Frobenius manifolds for generalized root systems of type D

a joint work with Akishi Ikeda, Takumi Otani and Yuuki Shiraishi

Atsushi TAKAHASHI

OSAKA UNIVERSITY

January 27, 2023
Homological Mirror Symmetry Conference at Miami

Motivation

Clarify a certain one-to-one correspondence among

- triangulated categories with full exceptional collections,
- generalized root systems,
- semi-simple Frobenius structures.

In particular, we are interested in the Hurwitz Frobenius structures and their relation to Bridgeland's space of stability conditions for Fukaya categories studied by Haiden–Katzarkov–Kontsevich.

Generalized root systems

Definition 1

A *root system* $\mathcal{R}$ of rank μ is a tuple $(\mathcal{N}, I, \Delta_{re})$ where

- $\mathcal{N}$ is a free $\mathbb{Z}$-module of rank μ,
- $I : \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{Z}$ is a symmetric $\mathbb{Z}$-bilinear form,
- Δ_{re} is a subset of $\mathcal{N}$, called the set of real roots,

satisfying the following properties:

1. $\mathcal{N} = \mathbb{Z}\Delta_{re}$.
2. For each $\alpha \in \Delta_{re}$, $I(\alpha, \alpha) = 2$.
3. For each $\alpha \in \Delta_{re}$, define a reflection $r_\alpha \in \mathrm{Aut}_{\mathbb{Z}}(\mathcal{N}, I)$ by $r_\alpha(\lambda) := \lambda - I(\alpha, \lambda)\alpha$. Then, $r_\alpha(\Delta_{re}) = \Delta_{re}$.

Definition 2

The group $W(\mathcal{R}) := \langle r_\alpha \,|\, \alpha \in \Delta_{re} \rangle$ is called the *Weyl group* of $\mathcal{R}$.

Remark 3

For simplicity, we only consider "simply-laced" root systems.

Remark 4

It is important that I is not assumed to be positive definite.

Remark 5

The Weyl group $W(\mathcal{R})$ is not necessary a Coxeter group.
Therefore, we need an intrinsic definition of a Coxeter element which does not depend on a particular presentation of $W(\mathcal{R})$.

Definition 6

The *signature* of $\mathcal{R}$ is the signature (μ_+, μ_0, μ_-) of $I_{\mathbb{R}}$ where μ_+ (resp. μ_0, μ_-) is the number of positive (resp. zero, negative) eigenvalues of $I_{\mathbb{R}}$.
In particular, $\mu_0 = \mathrm{rank}_{\mathbb{Z}}(\mathrm{rad}(I))$ where

$$\mathrm{rad}(I) := \{\lambda \in \mathcal{N} \mid I(\lambda, \lambda') = 0,\ \lambda' \in \mathcal{N}\}.$$

Proposition 7

Let $\mathcal{R} = (\mathcal{N}, I, \Delta_{re})$ be a root system of rank μ. The following are equivalent.

1. $(\mu_+, \mu_0, \mu_-) = (\mu, 0, 0)$.
2. Δ_{re} *is a finite set.*
3. $W(\mathcal{R})$ *is a finite group.*

$\mathcal{R}$ satisfying these three conditions is called a *finite root system*.

Remark 8

Other types of our interest are

- affine root systems: $(\mu_+, \mu_0, \mu_-) = (\mu - 1, 1, 0)$,
- elliptic root systems: $(\mu_+, \mu_0, \mu_-) = (\mu - 2, 2, 0)$,
- cuspidal root systems: $(\mu_+, \mu_0, \mu_-) = (\mu - 2, 1, 1)$,
- (no name) : $(\mu_+, \mu_0, \mu_-) = (\mu_+, \mu_0, 0)$.

Irreducibility, isomorphisms, ..., are naturally defined.

The following is well-known (cf. Bourbaki).

