Morse theory on moduli spaces of pairs and the Bogomolov–Miyaoka–Yau inequality

Paul Feehan

Department of Mathematics
Rutgers University, New Brunswick, New Jersey

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Outline

Collaborators and references

This talk is based on the following monographs [21, 23] and work in progress [25]:

- *Virtual Morse–Bott index, moduli spaces of pairs, and applications to topology of smooth four-manifolds* (with **Tom Leness**), Memoirs of the American Mathematical Society, in press, xiv+330 pages, arXiv:2010.15789
- *Białnicki–Birula theory, Morse–Bott theory, and resolution of singularities for analytic spaces*, xii+189 pages, arXiv:2206.14710
- *Moduli spaces of semistable pairs over complex projective surfaces and applications to the Bogomolov-Miyaoka-Yau inequality* (with **Tom Leness** and **Richard Wentworth**), in preparation.

Bogomolov–Miyaoka–Yau inequality: theorems and conjectures

We begin by recalling the

Theorem 1.1 (Bogomolov–Miyaoka–Yau inequality for complex surfaces of general type)

(See Miyaoka [45, Theorem 4] and Yau [65, Theorem 4].) If X is a compact, complex surface of general type, then

$$c_1(X)^2 \leq 3c_2(X). \tag{1}$$

Here, $c_1(X)$ and $c_2(X)$ are the Chern classes of the holomorphic tangent bundle, $\mathscr{T}_X \cong T^{1,0}X$.

In [45], Miyaoka proved Theorem 1.1 using algebraic geometry.

See Barth, Hulek, Peters, and Van de Ven [7, Section VII.4] for a simplification of Miyaoka's proof of Theorem 1.1.

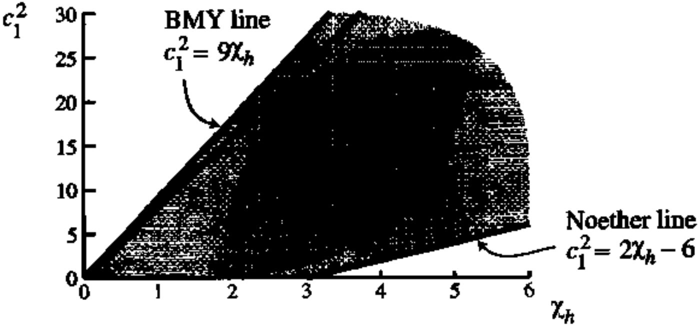

Figure 1.1: Geography of minimal complex surfaces of general type (from Gompf and Stipsicz [34, Section 3.4, Figure 3.3, p. 91]).

Bogomolov [9] proved a weaker version of (1), namely $c_1(X)^2 \leq 4c_2(X)$.

Yau proved (1) in a slightly more restricted setting than Theorem 1.1 as a consequence of his proof of the Calabi Conjectures.

Simpson [49, p. 871] proved Theorem 1.1 as a corollary of his main theorem [49, p. 870] on existence of a Hermitian–Einstein connection on a stable Higgs bundle of rank 3 over X and the following

Theorem 1.2 (Bogomolov–Gieseker inequality)

(See Kobayashi [40, Theorem 4.4.7] or Lübke and Teleman [44, Corollary 2.2.4].) Let (E, h) be a Hermitian vector bundle over of rank r over a compact, complex Kähler manifold of dimension $n \geq 2$. If (E, h) admits a Hermitian–Einstein connection, then

$$\int_X \left(2rc_2(E) - (r-1)c_1(E)^2\right) \wedge \omega^{n-2} \geq 0. \tag{2}$$

According to Bogomolov [9] and Gieseker [31], a version of inequality (2) holds for any **slope semi-stable, torsion-free sheaf** over a smooth complex projective surface (see Huybrechts and Lehn [39, Theorem 3.4.1, p. 80]).

For a closed topological four-manifold X, we define

$$c_1(X)^2 := 2e(X) + 3\sigma(X) \quad \text{and} \quad \chi_h(X) := \frac{1}{4}(e(X) + \sigma(X)),$$

where $e(X) = 2 - 2b_1(X) + b_2(X)$ and $\sigma(X) = b^+(X) - b^-(X)$ are the *Euler characteristic* and *signature* of X, respectively.

If Q_X is the intersection form on $H_2(X;\mathbb{Z})$, then $b^\pm(X)$ are the dimensions of the maximal positive and negative subspaces of Q_X on $H_2(X;\mathbb{R})$.

We call X **standard** if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

Conjecture 1 (Bogomolov–Miyaoka–Yau (BMY) inequality for four-manifolds with non-zero Seiberg–Witten invariants)

If X is a standard four-manifold of Seiberg–Witten simple type with a non-zero Seiberg–Witten invariant, then

$$c_1(X)^2 \leq 9\chi_h(X). \tag{3}$$

If X obeys the hypotheses of Conjecture 1, then it has an almost complex structure J and the inequality (3) is equivalent to (1), namely

$$c_1(X)^2 \leq 3c_2(X),$$

where the Chern classes are those of $T^{1,0}X$.

Conjecture 1 is based on Stern [50, Problem 4] (see also Kollár [41]), but often stated for simply connected, symplectic four-manifolds — see Gompf and Stipsicz [34, Remark 10.2.16 (c)] or Stern [50, Problem 2].

Taubes [55, 56] showed that symplectic four-manifolds have Seiberg–Witten simple type with non-zero Seiberg–Witten invariants.

Szabó [53] proved existence of four-dimensional, *non-symplectic*, smooth manifolds with non-zero Seiberg–Witten invariants.

Conjecture 1 has inspired contructions by topologists of examples to shed light on inequality (3), including work of Akhmedov, Hughes, and Park [1, 2, 3], Baldridge, Kirk, and Li [4, 5, 6], Bryan, Donagi, and Stipsicz [12], Fintushel and Stern [27], Gompf and Mrowka [32, 33], Hamenstädt [35], Park and Stipsicz [48, 51, 52], I. Smith, Torres [58], and others.

Conjecture 1 holds for all examples that satisfy the hypotheses, but the BMY inequality (3) can fail for four-manifolds with zero Seiberg–Witten invariants, such as a connected sum of two or more copies of $\mathbb{CP}^2$.

LeBrun [42] proved the BMY inequality (3) for Einstein four-manifolds with non-zero Seiberg–Witten invariants.

Conjecture 2 (Existence of ASD connections with small instanton number)

Assume the hypotheses of Conjecture 1 and let E be a complex rank two, Hermitian vector bundle over X whose associated SO(3) bundle $\mathfrak{su}(E)$ has first Pontrjagin number obeying the basic lower bound,

$$p_1(\mathfrak{su}(E)) \geq c_1(X)^2 - 12\chi_h(X). \tag{4}$$

Let g be a Riemannian metric on X that is generic in the sense of Freed and Uhlenbeck [17, 29]. Then there exists a smooth, projectively g-anti-self-dual Yang–Mills unitary connection A on E, so the curvature $F_A \in \Omega^2(\mathfrak{u}(E))$ obeys

$$(F_A^+)_0 = 0 \in \Omega^+(X; \mathfrak{su}(E)), \tag{5}$$

where $^+ : \wedge^2(T^*X) \to \wedge^+(T^*X)$ and $(\cdot)_0 : \mathfrak{u}(E) \to \mathfrak{su}(E)$ are orthogonal projections.

