{21} < 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1^{II} + A_2^{II} & -A_1^{II} \\ -A_1^{II} & A_1^{II} + A_3^{II} \end{pmatrix}$$







Atiyah flop example

$$\Omega = \begin{pmatrix} 1+\lambda & \lambda \\ \lambda & 1+\lambda \end{pmatrix}$$

$\lambda > 0$

$A_1^{I} = \lambda$

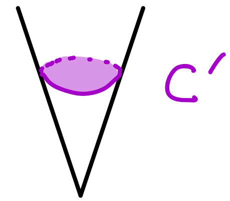
$A_2^{I} = A_3^{I} = 1$

$\lambda < 0$

$A_1^{II} = -\lambda$

$A_2^{II} = A_3^{II} = 1+2\lambda$

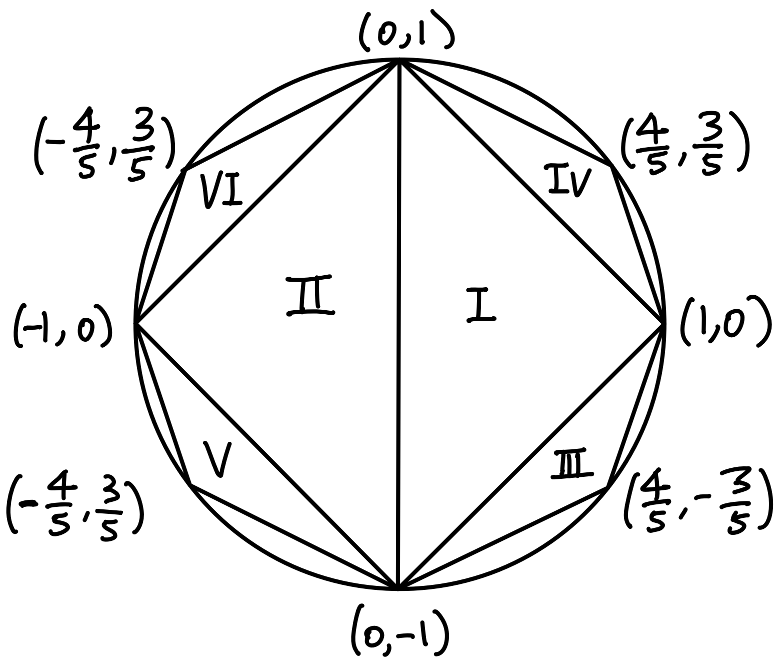
$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} z+y & x \\ x & z-y \end{pmatrix}$$



positive definite $\iff$ $\{(x,y,z) \in \mathbb{R}^3 \mid z > \sqrt{x^2+y^2}\}$

i.e. a cone over $C' \times \{1\} \subseteq \mathbb{R}^3$

$C' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$



* Same as Voronoi decomposition 1908

* Cannizzo's ray corresponds to the point $(\frac{1}{2}, 0)$ in C'_{I}.

$GL(2,\mathbb{Z}) \ni h$ action: $h\Omega = h\Omega h^T$

All chambers are in the same $GL(2,\mathbb{Z})$-orbit.

Homological Mirror Symmetry

$$\begin{tikzcd}
D^b_{\mathcal{L}}\,\mathrm{Coh}(V_\tau) \arrow[r, "L^*:\, \mathcal{L} \longmapsto \mathcal{L}|_{\Theta_\tau}"] \arrow[d] & D^b_{\mathcal{L}}\,\mathrm{Coh}(\Theta_\tau) \arrow[d] \\
H^0\,\mathrm{Fuk}(V_\tau^\vee) \arrow[r, "U:\, \ell \longmapsto U(\ell)"] & H^0\,\mathrm{FS}(Y_\tau, v_0)
\end{tikzcd}$$

[Polishchuk-Zaslow '98] $g=1$

[Fukaya 2002]

[Cannizzo 2020] $\tau = \frac{i}{2\pi} \log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

t large

[ACLL]

$\boxed{\text{HMS for the fiber abelian variety}}$

[Polishchuk–Zaslow 1998] $g=1$
[Fukaya 2002]

$(\xi, \eta, \theta, \theta_\eta) \in Y_\tau \qquad \xi = (\xi_1, \dots, \xi_g)\,,\quad \theta = (\theta_1, \dots, \theta_g)$