Proposition 9

An irreducible finite root system is isomorphic to one of the following types:

1. A_μ $(\mu \geq 1)$:
   - $\mathcal{N} := \{(n_1, \ldots, n_{\mu+1}) \in \mathbb{Z}^{\mu+1} \mid n_1 + \cdots + n_{\mu+1} = 0\}$,
   - I: *the restriction of the standard* $\mathbb{Z}$*-bilinear form on* $\mathbb{Z}^{\mu+1}$,
   - $\Delta_{re} := \{\alpha \in \mathcal{N} \mid I(\alpha, \alpha) = 2\}$.
2. D_μ $(\mu \geq 4)$:
   - $\mathcal{N} := \{(n_1, \ldots, n_\mu) \in \mathbb{Z}^\mu \mid n_1 + \cdots + n_\mu \in 2\mathbb{Z}\}$,
   - I: *the restriction of the standard* $\mathbb{Z}$*-bilinear form on* $\mathbb{Z}^\mu$,
   - $\Delta_{re} := \{\alpha \in \mathcal{N} \mid I(\alpha, \alpha) = 2\}$.
3. E_μ $(\mu = 6, 7, 8)$: *Omit today.*

The most important thing is the following

Definition 10

Let $\mathcal{R} = (\mathcal{N}, I, \Delta_{re})$ be a root system of rank μ.
A subset $B = \{\alpha_1, \dots, \alpha_\mu\}$ of Δ_{re} is a *root basis* if $\Delta_{re} = W_B \cdot B$ where $W_B := \langle r_{\alpha_1}, \dots, r_{\alpha_\mu} \rangle \subset W(\mathcal{R})$.

It follows that $\mathcal{N} = \mathbb{Z}B$ and $W(\mathcal{R}) = W_B$.
In general, $\mathcal{N} = \mathbb{Z}B$ does not imply that B is a root basis.

Definition 11

Let $\mathcal{R} = (\mathcal{N}, I, \Delta_{re})$ be a root system of rank μ. An element $c \in W(\mathcal{R})$ is called a *Coxeter element* of $\mathcal{R}$ if there exists a root basis $B = \{\alpha_1, \dots, \alpha_\mu\}$ such that $c = r_{\alpha_1} \dots r_{\alpha_\mu}$.

Definition 12

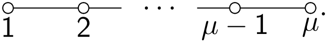
A pair $(\mathcal{R}, c)$ of a root system $\mathcal{R}$ and a Coxeter element c of $\mathcal{R}$ is called a *generalized root system*.

Classification of finite generalized root systems

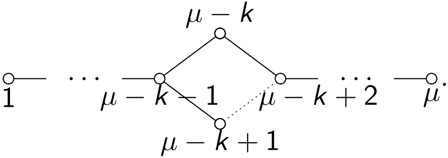
Theorem 13 (Nakamura–Shiraishi–T)

A generalized root system $(\mathcal{R}, c)$ *with irreducible finite* $\mathcal{R}$ *is isomorphic to one of the following types:*

1. A_μ $(\mu \geq 1)$ *whose Coxeter–Dynkin diagram is*



2. $D_{\mu,k}$ $(\mu \geq 4, 1 \leq k \leq [\mu/2])$ *whose Coxeter–Dynkin diagram is*



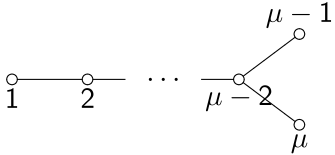
3. $E_{6,1}$, $E_{6,2}$, $E_{6,3}$, $E_{7,1}$, $\ldots$, $E_{7,5}$, $E_{8,1}$, $\ldots$, $E_{8,9}$. *(omit today)*

For a root basis $B = \{\alpha_1, \dots, \alpha_\mu\}$, the Coxeter–Dynkin diagram is given as follows:

- For each α_i, put a vertex $\circ_i$.
- " $\circ_i \qquad \circ_j$ " if $I(\alpha_i, \alpha_j) = 0$.
- " $\circ_i$ ——— $\circ_j$ " if $I(\alpha_i, \alpha_j) = -1$.
- " $\circ_i$ ·········· $\circ_j$ " if $I(\alpha_i, \alpha_j) = +1$.