One has $c_1(X)^2 - 12\chi_h(X) = -e(X) = -c_2(X)$ by [34, Section 1.4.1], so (4) $\iff$ the *instanton number* obeys $\kappa := -\frac{1}{4}p_1(\mathfrak{su}(E)) \leq \frac{1}{4}e(X)$.

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We now explain why Conjecture 2 $\implies$ Conjecture 1:

For $w \in H^2(X;\mathbb{Z})$ and $4\kappa \in \mathbb{Z}$, let (E,h) be a rank-2 Hermitian bundle over X with $c_1(E) = w$, fixed unitary connection A_d on $\det E$, and

$$p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E) = -4\kappa.$$

The moduli space of projectively anti-self-dual (ASD) connections on E is

$$M_\kappa^w(X,g) := \{A : (F_A^+)_0 = 0\}/\mathscr{G}_E.$$

$\mathscr{G}_E$ is the group of determinant-one, unitary automorphisms of (E,h).

The expected dimension of $M_\kappa^w(X,g)$ is given by [17]

$$\exp\dim M_\kappa^w(X,g) = -2p_1(\mathfrak{su}(E)) - 6\chi_h(X). \tag{6}$$

When g is *generic* in the sense of [17, 29], then $M_\kappa^w(X,g)$ is a smooth (usually non-compact) manifold if non-empty.

If Conjecture 2 holds, then $\mathfrak{su}(E)$ admits a g-anti-self-dual connection when the basic lower bound (4) holds and the metric g on X is generic.

The moduli space $M_\kappa^w(X,g)$ is thus a non-empty, smooth manifold and so

$$\operatorname{exp\,dim} M_\kappa^w(X,g) \geq 0.$$

This yields the Bogomolov–Miyaoka–Yau inequality (3) since

$$\begin{aligned}
0 &\leq \frac{1}{2}\operatorname{exp\,dim} M_\kappa^w(X,g) \\
&= -p_1(\mathfrak{su}(E)) - 3\chi_h(X) \quad \text{(by (6))} \\
&\leq -\left(c_1(X)^2 - 12\chi_h(X)\right) - 3\chi_h(X) \quad \text{(by (4))} \\
&= -c_1(X)^2 + 9\chi_h(X).
\end{aligned}$$

Taubes [54] proved existence of solutions to the ASD equation (5) only when the instanton number $\kappa(E) = -\frac{1}{4}p_1(\mathfrak{su}(E))$ is sufficiently large.

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The difficulty in proving Conjecture 2 is because the basic lower bound (4) implies that $\kappa(E)$ is small and Taubes' gluing method does not apply.

We aim to prove Conjecture 2 via existence of projectively anti-self-dual connections as absolute minima of a Hamiltonian function f for the circle action on the **singular** moduli space of non-Abelian monopoles.

Non-Abelian monopoles and the Bogomolov–Miyaoka–Yau inequality

Let (ρ, W) be a *spin^c structure* and (E, h) be a Hermitian vector bundle over an oriented, Riemannian four-manifold (X, g).

Consider the affine space of unitary connections A on E that induce a fixed unitary connection A_d on $\det E$ and sections Φ of $W^+ \otimes E$.

We call (A, Φ) a non-Abelian monopole if

$$\begin{aligned} (F_A^+)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A \Phi &= 0, \end{aligned} \tag{7}$$

where the section $(\Phi \otimes \Phi^*)_{00}$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ is the trace-free component of $\Phi \otimes \Phi^*$ of $\mathfrak{u}(W^+) \otimes \mathfrak{u}(E)$ and D_A is the Dirac operator and $\rho : \wedge^+(T^*X) \to \mathfrak{su}(W^+)$ is an isomorphism of SO(3) bundles.

The moduli space of non-Abelian monopoles is

$$\mathscr{M}_{\mathfrak{t}} := \{(A, \Phi) \text{ obeying } (7)\} / \mathscr{G}_E.$$

The space $\mathscr{M}_\mathfrak{t}$ has a decomposition as a disjoint union of subsets

$$\mathscr{M}_\mathfrak{t} = \mathscr{M}_\mathfrak{t}^{*,0} \sqcup \mathscr{M}_\mathfrak{t}^{\{\Phi\equiv 0\}} \sqcup \mathscr{M}_\mathfrak{t}^{\{A \text{ reducible}\}},$$

where $\mathscr{M}_\mathfrak{t}^{*,0} \subset \mathscr{M}_\mathfrak{t}$ is the subspace of *irreducible, non-zero-section pairs*, a *finite-dimensional smooth manifold* for generic geometric perturbations (see F. and Leness [24, 22] and Teleman [57]).

Our hypothesis in Conjecture 1 that X has a non-zero Seiberg–Witten invariant ensures that the subspace $\mathscr{M}_\mathfrak{t}^{*,0}$ is non-empty.

Multiplication by $\mathbb{C}^*$ on sections Φ induces an S^1 action on $\mathscr{M}_{\mathfrak{t}}$ with two types of fixed points, represented by pairs (A, Φ) such that

- $\Phi \equiv 0$, or
- A is a **reducible connection** for some splitting, $E = L_1 \oplus L_2$.

For points $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$, there are bijections between

- the subset of $\mathscr{M}_{\mathfrak{t}}^{\{\Phi\equiv 0\}}$, where $\Phi \equiv 0$, and the moduli space $M_\kappa^w(X, g)$ of anti-self-dual connections, and
- subsets of $\mathscr{M}_{\mathfrak{t}}^{\{A \text{ reducible}\}}$, where A is reducible for a splitting $E = L_1 \oplus L_2$, and a moduli space $M_{\mathfrak{s}}$ of Seiberg–Witten monopoles defined by a spin^c structure $\mathfrak{s} = (\rho, W \otimes L_1)$.

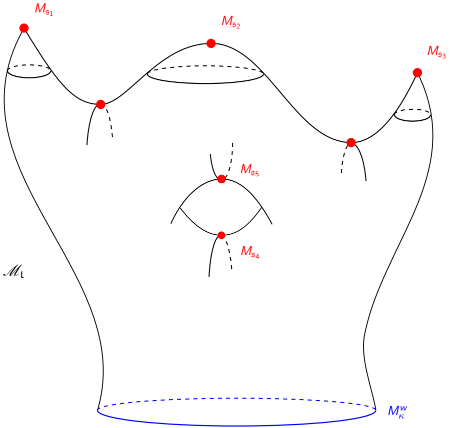

Figure 2.2: Non-Abelian monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with critical sets of the Hamiltonian function given by Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and the moduli subspace $M_\kappa^w(X, g)$ of anti-self-dual connections

We use a partial extension of Morse–Bott theory from smooth manifolds to singular analytic spaces to try to prove Conjecture 2 on existence of anti-self-dual connections on $\mathfrak{su}(E)$.

