1

2-torsions in singular instanton homology

joint w/ Deeparaj Bhat

Zhenkun Li

(erase those)

Kh = Khovanov

Shu = Shumakovitch

KM = Kronheimer-Mrowka

(write those later)

Thm A fibered K $I^\#(K)$ 2-tor

Thm B $g>1$ $I^\#(S^3_{\frac{1}{2}}(K))$ 2-tor

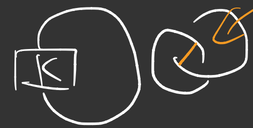
2

Thm (KM '11)

$\exists$ spectral sequences (over $\mathbb{Z}$)

$Kh(m(K)) \Rightarrow I^\#(K)$ sharp

$\widetilde{Kh}(m(K)) \Rightarrow I^\natural(K)$ natural

$I^\#(K) = I($  $)$ sw class

$I^\natural(K) = I($  $)$ earring

$I^\natural(K;\mathbb{C}) \cong KHI(K)$

singular suture

3

Thm (KM '10)

$\exists$ Alexander gr on $KHI(K)$

w/ top gr detects $g(K)$

Cor. $\widetilde{Kh}(K) \cong \mathbb{Z}$

(or $Kh(K) \cong \mathbb{Z}^2$)

iff $K = U$ = unknot

4

Fact $Kh(T_{2,3}) = \mathbb{Z}^2 \oplus (\mathbb{Z}^2 \oplus \mathbb{Z}/2)$

$= H^*(S^2) \oplus H^*(SO(3))$

$= H^*(R(T_{2,3}))$

$R(K) = \{ \rho : \pi_1(K) \to SU(2)$

$tr\, \rho(m_K) = 0 \}$

$\divideontimes$ no conjugation

$Kh(T_{2,2n+1}) = \mathbb{Z}^2 \oplus (\mathbb{Z}^2 \oplus \mathbb{Z}/2)^n$

$= H^*(R(T_{2,2n+1}))$

(From those exs, propose conj)

2

Conj (Shu'14) For $K \neq U$,

$Kh(K)$ has 2-torsion.

i.e. $x \in Kh$ s.t. $2x = 0$

- proved for alternating knots
- $\exists$ 2-tor iff $\dim Kh(\mathbb{Z}/2) > \dim Kh(\mathbb{C})$
- S.S. $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightsquigarrow \mathbb{Z}/2$

  $\mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \rightsquigarrow 0$

3

$I^\#(K) \,``="\, H(R(K), d_{Morse} + d_{instanton})$

$\exists$ Morse-Bott S.S. $H(R(K)) \Rightarrow I^\#(K)$

Q1 2-tor in $I^\#(K)$?

- not many comp $T_{2,n}$ (KM)

  Hedden-Herald-Kirk Poudel-Saveliev

Q2 2-tor in $I^\#(Y) = I^\natural(Y, U)$?

- computed over $\mathbb{C}$ or char $\neq 2$

  by Scaduto, Baldwin-Sivek, ...
- over $\mathbb{Z}$, $\Sigma_2(K)$ alternating K (Scaduto)

Any Q so far?

④

Thm A (Bhat-Li-Y.)

When $K \neq U$, fibered,

$I^\#(K)$ has 2-torsion

Thm B (BLY)

When $g(K) > 1$,

$I^\#(S^3_{1/2}(K))$ has 2-torsion

②

Conj (KM) over $\mathbb{C}$

$KHI(K) \cong \widehat{HFK}(K)$

$I^\#(Y) \cong \widehat{HF}(Y)$

$\Bigg\}$ Heegaard Floer

Conj (By examples) (come back if have time)

$\widehat{HFK}$, $\widehat{HF}$ no torsion.