<u>On a fiber</u> $\ni (\xi, \theta)$ $\quad$ η = function of ξ, θ_η = constant

$$\omega_\tau = \sum_{k=1}^{g} d\xi_k \wedge d\theta_k = \sum_{j,k=1}^{g} \Omega_{jk}\, dr_j \wedge d\theta_k \qquad \xi = \Omega r$$

Complex side

$V_\tau^+ = \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$

$\tau = B + i\Omega$

$$\mathcal{L}_{\mathbb{k},[z]} := \mathcal{L}_\tau^{\otimes \mathbb{k}} \otimes \mathbb{L}_{[z]}$$

$z = a + \tau b \in V_\tau^+\,,\quad a, b \in \mathbb{R}^g$

$V_\tau^+ \xrightarrow{\cong} \mathrm{Pic}^0(V_\tau^+)\,,\quad [z] \mapsto \mathbb{L}_{[z]} = T_{[z]}^* \mathcal{L}_\tau \otimes \mathcal{L}_\tau^{-1}$

$T_{[z]} : V_\tau^+ \to V_\tau^+\,,\quad [u] \mapsto [u+z]$

Symplectic side

$V_\tau^\vee \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \ni (r, \theta)$

$$\omega_\tau^{\mathbb{C}} = \sum_{j,k=1}^{g} (B_{jk} + i\Omega_{jk})\, dr_j \wedge d\theta_k$$

$$\hat{\ell}_{\mathbb{k},[z]} := (\ell_{\mathbb{k},[b]}, \mathcal{E}_{[a]})$$

$[a], [b] \in (\mathbb{R}/\mathbb{Z})^g$

$\ell_{\mathbb{k},[b]} := \{(r, \theta) \in \mathbb{R}^{2g}/\mathbb{Z}^{2g} \mid \theta = b - \mathbb{k} r\}$

$\mathcal{E}_{[a]}$ trivial line bundle $\ell_{\mathbb{k},[b]} \times \mathbb{C}$
with flat $U(1)$ connection
$\nabla_{[a]} = d - 2\pi i\, a\, dr$

Fukaya-Seidel category of (Y_τ, v_0)

Generating objects: $\widehat{L}_{\mathbb{K},b} = (L_{\mathbb{K},b}, E_a)$

$$L_{\mathbb{K},b} := \bigcup_{t\in\mathbb{R}} \Phi^t_{\gamma_L}(\ell_{\mathbb{K},b})$$

$\Phi^t_{\gamma_L}$:= parallel transport along U-shaped $\gamma_L(t)$

$\ell_{\mathbb{K},b}$ linear Lagrangian in fiber $v_0^{-1}(\epsilon)$

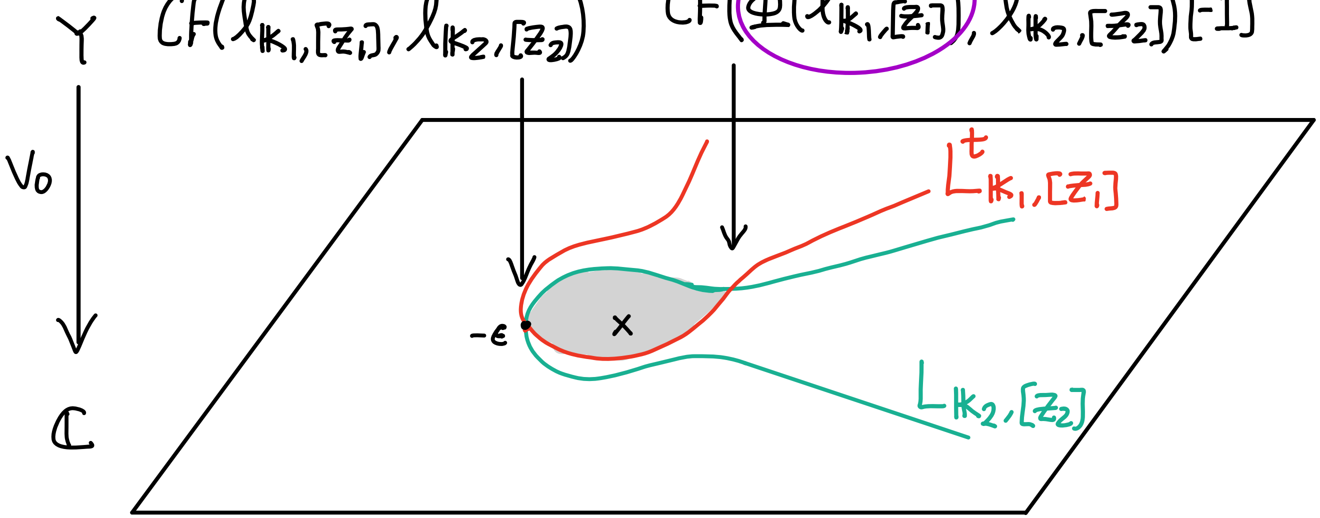
E_a = trivial line bundle with $E_a|_{v_0^{-1}(\epsilon)} = \mathcal{E}_a$