To motivate our version of Morse–Bott theory, we describe an idealized model case. Hitchin's Hamiltonian function,

$$f : \mathscr{M}_\mathfrak{t} \ni [A, \Phi] \mapsto f[A, \Phi] := \frac{1}{2}\|\Phi\|^2_{L^2(X)} \in \mathbb{R}, \qquad (8)$$

is continuous and smooth on smooth strata of $\mathscr{M}_\mathfrak{t}$ and attains its absolute minimum value of zero on the moduli subspace $M_\kappa^w(X, g)$, **if non-empty**.

(Hitchin used Morse–Bott theory for f in (8) in his analysis [36] of the topology of the moduli space of Higgs pairs over a Riemann surface.)

We temporarily assume that $\mathscr{M}_\mathfrak{t}$ is a smooth manifold (usually false), in which case f is also smooth, and that $\mathscr{M}_\mathfrak{t}$ is compact (usually false).

The moduli space $\mathscr{M}_\mathfrak{t}$ is equipped with the L^2 Riemannian metric. Assume further that f is Morse–Bott on $\mathscr{M}_\mathfrak{t}$ and that its critical submanifolds comprise the moduli subspace $M_\kappa^w(X,g)$ of anti-self-dual connections (if non-empty) and the moduli subspaces $M_{\mathfrak{s}_i}$ of Seiberg–Witten monopoles.

Because f is Morse–Bott on $\mathscr{M}_\mathfrak{t}$, if $[A,\Phi] \in \mathscr{M}_\mathfrak{t}$ is a critical point, so

$$\operatorname{Ker} df[A,\Phi] = T_{[A,\Phi]}\mathscr{M}_\mathfrak{t},$$

then the Hessian of f (defined by the L^2 metric) obeys

$$\operatorname{Ker} \operatorname{Hess} f[A,\Phi] = T_{[A,\Phi]} \operatorname{Crit} f,$$

and the tangent space $T_{[A,\Phi]}\mathscr{M}_\mathfrak{t}$ has an orthogonal splitting,

$$T_{[A,\Phi]}\mathscr{M}_\mathfrak{t} = T^+_{[A,\Phi]}\mathscr{M}_\mathfrak{t} \oplus T^-_{[A,\Phi]}\mathscr{M}_\mathfrak{t} \oplus T^0_{[A,\Phi]}\mathscr{M}_\mathfrak{t}.$$

The subspaces $T^{\pm}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}$ where $\operatorname{Hess} f[A,\Phi]$ is *positive* or *negative definite* are tangent spaces to the stable and unstable manifolds through $[A,\Phi]$.

The subspace $T^{0}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}$ where $\operatorname{Hess} f[A,\Phi]$ is zero is the *tangent space to the critical submanifold* $\operatorname{Crit} f$.

The Morse–Bott signature of the critical point $[A,\Phi]$ is given by

$$\lambda^{\pm}_{[A,\Phi]}(f) := \dim T^{\pm}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}} \quad \text{and} \quad \lambda^{0}_{[A,\Phi]}(f) := \dim T^{0}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}},$$

comprising the Morse–Bott index, co-index, and nullity.

Observation 2.1 (Positive Morse–Bott indices for Seiberg–Witten critical points $\Longrightarrow$ existence of anti-self-dual connections)

If the Morse–Bott index of every Seiberg–Witten critical submanifold is positive, then the critical submanifold given by the moduli space of anti-self-dual connections is non-empty.

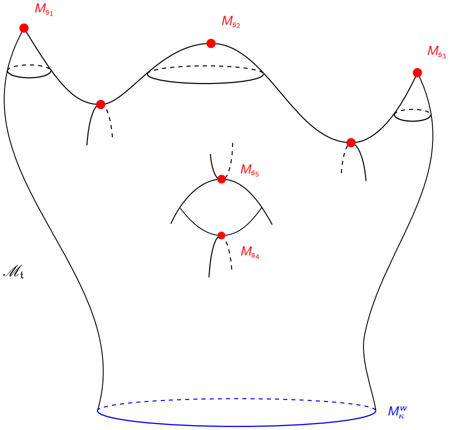

Figure 2.3: Non-Abelian monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with critical sets of the Hamiltonian function given by Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and the moduli subspace $M_\kappa^w(X, g)$ of anti-self-dual connections

One can compute the Morse–Bott index of a critical point using Frankel's Theorem [28], used by Hitchin [36] for the moduli space of Higgs monopoles on a Hermitian bundle (E, h), whose rank and degree are coprime, over a Riemann surface.

Suppose that (M, g, J) is an almost Hermitian manifold that admits a smooth circle action $\rho : S^1 \times M \to M$ and a circle-invariant, non-degenerate two-form, $\omega = g(\cdot, J\cdot)$.

Let f be a Hamiltonian function for the circle action, so

$$df = \iota_\xi \omega,$$

where the smooth vector field ξ on M is the generator of the S^1 action.

Theorem 2.2 (Frankel's theorem for almost Hermitian manifolds)

(See Frankel [28, Section 3] for complex, Kähler manifolds and [23, Theorem 2] for almost Hermitian manifolds.)

1. *A point $p \in M$ is a critical point of $f \iff p$ is a fixed point of the circle action ρ on M.*
2. *The Hamiltonian, f, is Morse–Bott at each critical point p, with Morse–Bott signature $(\lambda_p^+(f), \lambda_p^-(f), \lambda_p^0(f))$ given by the dimensions $(\lambda_p^+(\rho), \lambda_p^-(\rho), \lambda_p^0(\rho))$ of the positive, negative, and zero weight spaces for the circle action ρ_* on the tangent space T_pM.*

If X is a compact, complex Kähler surface, then the subspace $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ of smooth points is a complex Kähler manifold with circle-invariant Kähler form ω and f in (8) is a Hamiltonian function for this circle action.

Thus, if X is Kähler, then the following are equivalent for $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$:

- $[A,\Phi]$ is a critical point of f,
- $[A,\Phi]$ is a fixed point of the circle action on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$,
- A is reducible, so (A,Φ) is a Seiberg–Witten monopole, or $\Phi \equiv 0$ and A is projectively anti-self-dual.

The preceding ideas extend to the case of fixed points $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}$ that are **singular points** of the moduli space.

In [23], we apply the *Hirzebruch–Riemann–Roch Theorem* to compute a virtual Morse–Bott signature for each fixed point $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}$ represented by a Seiberg–Witten monopole and show that its virtual Morse–Bott index,

$$\lambda^{-}_{[A,\Phi]}(f) := \dim \mathbf{H}^{-,1}_{A,\phi} - \dim \mathbf{H}^{-,2}_{A,\phi},$$

is positive and thus *cannot be a local minimum*.

Few of our assumptions for the idealized model hold in practice:

1. Singular critical points. The moduli subspace $M_\kappa^w(X)$ of anti-self-dual connections and moduli subspaces $M_{\mathfrak{s}_i}$ of Seiberg–Witten monopoles are *singularities* in the moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles (even when those subspaces are smooth manifolds).
2. Non-compact. The moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles is non-compact due to *Uhlenbeck energy bubbling* [59, 60].
3. Non-Kähler. The moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles is *not necessarily a complex Kähler manifold* (away from singularities) when the almost complex structure J on X is not assumed integrable and the fundamental two-form $\omega = g(\cdot, J\cdot)$ is not assumed closed.