Remark 14

$D_{\mu,1}$ is a root system of type D_μ with the standard/usual Coxeter element, which will be denoted simply by D_μ:



Problems

The characteristic polynomial of c of $D_{\mu,k}$ is given by

$$\phi_{D_{\mu,k}}(t) = (t^{\mu-k}+1)(t^k+1).$$

Therefore, D_μ and $D_{\mu,k}$ $(k \geq 2)$ are not isomorphic.

Problem 15

Construct $D_{\mu,k}$ $(k \geq 2)$ geometrically.

Problem 16

Construct a Frobenius structure “compatible with $\phi_{D_{\mu,k}}(t)$”.

Our purpose is to give an affirmative answer to these problems based on the idea that "type D_μ is an type A with an involution".

Geometric construction of $D_{\mu,k}$

For $1 \leq k \leq [\mu/2]$, consider a holomorphic map

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \quad z \mapsto f(z) = z^{2(\mu-k)} + z^{-2k},$$

and an involution $\iota : \mathbb{C}^* \longrightarrow \mathbb{C}^*$, $z \mapsto -z$.
Then we have a short exact sequence

$$0 \longrightarrow H_1(\mathbb{C}^*; \mathbb{Z}) \longrightarrow H_1(\mathbb{C}^*, f^{-1}(R); \mathbb{Z}) \xrightarrow{\partial} \widetilde{H}_0(f^{-1}(R); \mathbb{Z}) \longrightarrow 0,$$

where $0 \ll R \in \mathbb{R}$ and $\widetilde{H}$ denotes the reduced homology

$$\widetilde{H}_0(f^{-1}(R); \mathbb{Z}) = \mathrm{Ker}(H_0(f^{-1}(R); \mathbb{Z}) \longrightarrow H_0(\mathbb{C}^*; \mathbb{Z})).$$

Remark 17

The intersection form $I_{\widetilde{H}_0}$ on $\widetilde{H}_0(f^{-1}(R); \mathbb{Z})$ naturally induces a finite root system of type $A_{2\mu-1}$ (exactly the same description as above).

Theorem 18 (Ikeda–Otani–Shiraishi–T)

The tuple $(\mathcal{N}, I, \Delta_{re}, c)$ *where*

- $\mathcal{N} := \{\lambda \in H_1(\mathbb{C}^*, f^{-1}(R); \mathbb{Z}) \mid \iota_*(\lambda) = -\lambda\}$,
- $I(\lambda, \lambda') := \frac{1}{2} I_{\widetilde{H}_0}(\partial\lambda, \partial\lambda')$, $\lambda, \lambda' \in \mathcal{N}$,
- $\Delta_{re} := \{\alpha \in \mathcal{N} \mid I(\alpha, \alpha) = 2\}$,
- $c \in \mathrm{Aut}_{\mathbb{Z}}(\mathcal{N}, I)$ *is the automorphism induced by the monodromy around the circle* $\{w \in \mathbb{C} \mid |w| = R\}$,

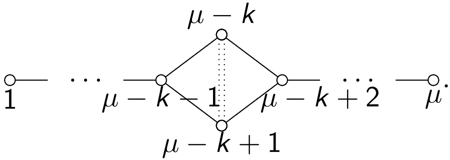
is the generalized root system of type $D_{\mu,k}$.

Remark 19

The tuple $(\mathcal{N}', I', \Delta'_{re}, c')$ where

- $\mathcal{N}' := H_1(\mathbb{C}^*, f^{-1}(R); \mathbb{Z})$,
- $I'(\lambda, \lambda') := I_{\widetilde{H}_0}(\partial\lambda, \partial\lambda')$, $\lambda, \lambda' \in \mathcal{N}'$,
- $\Delta'_{re} := \{\alpha \in \mathcal{N}' \mid I'(\alpha, \alpha) = 2\}$,
- $c \in \mathrm{Aut}_{\mathbb{Z}}(\mathcal{N}', I')$ is the automorphism induced by the monodromy around the circle $\{w \in \mathbb{C} \mid |w| = R\}$,

is the generalized affine root system of type $\widetilde{A}_{2(\mu-k),2k}$, whose "$\iota$-invariant generalized root system" is of type $\widetilde{A}_{\mu-k,k}$:



Idea of proof

There exist μ distinct points $x_1, \dots, x_\mu$ on $\mathbb{C}^*$ such that zeros of f are $\{x_1, -x_1, \dots, x_\mu, -x_\mu\}$.