- $I^\#(P)$ has 2-tor (Ali Mike Chris)
- Don't expect $\widehat{HF} \cong I^\#$ over $\mathbb{Z}$

3

Rem 1 $g=1$, tech applies to $K \# T_{2,3}$

$$S^3_{\frac{1}{2}}(T_{2,3}) = -\Sigma(2,3,11)$$

Rem 2 Def $\#_2 = \#$ 2-tor

$$:= \frac{1}{2}\left(I^{\#}(\mathbb{Z}/2) - I^{\#}(\mathbb{C})\right).$$

$$\#_2 I^{\#}(S_{\frac{1}{2}}) \geq 2\,KHI(g) - 1 + 2\#_2 I^{\#}(K_1)$$

when $g>1$ Alex gr

Q?? Write Thm A, B in 1

Rotate board

4 or 2

Idea: Compare $\mathbb{Z}/2$, $\mathbb{C}$

Thm (Bhat) $\exists$ exact $\Delta(\mathbb{Z}, \mathbb{Z}/2, \mathbb{C})$

$$\begin{tikzcd} I^{\#}(S_0 \overset{\text{``}S^3_0(K)}{}) \arrow[rr] & & I^{\#}(S_2) \arrow[dl] \\ & I^{\#}(K) \arrow[ul] & \end{tikzcd}$$

Change framing $\mu' = \mu + \lambda$ $\lambda' = -\mu$

$$2\mu' + \lambda' = \mu + 2\lambda$$

③ or another blackboard.

(Bhat)

$$\begin{tikzcd}[column sep=small]
I^\#(S^3;\mathbb{Z}/2) \arrow[rr] & & I^\#(S^3_{\frac{1}{2}}(K);\mathbb{Z}/2) \arrow[dl] \\
& I^\#(S^3_1(K), K_1;\mathbb{Z}/2) \arrow[ul] &
\end{tikzcd}$$

dual knot

(omit dim)

$I^\#(S^3)$

① $I^\#(S_{\frac{1}{2}};\mathbb{Z}/2) \geq I^\#(K_1;\mathbb{Z}/2)-1$

④

(Li-Y.)

$$\begin{tikzcd}[column sep=small]
I^\#(S^3;\mathbb{C}) \arrow[rr] & & I^\#(S^3_{\frac{1}{2}}(K);\mathbb{C}) \arrow[dl] \\
& \mathrm{Cone}(\alpha'\circ\beta' : KHI(K_1) \circlearrowleft) \arrow[ul] &
\end{tikzcd}$$

② $I^\#(S_{\frac{1}{2}};\mathbb{C}) \leq \mathrm{Cone}+1$

③ (KM'18) $I^\#(K_1;\mathbb{Z}/2) = 2\,I^\natural(K_1;\mathbb{Z}/2)$ ✓

④ (UCT) $I^\natural(K_1;\mathbb{Z}/2) = I^\natural(K_1;\mathbb{C}) + 2\,\#_2 I^\natural(K_1)$ ✓

⑤ (KM'11) $I^\natural(K_1;\mathbb{C}) = KHI(K_1)$ ✓

⑥ (Li-Y.) $\mathrm{rk}(\alpha'\circ\beta') \geq 2KHI(K_1,g) = 2KHI(K,g)$ when $g>1$.

$\Rightarrow$ $\boxed{\#_2 I^\#(S_{\frac{1}{2}}) \geq 2KHI(g)-1+2\,\#_2 I^\natural(K_1)}$ $\Rightarrow$ Thm B.

Over $\mathbb{C}$

Prop (Li-Y. '20) $\exists\ \Delta$

$$\begin{tikzcd}
I^{\#}(S_0) \arrow[r] & KHI(K_1) \arrow[r] \arrow[dl, "\alpha"] & I^{\#}(S_2) \arrow[dl] \\
KHI(K) \arrow[u] & KHI(K) \arrow[u, "\beta"] &
\end{tikzcd}$$

By octa lem, $\exists\ \Delta$

$$\begin{tikzcd}
I^{\#}(S_0) \arrow[rr] & & I^{\#}(S_2) \arrow[dl] \\
& \mathrm{Cone}(\alpha\circ\beta) \arrow[ul] &
\end{tikzcd}$$

Prop (Li-Y. '22)

$\alpha\circ\beta = d_{1,+} + d_{1,-}$.

$gr(d_{1,\pm}) = \pm 1$ on $KHI(K)$

when $g>1$

$\mathrm{rk}\,\alpha\circ\beta \geq \mathrm{rk}\, d_{1,+}|_{g-1}$ ← Alex gr

$+\mathrm{rk}\, d_{1,-}|_{1-g}$

Change framing $K \to K_1$

$\alpha \circ \beta \to \alpha' \circ \beta'$

$d_{1,\pm} \to d'_{1,\pm}$

Prop (Y.) when $g > 1$.