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The non-compactness of $\mathscr{M}_{\mathfrak{t}}$ can be addressed in two ways:

When X is a smooth, complex projective surface, the Hitchin–Kobayashi correspondence gives a real analytic isomorphism between $\mathscr{M}_{\mathfrak{t}}^0$ and the moduli space $\mathfrak{M}^0(E,\omega)$ of Bradlow stable, holomorphic pairs.

$\mathfrak{M}^0(E,\omega)$ has a Gieseker compactification, a moduli space $\mathfrak{M}_{\mathrm{ss}}(E,\omega)$ of *pairs* of coherent sheaves and sections that are semistable in the sense of Bradlow, Gieseker, and Maruyama (see Dowker [18], Huybrechts and Lehn [38, 37], Lin [43], Okonek, Teleman, and Schmitt [47], and Wandel [61]).

When X is a smooth Riemannian four-manifold, then $\mathscr{M}_{\mathfrak{t}}$ admits an Uhlenbeck (or bubble-tree) compactification $\bar{\mathscr{M}}_{\mathfrak{t}}$ given by the Uhlenbeck closure of $\mathscr{M}_{\mathfrak{t}}$ in the space of *ideal non-Abelian monopoles*,

$$\mathscr{I}\mathscr{M}_{\mathfrak{t}} := \bigsqcup_{\ell=0}^{\infty} \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X)\right), \tag{9}$$

where $\mathfrak{t}(\ell) = (\rho, W \otimes E_\ell)$ and (E_ℓ, h_ℓ) is a rank-2 Hermitian vector bundle over X with fixed unitary connection A_d on $\det E_\ell \cong \det E$ and

$$c_1(E_\ell) = c_1(E), \quad c_2(E_\ell) = c_2(E) - \ell, \quad p_1(\mathfrak{su}(E_\ell)) = p_1(\mathfrak{su}(E)) + 4\ell.$$

We call the intersection of $\bar{\mathscr{M}}_{\mathfrak{t}}$ with $\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)$ its ℓ-th level.

Either choice of compactification (Gieseker or Uhlenbeck) introduces more singularities and leads back to the first difficulty that the moduli space $\mathscr{M}_{\mathfrak{t}}$ of non-Abelian monopoles (and any compactification) has singularities.

We summarize our program to prove Conjecture 1:

1. Prove existence of feasible spin^u structure $\mathfrak{t} = (\rho, W \otimes E)$ with
   - $p_1(\mathfrak{su}(E))$ obeying the basic lower bound (4);
   - Moduli subspace $\mathscr{M}_{\mathfrak{t}}^{*,0}$ of irreducible, non-zero-section non-Abelian monopoles is non-empty.
2. Prove that all critical points of Hitchin's function on $\bar{\mathscr{M}}_{\mathfrak{t}}$ are
   - points in the anti-self-dual moduli subspace $M_\kappa^w(X, g) \subset \mathscr{M}_{\mathfrak{t}}$; or
   - points in moduli subspaces $M_{\mathfrak{s}} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ of Seiberg–Witten monopoles.
3. Prove that all points in moduli subspaces $M_{\mathfrak{s}} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ of Seiberg–Witten monopoles have positive virtual Morse–Bott index.

The above three steps in our program are completed in our monograph [23] for $\mathscr{M}_{\mathfrak{t}}$ when X is Kähler and almost completed for $\mathfrak{M}_{ss}(E, \omega)$ when X is smooth, complex projective, but **not** yet for $\bar{\mathscr{M}}_{\mathfrak{t}}$ when X smooth.

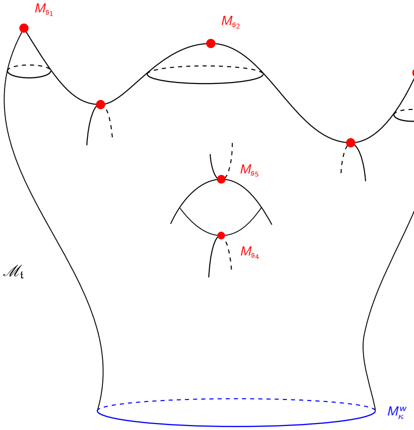

Figure 2.4: Non-Abelian monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with critical sets of the Hamiltonian function given by Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and the moduli subspace $M_{\kappa}^{w}(X, g)$ of anti-self-dual connections

Virtual Morse–Bott indices for Hamiltonian functions of circle actions on complex analytic subspaces of complex, Kähler manifolds with holomorphic $\mathbb{C}^*$ actions

Inspired by Hitchin [36], we extend the definition of the index of a Morse–Bott function at a critical point in a smooth manifold to the case of

> *A critical point of a Hamiltonian function for the circle action on $\mathbb{C}^*$-invariant, closed, complex analytic subspace of a complex, Kähler manifold with a holomorphic $\mathbb{C}^*$ action.*

Complex analytic spaces with circle actions are pervasive in gauge theory over complex Kähler manifolds or smooth complex, projective varieties:

- Moduli spaces of Higgs bundles (Hitchin–Simpson pairs),
- Moduli spaces of projective vortices (Bradlow pairs),
- Moduli spaces of non-Abelian monopoles,
- Moduli spaces of stable pairs of holomorphic bundles and sections.

Białynicki–Birula theory for holomorphic $\mathbb{C}^*$ actions on complex manifolds

Based on results due to Białynicki–Birula [8] for torus actions on smooth algebraic varieties (also Carrell and Sommese [13], Fujiki [30]), we have

Definition 3 (Białynicki–Birula decompositions for complex manifolds)

Let X be a complex manifold and $\mathbb{C}^* \times X \to X$ be a holomorphic $\mathbb{C}^*$ action such that the subset $X^0 := X^{\mathbb{C}^*} \subset X$ of fixed points of the $\mathbb{C}^*$ action is non-empty with at most countably many connected components, X_α^0 for $\alpha \in \mathscr{A}$, that are embedded complex submanifolds of X. For each $\alpha \in \mathscr{A}$, define

$$X_\alpha^+ := \left\{z : \lim_{\lambda\to 0} \lambda \cdot z \in X_\alpha^0\right\} \quad \text{and} \quad X_\alpha^- := \left\{z : \lim_{\lambda\to\infty} \lambda \cdot z \in X_\alpha^0\right\}, \qquad (10)$$

so the subsets $X_\alpha^+ \subset X$ are $\mathbb{C}^*$-invariant and mutually disjoint for all $\alpha \in \mathscr{A}$ and similarly for the the subsets $X_\alpha^- \subset X$ for all $\alpha \in \mathscr{A}$, and

$$\pi_\alpha^+(z) := \lim_{\lambda\to 0} \lambda\cdot z, \quad \text{for all } z \in X_\alpha^+, \quad \text{and} \quad \pi_\alpha^-(z) := \lim_{\lambda\to\infty} \lambda\cdot z, \quad \text{for all } z \in X_\alpha^-. \qquad (11)$$

Definition 3 (Białynicki–Birula decompositions for complex manifolds)