For each i, choose a path p_i from $-x_i$ to x_i traveling counterclockwise around the origin. Then $\iota_*([p_i]) = -[p_i] + \delta$ where δ is the image of the generator of $H_1(\mathbb{C}^*; \mathbb{Z})$.

Note that $\{[p_1] - \delta/2, \dots, [p_\mu] - \delta/2\}$ is a $\mathbb{Q}$-basis of $\mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ and also that for $(n_1, \dots, n_\mu) \in \mathbb{Z}^\mu$ we have

$$\sum_{i=1}^{\mu} n_i \left([p_i] - \frac{1}{2}\delta \right) \in \mathcal{N} \Longleftrightarrow \sum_{i=1}^{\mu} n_i \in 2\mathbb{Z},$$

which is nothing the description of the D_μ-lattice.

It is easy to find the expected root basis.

ι-invariant deformation of f and the Frobenius structure

Consider the following deformation $F_{D_{\mu,k}}$ of f

$$F_{D_{\mu,k}}(z;\mathbf{s}) := z^{2(\mu-k)} + \sum_{i=1}^{\mu-k} s_{\infty,i} z^{2i-2} + \sum_{j=1}^{k} s_{0,k}^{2j-1} s_{0,j} \frac{1}{z^{2j}},$$

over the parameter space

$$\begin{aligned} M_{D_{\mu,k}} &:= \{(s_{\infty,1},\dots,s_{\infty,\mu-k},s_{0,1},\dots,s_{0,k-1},s_{0,k})\} \\ &= \mathbb{C}^{\mu-k}\times\mathbb{C}^{k-1}\times\mathbb{C}^{*}. \end{aligned}$$

$F_{D_{\mu,k}}$ is ι-invariant, $F_{D_{\mu,k}}(-z;\mathbf{s}) = F_{D_{\mu,k}}(z;\mathbf{s})$.

Example 20

$F_{D_{4,1}}(z;\mathbf{s}) := z^6 + s_{\infty,1} + s_{\infty,2}z^2 + s_{\infty,3}z^4 + s_{0,1}^2 z^{-2}.$
$F_{D_{4,2}}(z;\mathbf{s}) := z^4 + s_{\infty,1} + s_{\infty,2}z^2 + s_{0,2}s_{0,1}z^{-2} + s_{0,2}^4 z^{-4}.$

Frobenius structure on M is a tuple $(\circ, \eta, e, E)$ where $\circ : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \longrightarrow \mathcal{T}_M$, $\eta : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \longrightarrow \mathcal{O}_M$ and $e, E \in \Gamma(M, \mathcal{T}_M)$ satisfying several "flatness conditions" and "homogeneity conditions". It is locally described by *flat coordinates* and a local holomorphic function called the *Frobenius potential*.

There is a systematic construction of a Frobenius structure by the use of a primitive form (cf. Saito–T).

A key step is the following Kodaira–Spencer isomorphism.

Proposition 21

There exists an isomorphism of $\mathcal{O}_{M_{D_{\mu,k}}}$-free modules

$$KS_\iota : \mathcal{T}_{M_{D_{\mu,k}}} \cong \left(\mathcal{O}_{M_{D_{\mu,k}}}[z, z^{-1}] \Big/ \left(\frac{\partial F_{D_{\mu,k}}}{\partial z} \right) \right)^\iota .$$

By this isomorphism, a product structure $\circ$ on $\mathcal{T}_{M_{D_{\mu,k}}}$ is induced from the one on the RHS and we can define $e := KS_\iota^{-1}([1]) = \partial/\partial s_{\infty,1}$,

$$E := KS_\iota^{-1}([F_{D_{\mu,k}}]) = \sum_{i=1}^{\mu-k} d_{\infty,i} s_{\infty,i} \frac{\partial}{\partial s_{\infty,i}} + \sum_{j=1}^{k} d_{0,j} s_{0,j} \frac{\partial}{\partial s_{0,j}},$$

$$d_{\infty,i} := \frac{\mu - k - i + 1}{\mu - k}, \quad d_{0,j} := \frac{2k - 2j + 1}{2k} + \frac{1}{2(\mu - k)}.$$

Note that $d_{\infty,i}$ and $d_{0,j}$ are positive.