$d_{1,\pm}|_{\pm(g-1)}$ surjective $\to$ 6

$\square$

(−1 surgery on K_1 gives S^3)

integral surgery formula Li-Y. '22)

Prop (Bhat-Li-Y.)

$I^\#(K;\mathbb{C}) \cong \mathrm{Cone}(d_{1,+} + d_{1,-} : KHI(K) \circlearrowleft)$

Prop (Baldwin-Sivek '22)

K fibered $\mathrm{rk}\, d_{1,\pm}|_{\pm(g-1)} > 1$

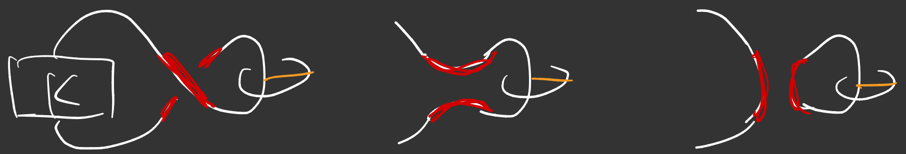
$\#_2 I^\#(K) = \mathrm{rk}(d_{1,+} + d_{1,-}) \Rightarrow$ Thm A $\square$

Rem. Can compute $I^\#(T_{p,q};\mathbb{C})$

More comments.

Prop (KM '10) $\exists$ exact $\triangle$

$$\begin{tikzcd}
I^\natural(K) \arrow[rr, "2\Theta"] & & I^\natural(K) \arrow[dl] \\
& I^\#(K) \arrow[ul] &
\end{tikzcd}$$



Q. (in progress) commute?

$$\begin{tikzcd}
I^\natural(K;\mathbb{C}) \arrow[r, "2\Theta"] \arrow[d, "\cong (KM)"] & I^\natural(K;\mathbb{C}) \arrow[r] \arrow[d, "\cong (KM)"] & I^\#(K;\mathbb{C}) \arrow[d, "="] \\
KHI(K) \arrow[r, "\alpha\circ\beta"', "d_{1,+}+d_{1,-}"'] & KHI(K) \arrow[r] & I^\#(K;\mathbb{C})
\end{tikzcd}$$

rk(2Θ) hard   rk $d_{1,\pm}$ easy.

Q: How to define $HFK^\#$?

s.t.

- $HFK^\#(U) = \mathbb{Z}^2$
- $HFK^\#(\mathbb{Z}/2) = 2\,\widehat{HFK}(\mathbb{Z}/2)$
- $$\begin{tikzcd} \widehat{HFK}(C) \arrow[rr, "d_{1,z}+d_{1,w}"] & & \widehat{HFK}(C) \arrow[dl] \\ & HFK^\#(C) \arrow[ul] & \end{tikzcd}$$

Baldwin - Levine - Sarkar, Dowlin

Ozsváth - Stipsicz - Szabó, Zemke

Gong - Marengon

CFK: $dx = \sum \# M(\phi)\, U^{n_w(\phi)} V^{n_z(\phi)} y$

Set $U = V = 2H$

$$\widehat{CFK} = CFK / H = 0$$

$$CFK^\# = CFK / H^2 = 0$$

$$= \mathrm{Cone}\left(\widehat{CFK} \xrightarrow{2(d_{1,+} + d_{1,-})} H \widehat{CFK}\right)$$

Q : How to decompose 2Θ in Kh?

$$
\begin{tikzcd}
\widetilde{Kh}(K) \arrow[rr, "2\Theta"] & & \widetilde{Kh}(K) \arrow[dl] \\
& Kh(K) \arrow[ul] &
\end{tikzcd}
$$

$\widetilde{BN}(K)$ $\quad x^2 + Hx = 0$

over $\mathbb{C}$ (Kotelskiy-Watson-Zibrowius '19)

$$\widetilde{Kh} = \mathrm{Cone}(\widetilde{BN} \xrightarrow{H} \widetilde{BN})$$

$$Kh = \mathrm{Cone}(\widetilde{BN} \xrightarrow{H^2} \widetilde{BN})$$