Then X has a *(mixed, plus, or minus) Białynicki–Birula decomposition* if the following hold:

1. Each X_α^+ is an embedded complex submanifold of X;
2. The natural map $\pi_\alpha^+ : X_\alpha^+ \to X_\alpha^0$ is a $\mathbb{C}^*$-equivariant, holomorphic, maximal-rank surjection;

and the analogous properties hold for the subsets X_α^- and for the maps $\pi_\alpha^- : X_\alpha^- \to X_\alpha^0$. Furthermore, we require that:

3. The normal bundles $N_{X_\alpha^0/X_\alpha^+}$ of X_α^0 in X_α^+ and $N_{X_\alpha^0/X_\alpha^-}$ of X_α^0 in X_α^- are subbundles of the normal bundle $N_{X_\alpha^0/X}$ of X_α^0 in X. There is a weight-sign decomposition defined by the S^1 action on X induced by the $\mathbb{C}^*$ action,

$$TX \restriction X_\alpha^0 = T^0X_\alpha \oplus N^+_{X_\alpha^0/X} \oplus N^-_{X_\alpha^0/X}; \tag{12}$$

Definition 3 (Białynicki–Birula decompositions for complex manifolds)

4. For some $A \subset \mathscr{A} \times \{+, -\}$, the space X is expressed as a disjoint union,

$$X = \bigsqcup_{(\alpha,j)\in A} X_\alpha^j. \tag{13}$$

If one can express X as

$$X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha^+ \quad \text{or} \quad X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha^-, \tag{14}$$

where the union is disjoint, then we say that X has a *plus* or *minus decomposition*, respectively and, otherwise, if X is expressed as in (13), that it has a *mixed decomposition*.

One has the following result due to Białynicki–Birula [8], Carrell and Sommese [13, 14, 15], Fujiki [30], and Yang [64] (see F. [21, Theorem 3] for a generalization).

Theorem 4 (Białynicki–Birula decomposition for Kähler manifolds)

If X is a compact, complex Kähler manifold with a holomorphic action $\mathbb{C}^* \times X \to X$, then it admits plus and minus Białynicki–Birula decompositions in the sense of Definition 3.

Białynicki–Birula theory for holomorphic $\mathbb{C}^*$ actions on complex analytic spaces

We have the following generalization of Definition 3.

Definition 5 (Białynicki–Birula decompositions for complex analytic spaces)

Let $(X, \mathscr{O}_X)$ be a complex analytic space and $\mathbb{C}^* \times X \to X$ be a holomorphic action such that the subset $X^0 \subset X$ of fixed points of the $\mathbb{C}^*$ action is non-empty with at most countably many connected components, X^0_α for $\alpha \in \mathscr{A}$, that are locally closed complex analytic subspaces of X. For each $\alpha \in \mathscr{A}$, define $X^\pm_\alpha$ as in (10) and the natural maps $\pi^\pm_\alpha$ as in (11). Then X has a *(mixed, plus, or minus) Białynicki–Birula decomposition* if the following hold:

1. Each X^+_α is a locally closed, complex analytic subspace of X;
2. The map $\pi^+_\alpha : X^+_\alpha \to X^0_\alpha$ is a $\mathbb{C}^*$-equivariant epimorphism of complex analytic spaces;

and the analogous properties hold for the subsets X^-_α and for the maps $\pi^-_\alpha : X^-_\alpha \to X^0_\alpha$.

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Definition 5 (Białynicki–Birula decompositions for complex analytic spaces)

Furthermore, we require that:

3. X may be expressed as a disjoint union as in (13), for some subset $A \subset \mathscr{A} \times \{+, -\}$.

If one can express X as a disjoint union as in (14), then we say that X has a *plus* or *minus decomposition*, respectively and, otherwise, if X is expressed as in (13), that it has a *mixed decomposition*.

Weber [62, Section 2, p. 539] studied the Białynicki–Birula decomposition for singular complex algebraic varieties with $\mathbb{C}^*$ actions that are $\mathbb{C}^*$-equivariantly embedded in smooth, complex algebraic varieties.

Drinfeld provides a more general framework in a 2013 preprint [19].

One can define analogues of Morse–Bott index, co-index, and nullity:

Definition 6 (Stable and unstable subspaces of a complex analytic space and Białynicki–Birula index, co-index, and nullity)

The locally closed complex analytic subspace X_α^+ (respectively, X_α^-) is called the *stable* (respectively, *unstable*) *subspace* for the fixed-point subspace X_α^0.
For each point $p \in X_\alpha^0$, the Krull dimensions $\beta_X^0(p)$, $\beta_X^+(p)$, and $\beta_X^-(p)$, of the local rings $\mathscr{O}_{X_\alpha^0,p}$, $\mathscr{O}_{X_p^+,p}$, and $\mathscr{O}_{X_p^-,p}$ are called the *Białynicki–Birula nullity*, *co-index*, and *index*, respectively, of the point p in X defined by the $\mathbb{C}^*$ action, where we write $X_p^\pm = X_\alpha^\pm|_p$ for the fibers of $X_\alpha^\pm$ over a point $p \in X_\alpha^0$.

We have the following generalization of Theorem 4.

Theorem 7 (Białynicki–Birula decomposition for a complex analytic space)

Let X be a finite-dimensional complex manifold, $(Y, \mathscr{O}_Y)$ be a closed complex analytic subspace of X, and $\mathbb{C}^* \times X \to X$ be a holomorphic action on X that leaves Y invariant with at least one fixed point in Y.

1. *If X has a plus (respectively, minus or mixed) Białynicki–Birula decomposition as in Definition 3 with subsets X^0, $X^\pm$, $X^\pm_\alpha$, then Y inherits a plus (respectively, minus or mixed) Białynicki–Birula decomposition as in Definition 5 with locally closed complex analytic subspaces:*

$$Y^0 = Y \cap X^0, \quad Y^\pm = Y \cap X^\pm, \quad Y^\pm_p = Y \cap X^\pm_p, \quad \text{for all } p \in Y^0. \tag{15}$$

2. *If $p \in Y$, then there is an open neighborhood $U \subset Y$ of p such that $\dim(Y_{\mathrm{sm}} \cap U) = \dim \mathscr{O}_{Y,p}$, where $Y_{\mathrm{sm}} \subset Y$ is the subset of smooth points.*

Theorem 7 (Białynicki–Birula decomposition for a complex analytic space)

3. *If $p \in Y^0$ then, after possibly shrinking U, the Białynicki–Birula nullity, co-index, and index of p in Y in the sense of Definition 6 are given by*

$$\beta^0_Y(p) = \dim \mathscr{O}_{Y^0,p} = \dim (Y^0)_{\mathrm{sm}} \cap U, \tag{16a}$$

$$\beta^+_Y(p) = \dim \mathscr{O}_{Y^+_p,p} = \dim (Y^+_p)_{\mathrm{sm}} \cap U, \tag{16b}$$

$$\beta^-_Y(p) = \dim \mathscr{O}_{Y^-_p,p} = \dim (Y^+_p)_{\mathrm{sm}} \cap U, \tag{16c}$$

where $(Y^0)_{\mathrm{sm}} \subset Y^0$ and $(Y^\pm_p)_{\mathrm{sm}} \subset Y^\pm_p$ denote subsets of smooth points.