Let $\Omega_{F_{D_{\mu,k}},-\iota} := \left(\mathcal{O}_{M_{D_{\mu,k}}}[z, z^{-1}]dz \,/\, dF_{D_{\mu,k}}\right)^{-\iota}$.
There exists a non-degenerate symmetric $\mathcal{O}_{M_{D_{\mu,k}}}$-bilinear form

$$J_{F_{D_{\mu,k}},-\iota} : \Omega_{F_{D_{\mu,k}},-\iota} \times \Omega_{F_{D_{\mu,k}},-\iota} \longrightarrow \mathcal{O}_{M_{D_{\mu,k}}},$$

$$([\phi_1 dz], [\phi_2 dz]) \mapsto \frac{1}{2\pi\sqrt{-1}} \oint_{\left|\frac{\partial F_{D_{\mu,k}}}{\partial z}\right|=\epsilon} \frac{\phi_1 \phi_2}{\frac{\partial F_{D_{\mu,k}}}{\partial z}} dz.$$

A nowhere vanishing 1-form ζ yields the $\mathcal{O}_{M_{D_{\mu,k}}}$-isomorphism

$$\mathcal{T}_{M_{D_{\mu,k}}} \overset{\zeta}{\cong} \Omega_{F_{D_{\mu,k}},-\iota}, \quad X \mapsto [XF_{D_{\mu,k}} \cdot \zeta],$$

which enables one to define η on $\mathcal{T}_{M_{D_{\mu,k}}}$. If ζ is "good enough", one can show the tuple $(\circ, \eta, e, E)$ satisfies "flatness conditions" and "homogeneity conditions" and hence one obtains a Frobenius structure.

There exist a filtered de Rham cohomology group $\mathcal{H}^{(0)}_{F_{D_{\mu,k}},-\iota}$, the Gauß–Manin connection ∇ and the higher residue pairing $K_{F_{D_{\mu,k}},-\iota}$.

In particular, $[dz] \in \mathcal{H}^{(0)}_{F_{D_{\mu,k}},-\iota}$ is a primitive form with the minimal exponent $\frac{1}{2(\mu-k)}$, which induces on $M_{D_{\mu,k}}$ a Frobenius structure of rank μ and conformal dimension $1-\frac{2}{2(\mu-k)}$.

(From results by Milanov–Tseng, Ishibashi–Shiraishi–T, Milanov)

Theorem 22 (IOST)

1. *The Frobenius potential $\mathcal{F}$ is an element of*

$$\mathbb{Q}[t_{0,k}, t_{0,k}^{-1}][t_{\infty,1}, \dots, t_{\infty,\mu-k}, t_{0,1}, \dots, t_{0,k-1}],$$

   where $t_{*,*}$ is the flat coordinate corresponding to $s_{*,*}$.

2. *Exponents of the Frobenius structure are exponents of the characteristic polynomial $\phi_{D_{\mu,k}}(t)$ of the Coxeter element. Namely, we have*

$$\phi_{D_{\mu,k}}\left(e^{2\pi\sqrt{-1}\left(d_{*,*}-\frac{1}{2(\mu-k)}\right)}\right)=0.$$

Since $1 \leq k \leq \mu - k$, $d_{\infty,\mu-k} - \frac{1}{2(\mu-k)} = \frac{1}{2(\mu-k)}$ is indeed the "minimal" exponent, the smallest one with respect to the natural ordering $<$ on $\mathbb{Q}$. This shows that this Frobenius structure is the most natural one.