equipped with $U(1)$ connection ∇_a with curvature $d\nabla_a = -2\pi i B|_{L_{\mathbb{K},b}}$

Ham. iso. $\cong \ell_{\mathbb{K}_1+1,[z_1]}$

$CF(\ell_{\mathbb{K}_1,[z_1]}, \ell_{\mathbb{K}_2,[z_2]})$ $\qquad$ $CF(\Phi(\ell_{\mathbb{K}_1,[z_1]}), \ell_{\mathbb{K}_2,[z_2]})[-1]$

Φ = monodromy mirror to $- \otimes \mathcal{O}_{\Theta_\tau}$



Morphism

$\Delta\mathbb{k} = \mathbb{k}_2 - \mathbb{k}_1$, $\Delta z = z_2 - z_1$

$$\left(0 \longrightarrow \mathcal{L}_\tau^{-1} \longrightarrow \mathcal{O}_{V_\tau} \longrightarrow \mathcal{O}_{\Theta_\tau}\right) \otimes \mathcal{L}_{\Delta\mathbb{k},[\Delta z]}$$

$$\begin{tikzcd}
H^0(V_\tau, \mathcal{L}_{\Delta\mathbb{k}-1,[\Delta z]}) \arrow[d, equal] & H^0(V_\tau, \mathcal{L}_{\Delta\mathbb{k},[\Delta z]}) \arrow[d, equal] & H^0(\Theta_\tau, \iota^*\mathcal{L}_{\Delta\mathbb{k},[\Delta z]}) \arrow[d, equal] & \\
\mathrm{Hom}(\mathcal{L}_{\mathbb{k}_1+1,[z_1]}, \mathcal{L}_{\mathbb{k}_2,[z_2]}) \arrow[r, "\vartheta"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\mathcal{L}_{\mathbb{k}_1,[z_1]}, \mathcal{L}_{\mathbb{k}_2,[z_2]}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathcal{L}_{\mathbb{k}_1,[z_1]}, \mathcal{L}_{\mathbb{k}_2,[z_2]} \otimes \iota_*\mathcal{O}_{\Theta_\tau}) \arrow[r] \arrow[d] & 0 \\
CF(\underbrace{\hat\ell_{\mathbb{k}_1+1,[z_1]}}, \hat\ell_{\mathbb{k}_2,[z_2]})[-1] \arrow[r, "\partial"] & CF(\hat\ell_{\mathbb{k}_1,[z_1]}, \hat\ell_{\mathbb{k}_2,[z_2]}) \arrow[r] & HF(\hat{L}_{\mathbb{k}_1,[z_1]}, \hat{L}_{\mathbb{k}_2,[z_2]}) \arrow[r] & 0
\end{tikzcd}$$

Hamiltonian isotopic to $\Phi(\hat\ell_{\mathbb{k}_1,[z_1]})$, Φ = monodromy

$$HF(\hat{L}_{\mathbb{k}_1,[z_1]}, \hat{L}_{\mathbb{k}_2,[z_2]}) = CF(\Phi(\ell_{\mathbb{k}_1,[z_1]}), \hat\ell_{\mathbb{k}_2,[z_2]})[-1] \oplus CF(\hat\ell_{\mathbb{k}_1,[z_1]}, \hat\ell_{\mathbb{k}_2,[z_2]})$$

$\partial = \cdot\,\vartheta(\tau, x)$ (up to a scale factor)

↑ defining function of Θ_τ