4. *If $\beta^0_Y(p) > 0$ (respectively, $\beta^+_Y(p) > 0$ or $\beta^-_Y(p) > 0$), then $(Y^0)_{\mathrm{sm}} \cap U$ (respectively, $(Y^+_p)_{\mathrm{sm}} \cap U$ or $(Y^-_p)_{\mathrm{sm}} \cap U$) is non-empty.*

5. *If the induced circle action $S^1 \times X \to X$ has a Hamiltonian function $f : X \to \mathbb{R}$ and $\beta^-_Y(p) > 0$ (respectively, $\beta^+_Y(p) > 0$), then p is not a local minimum (respectively, maximum) of the restriction $f : Y_{\mathrm{sm}} \cup \{p\} \to \mathbb{R}$.*

Krull dimensions are difficult to compute, but they may be estimated.

Suppose $(U, \mathscr{O}_U)$ is a local model space for an open neighborhood of a point p in a complex analytic space $(X, \mathscr{O}_X)$, so U is the topological support of $\mathscr{O}_D/\mathscr{I}$ with a domain $D \subset \mathbb{C}^n$ and ideal $\mathscr{I} \subset \mathscr{O}_D$ with generators $f_1, \ldots, f_r$ and structure sheaf $\mathscr{O}_U := (\mathscr{O}_D/\mathscr{I}) \restriction U$. One has

$$\dim \mathscr{O}_{X,p} \geq n - r,$$

where $\exp\dim_p X := n - r$ is the expected dimension of X at p. When r is equal to the *minimal number* of generators of $\mathscr{I}_p \subset \mathscr{O}_{U,p}$, then

$$\dim \mathscr{O}_{X,p} = n - r.$$

Local models for moduli spaces are Kuranishi models and expected dimensions are computable via the Hirzebruch–Riemann–Roch Theorem.

Such a lower bound for the Białynicki–Birula index is called the virtual Białynicki–Birula index (or virtual Morse–Bott index).

Selected results, mostly for smooth Kähler surfaces

Theorem 8 (Feasibility of spin^u structures)

(See F. and Leness [23, Theorem 3].) Let X be a standard four-manifold with $b^-(X) \geq 2$ and Seiberg–Witten simple type. Let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ denote the smooth blow-up of X and let $\tilde{g}$ be a smooth Riemannian metric on $\widetilde{X}$. Then there exists a complex rank two vector bundle E over $\widetilde{X}$ and spin^u structure $\tilde{\mathfrak{t}} = (\rho, W \otimes E)$ over $\widetilde{X}$ such the following hold:

1. *The moduli space $\mathscr{M}_{\tilde{\mathfrak{t}}}^{*,0}$ of irreducible, non-zero section non-Abelian monopoles is non-empty for generic Riemannian metrics.*
2. *The bundle E over $\widetilde{X}$ obeys the basic lower bound,*

$$p_1(\mathfrak{su}(E)) \geq c_1(\widetilde{X})^2 + 1 - 12\chi_h(\widetilde{X}). \tag{17}$$

Theorem 8 (Feasibility of spin^u structures)

3. *The expected dimension of the moduli space $M_\kappa^w(\widetilde{X}, \tilde{g})$ of anti-self-dual connections obeys the following inequality:*

$$\frac{1}{2}\exp\dim M_\kappa^w(\widetilde{X}, \tilde{g}) \leq -c_1(X)^2 + 9\chi_h(X). \tag{18}$$

4. *For all spin^c structures $\tilde{\mathfrak{s}}$ for which $M_{\tilde{\mathfrak{s}}}$ continuously embeds in $\mathscr{M}_{\tilde{\mathfrak{t}}}$, the formal Morse–Bott index is positive:*

$$\lambda^-(\tilde{\mathfrak{t}}, \tilde{\mathfrak{s}}) := -2\chi_h(\widetilde{X}) - \left(c_1(\tilde{\mathfrak{s}}) - c_1(\tilde{\mathfrak{t}})\right) \cdot c_1(\widetilde{X}) - \left(c_1(\tilde{\mathfrak{s}}) - c_1(\tilde{\mathfrak{t}})\right)^2 > 0. \tag{19}$$

The expression (19) has the following motivation.

Suppose that $[A, \Phi]$ is a reducible, type 1 non-Abelian monopole with $\Phi \not\equiv 0$, thus a fixed point of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$ and a critical point of the restriction of Hitchin's Hamiltonian function f on the configuration space $\mathscr{C}_{\mathfrak{t}}^0$ of all non-zero section, unitary pairs in the sense that

$$H^1_{A,\Phi} = T_{A,\Phi}\mathscr{M}_{\mathfrak{t}} \subseteq \operatorname{Ker} df[A,\Phi].$$

The point $[\bar\partial_A, \varphi] \in \mathfrak{M}^0(E,\omega)$ defined by the Hitchin–Kobayashi correspondence, where $\bar\partial_A$ is a holomorphic structure on E and $\Phi = (\varphi, 0) \in \Omega^0(E) \oplus \Omega^{0,2}(E)$, is a fixed point of $\mathbb{C}^*$ action on $\mathfrak{M}^0(E,\omega)$.

One has the following weight splittings of Zariski tangent spaces and obstruction spaces defined by the S^1 action:

$$H^1_{A,\Phi} = H^{0,1}_{A,\Phi} \oplus H^{+,1}_{A,\Phi} \oplus H^{-,1}_{A,\Phi}, \quad \text{and} \quad H^2_{A,\Phi} = H^{0,2}_{A,\Phi} \oplus H^{+,2}_{A,\Phi} \oplus H^{-,2}_{A,\Phi},$$
$$H^1_{\bar\partial_A,\varphi} = H^{0,1}_{\bar\partial_A,\varphi} \oplus H^{+,1}_{\bar\partial_A,\varphi} \oplus H^{-,1}_{\bar\partial_A,\varphi}, \quad \text{and} \quad H^2_{\bar\partial_A,\varphi} = H^{0,2}_{\bar\partial_A,\varphi} \oplus H^{+,2}_{\bar\partial_A,\varphi} \oplus H^{-,2}_{\bar\partial_A,\varphi}.$$

One has $H^0_{A,\Phi} = (0)$ and $H^0_{\bar\partial_A,\varphi} = (0)$ since $\varphi \not\equiv 0$ and one can also show that $H^3_{\bar\partial_A,\varphi} = (0)$.

Euler characteristics of the negative-weight complexes, $H^{-,\bullet}_{A,\Phi}$ and $H^{\bullet,-}_{\bar\partial_{A,\varphi}}$, are equal and computable by the Hirzebruch–Riemann–Roch Theorem:

$$\mathrm{Euler}\left(\bar\partial^{-,\bullet}_{A,\varphi}\right) := \sum_{k=0}^{3}(-1)^k h^{-,k}_{\bar\partial_{A,\varphi}} = \sum_{k=0}^{3}(-1)^k h^{-,k}_{A,\Phi} =: \mathrm{Euler}\left(d^{-,\bullet}_{A,\Phi}\right).$$

Hence, the virtual Morse–Bott index for f at $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}$ or equivalently, the virtual Białynicki–Birula index for the fixed point $[\bar\partial_A, \varphi] \in \mathfrak{M}^0(E,\omega)$ of the $\mathbb{C}^*$ action ρ are given by

$$\begin{aligned}\lambda^-_{[A,\Phi]}(f) &:= h^{-,1}_{A,\Phi} - h^{-,2}_{A,\Phi} = -\,\mathrm{Euler}\left(d^{-,\bullet}_{A,\Phi}\right) \\ &= -\,\mathrm{Euler}\left(\bar\partial^{-,\bullet}_{A,\varphi}\right) = h^{-,1}_{\bar\partial_{A,\varphi}} - h^{-,2}_{\bar\partial_{A,\varphi}} =: \lambda^-_{\bar\partial_{A,\varphi}}(\rho). \qquad (20)\end{aligned}$$

The forthcoming Theorem 9 and Corollary 10 show that the expression $\lambda^-(\tilde{\mathfrak{t}}, \tilde{\mathfrak{s}})$ in (19) is *equal* to the virtual Morse–Bott index $\lambda^-_{[A,\Phi]}(f)$.

Theorem 9 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a reducible non-Abelian monopole)

(See F. and Leness [23, Theorem 5].) Let $(\rho_{\mathrm{can}}, W_{\mathrm{can}})$ *be the canonical* spin^c *structure over a closed, complex Kähler surface* X*, and* E *be a complex rank two Hermitian vector bundle over* X *that admits a splitting* $E = L_1 \oplus L_2$ *as a direct sum of Hermitian line bundles, and* $\mathfrak{t} = (\rho, W_{\mathrm{can}} \otimes E)$ *be the corresponding* spin^u *structure. Assume that* (A, Φ) *is a type 1 non-Abelian monopole on* $\mathfrak{t}$ *that is reducible with respect to the splitting* $E = L_1 \oplus L_2$ *as a direct sum of Hermitian line bundles, with* $\Phi = (\Phi_1, 0)$ *and* $\Phi_1 \in \Omega^0(W^+_{\mathrm{can}} \otimes L_1)$ *non-zero. Then the virtual Morse–Bott index* (20) *of Hitchin's Hamiltonian function* f *in* (8) *at the point* $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ *is given by minus twice the Euler characteristic of the negative-weight elliptic complex for the holomorphic pair* $(\bar\partial_A, \varphi)$*, where* $\Phi_1 = (\varphi, 0) \in \Omega^0(L_1) \oplus \Omega^{0,2}(L_1)$*, and equals*

$$\lambda^-_{[A,\Phi]}(f) = -2\chi_h(X) - (c_1(L_1) - c_1(L_2)) \cdot c_1(X) - (c_1(L_1) - c_1(L_2))^2 . \qquad (21)$$

Corollary 10 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a point represented by a Seiberg–Witten monopole)

(See F. and Leness [23, Corollary 6].) Let X be a closed, complex Kähler surface, $\mathfrak{t}$ be a spin^u structure over X, and $\mathfrak{s}$ be a spin^c structure over X. If $[A,\Phi] \in \mathscr{M}_\mathfrak{t}$ is a point represented by a reducible, non-Abelian type 1 monopole in the image of the embedding of the moduli space $M_\mathfrak{s}$ of Seiberg–Witten monopoles on $\mathfrak{s}$ into the moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles on $\mathfrak{t}$, then the virtual Morse–Bott index (20) *of Hitchin's Hamiltonian function f in* (8) *on the moduli space $\mathscr{M}_\mathfrak{t}$ at $[A,\Phi]$ is given by*

$$\lambda^-_{[A,\Phi]}(f) = -2\chi_h(X) - (c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \cdot c_1(X) - (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2, \qquad (22)$$

where $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ and $c_1(\mathfrak{t}) := c_1(E) + c_1(W^+) \in H^2(X;\mathbb{Z})$.

Corollary 11 (Positivity of virtual Morse–Bott index of Hitchin's Hamiltonian at point represented by Seiberg–Witten monopole for a feasible spin^u structure)

(See F. and Leness [23, Corollary 7].) Let X be a closed, complex Kähler surface with $b_1(X) = 0$, $b^-(X) \geq 2$, and $b^+(X) \geq 3$. If $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ is the blow-up of X and $\tilde{\mathfrak{t}}$ is the spin^u structure on $\tilde{X}$ constructed in Theorem 8, then for all non-empty Seiberg–Witten moduli subspaces $M_{\tilde{\mathfrak{s}}}$ that are continuously embedded in $\mathscr{M}_{\tilde{\mathfrak{t}}}$ as type 1 reducible, non-Abelian monopoles, the virtual Morse–Bott index (20) of Hitchin's Hamiltonian f in (8) on $\mathscr{M}_{\tilde{\mathfrak{t}}}$ is positive at all points in $M_{\tilde{\mathfrak{s}}}$.

Compact, connected, minimal complex surfaces: Test case

Compact, connected, minimal complex surfaces have the rough Kodaira classification according their **Kodaira dimension** (namely, $\mathrm{kod}(X) = -\infty$, 0, 1, or 2) and the more refined, Kodaira–Enriques classification (see Figure 5.5) within each value of $\mathrm{kod}(X)$.

A compact, complex surface X is projective if and only if there exists a holomorphic line bundle $\mathscr{L}$ over X with $c_1^2(\mathscr{L}) > 0$ (see [7, Section IV.6, Theorem 6.2, p. 159]).

In their proof of [7, Section IV.6, Theorem 6.2, p. 159], Barth, Hulek, Peters, and Van de Ven prove that $c_1^2(\mathscr{L}) > 0$ is equivalent to $\mathscr{L}$ being an ample line bundle, so $\mathscr{L}^{\otimes n}$ is very ample for some $n \geq 1$.

By definition, a line bundle $\mathscr{M}$ over a complex analytic space Z is **very ample** if its space of holomorphic sections, $H^0(Z, \mathscr{M})$, yields a holomorphic embedding of Z into $\mathbb{CP}^N$ for some $N \geq 1$ (see [7, Section I.19, p. 57]).

Class of X	$\mathrm{kod}(X)$	smallest $n>0$ with $\mathcal{K}_X^{\otimes n} = \mathcal{O}_X$	$b_1(X)$	possible value of $\mathrm{a}(X)$	c_1^2	c_2
1) minimal rational surfaces			0	2	8 or 9	4 or 3
2) minimal surfaces of class VII	$-\infty$		1	$0, 1$	≤ 0	≥ 0
3) ruled surfaces of genus $g \geq 1$			$2g$	2	$8(1-g)$	$4(1-g)$
4) Enriques surfaces		2	0	2	0	12
5) bi-elliptic surfaces		$2, 3, 4, 6$	2	2	0	0
6) Kodaira surfaces	0					
a) primary		1	3	1	0	0
b) secondary		$2, 3, 4, 6$	1	1	0	0
6) K 3-surfaces		1	0	$0, 1, 2$	0	24
8) tori		1	4	$0, 1, 2$	0	0
9) minimal properly elliptic surfaces	1			$1, 2$	0	≥ 0
10) minimal surfaces of general type	2		$\equiv 0(2)$	2	> 0	> 0

Figure 5.5: Kodaira–Enriques classification of compact, connected, minimal complex surfaces (from Barth, Hulek, Peters, and Van de Ven [7, Section VI.1, Table 10, p. 244]).

Minimal compact, complex surfaces of general type have $c_1^2(X) = K_X^2 > 0$ (where $K_X := \wedge^2 \mathscr{T}_X^*$ is the canonical line bundle and $\mathscr{T}_X \cong T^{1,0}X$ is the holomorphic tangent bundle), so they are **projective** (see [7, Section IV.6, Corollary 6.3, p. 160]).

For smooth complex projective surfaces, we may construct a **compactification** of the moduli space of non-Abelian monopoles, $\mathscr{M}_{\mathfrak{t}}$, as a **complex, projective variety**.

Smooth complex projective surfaces are Kähler (restrict the Fubini–Study metric on complex projective space to the embedded surface).

An analogue of Witten's dichotomy for Seiberg–Witten monopoles over complex, Kähler surfaces (see Morgan [46] or Witten [63]) shows that non-Abelian monopoles are always of "type 1" or "type 2" and that it suffices to only consider non-Abelian monopoles of type 1, that is, **projective vortices**, in partial analogy with Bradlow [10, 11].

We call a pair (A, φ) a projective vortex if A is a unitary connection on a Hermitian vector bundle E over a complex, Kähler surface (X, ω) that induces a fixed unitary connection, A_d, on the Hermitian line bundle $\det E$ and φ is a section of E such that

$$\Lambda_\omega (F_A)_0 = \frac{i}{2}(\varphi \otimes \varphi^*)_0,$$
$$(F_A^{0,2})_0 = 0,$$
$$\bar{\partial}_A \varphi = 0,$$

where Λ_ω is the adjoint of $\omega \wedge \cdot : \Omega^0(X) \to \Omega^{1,1}(X)$.

When $F_{A_d}^{0,2} = 0$, a version of the **Hitchin–Kobayashi correspondence** essentially due to Bradlow [11] shows that there is a bijection (up to gauge equivalence) between projective vortices and (poly-)stable holomorphic pairs, namely pairs $(\bar\partial_E, \varphi)$ of holomorphic structures $\bar\partial_E$ on E and holomorphic sections φ of E,

$$F_{\bar\partial_E} = 0,$$
$$\bar\partial_E \varphi = 0,$$

where $F_{\bar\partial_E} := \bar\partial_E \circ \bar\partial_E$, that obey Bradlow's (poly-)stability criterion.

When E has rank two, there is a set-theoretic bijection,

$$\mathscr{M}_{\mathrm{t},1} \cong \mathfrak{M}_{\mathrm{ps}}(E, \omega),$$

between the moduli space of type 1 non-Abelian monopoles and the moduli space of **polystable** holomorphic pairs.

Moreover, there is an isomorphism (in the sense of real analytic spaces),

$$\mathscr{M}^0_{\mathrm{t},1} \cong \mathfrak{M}^0(E, \omega),$$

between the moduli space of non-zero-section, type 1 non-Abelian monopoles and the moduli space of non-zero-section, **stable** holomorphic pairs.

By adapting the proofs of previous related results due to Huybrechts and Lehn [37, 38], Yinbang Lin [43], Okonek, Schmitt, and Teleman [47], and Wandel [61], we can prove that there exists a **complex projective moduli space** $\mathfrak{M}_{\mathrm{ss}}(E,\omega)$ of holomorphic pairs $(\mathscr{E},\varphi)$ of sheaves and sections that are **semistable** in the sense Bradlow, Gieseker, and Maruyama (see F, Leness, and Wentworth [25]).

The complex projective moduli space $\mathfrak{M}_{\mathrm{ss}}(E,\omega)$ contains $\mathfrak{M}^0(E,\omega)$ as an open subspace and the isomorphism and embeddings,

$$\mathscr{M}_{\mathrm{t},1}^0 \cong \mathfrak{M}^0(E,\omega) \hookrightarrow \mathfrak{M}_{\mathrm{ss}}(E,\omega) \hookrightarrow \mathbb{CP}^N,$$

are circle-equivariant.

Therefore, by combining the

- Białynicki–Birula and (virtual) Morse–Bott theories due to F in [21],
- Methods of F and Leness in [23], and
- Deformation theory and calculations of virtual Morse–Bott indices by F, Leness, and Wentworth in [25],

we obtain a new, gauge-theoretic proof of the known Bogomolov–Miyaoka–Yau inequality (1) for compact, complex surfaces of general type.

Next steps

We have been exploring two main directions:

Uhlenbeck bubbling

Extend our calculation of virtual Morse–Bott indices for points $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}$ represented by Seiberg–Witten monopoles to points $[A,\Phi] \in \bar{\mathscr{M}}_{\mathfrak{t}}$ that lie in $\mathrm{Sym}^{\ell}(X) \times \mathscr{M}_{\mathfrak{t}(\ell)}$, that is, allow for bubbling.

There are two ways to address this:

1. Replace $\bar{\mathscr{M}}_{\mathfrak{t}}$ by the Gieseker compactification $\mathfrak{M}_{\mathrm{ss}}(E,\omega)$ when X is a compact, complex projective surface of general type;
2. Use gluing to construct local models for Uhlenbeck boundary points.

The second approach should also lead to gauge-theoretic proofs of the Bogomolov–Miyaoka–Yau inequality for compact, complex surfaces of general type, but may extend the scope to include smooth four-manifolds of Seiberg–Witten simple type.

The virtual Morse–Bott index of a Seiberg–Witten fixed point **decreases** as an instanton bubble of multiplicity ℓ forms, but a version of the Bogomolov–Gieseker inquality due to Bradlow [11, Theorem 4.1, p. 208] appears to prevent ℓ from becoming so large that the virtual Morse–Bott index of a Seiberg–Witten fixed point is non-positive.

Non-integrable almost complex structures and non-Kähler metrics

Extend our calculation of virtual Morse–Bott indices when X is a Kähler surface by allowing X to be an almost Hermitian, smooth four-manifold.

For this purpose, a conjecture due to Tobias Shin [26] appears useful:

> *If (X, g, J) is an almost Hermitian manifold, then there is a sequence $\{J_i\}_{i\in\mathbb{N}}$ of almost complex structures J_i on X such that the C^0-norms of the Nijenhuis tensors N_{J_i} become arbitrarily small as $i \to \infty$.*

Shin's conjecture holds for examples of four-manifolds that are **almost complex but not complex** (see Fernandez, Shin, and Wilson [26]).

If Shin's conjecture is true, methods of *holomorphic approximation* (in the spirit of Auroux, Donaldson, and Taubes) in combination with *gluing* should allow us to extend our calculations to the general case of Conjecture 1, where X is a standard four-manifold of Seiberg–Witten simple type.

A weaker version of Shin's conjecture has been proved by Johnny Evans [20] for symplectic manifolds (using results due to Donaldson [16]) when the C^0 norm is replaced L^2 norm.

Thank you for your attention!

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