Example 23 ($D_4 = D_{4,1}$)

$$e = \frac{\partial}{\partial t_{\infty,1}},\ E = t_{\infty,1}\frac{\partial}{\partial t_{\infty,1}} + \frac{2}{3}t_{\infty,2}\frac{\partial}{\partial t_{\infty,2}} + \frac{1}{3}t_{\infty,3}\frac{\partial}{\partial t_{\infty,3}} + \frac{2}{3}t_{0,1}\frac{\partial}{\partial t_{0,1}},$$

$$\phi_{D_{4,1}}(t) = (t^3+1)(t+1), \quad \text{exponents: } \left\{\frac{1}{6}, \frac{3}{6}, \frac{3}{6}, \frac{5}{6}\right\},$$

$$\begin{aligned} \mathcal{F}_{D_{4,1}} &= \frac{1}{12}t_{\infty,1}^2 t_{\infty,3} + \frac{1}{12}t_{\infty,1}t_{\infty,2}^2 + t_{\infty,1}t_{0,1}^2 \\ &\quad - \frac{1}{216}t_{\infty,2}^3 t_{\infty,3} + \frac{1}{1632460}t_{\infty,3}^7. \end{aligned}$$

Example 24 ($D_{4,2}$)

$$e = \frac{\partial}{\partial t_{\infty,1}}, \ E = t_{\infty,1}\frac{\partial}{\partial t_{\infty,1}} + \frac{1}{2}t_{\infty,2}\frac{\partial}{\partial t_{\infty,2}} + t_{0,1}\frac{\partial}{\partial t_{0,1}} + \frac{1}{2}t_{0,2}\frac{\partial}{\partial t_{0,2}},$$

$$\phi_{D_{4,2}}(t) = (t^2+1)^2, \quad \text{exponents: } \left\{\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right\},$$

$$\begin{aligned}\mathcal{F}_{D_{4,2}} &= \frac{1}{8}t_{\infty,1}^2 t_{\infty,2} + \frac{1}{4}t_{\infty,1}t_{0,1}t_{0,2} \\ &\quad + \frac{1}{3}t_{\infty,2}t_{0,2}^4 + \frac{1}{32}t_{\infty,2}^2 t_{0,1}t_{0,2} + \frac{1}{3840}t_{\infty,2}^5 + \frac{1}{1536}\frac{t_{0,1}^3}{t_{0,2}}.\end{aligned}$$

Remark 25

If $k = 1$, then $\mathcal{F}$ is a polynomial. In particular, we have a natural identification

$$M_{D_\mu} \cup \{s_{0,k} = 0\} = \mathfrak{h}/W(D_\mu)$$

as expected, where $\mathfrak{h} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, \mathbb{C})$.

Remark 26

Dubrovin expected that an irreducible semi-simple Frobenius structure with positive degrees whose Frobenius potential is an algebraic function corresponds to an irreducible Coxeter group and its "quasi-Coxeter element". In particular, he expected that the cases for the "standard" Coxeter elements correspond to polynomial Frobenius structures, that is proven by Hertling.
Our result support Dubrovin's conjecture for D_μ.

Remark 27

Consider the map

$$M_{D_{\mu,k}} \longrightarrow M_{\widetilde{A}_{\mu-k,k}} := \mathbb{C}^{\mu-k} \times \mathbb{C}^{k-1} \times \mathbb{C}^*,$$

$$(\mathbf{s}_\infty, s_{0,1}, \ldots, s_{0,k-1}, s_{0,k}) \mapsto (\mathbf{s}_\infty, \frac{s_{0,1}}{s_{0,k}}, \ldots, \frac{s_{0,k-1}}{s_{0,k}}, s_{0,k}^2),$$

and the "ι-invariant version" of the story. Then $[dz/z] \in \mathcal{H}^{(0)}_{F_{D_{\mu,k}},\iota}$ is a primitive form with the minimal exponent 0, which induces on $M_{\widetilde{A}_{\mu-k,k}}$ a Frobenius structure of rank μ and conformal dimension 1 which is isomorphic to the one from orbifold Gromov–Witten theory for the orbifold $\mathbb{P}^1_{\mu-k,k}$.

Remark 28

The degree of the Lyashko–Looijenga map

$$LL : M_{\widetilde{A}_{\mu-k,k}} \setminus B \longrightarrow (\mathbb{C}^{\mu} \setminus diag) / S_{\mu},$$

where B is the bifurcation set, is calculated by Arnold and Dubrovin as

$$\frac{(\mu-1)!}{(\mu-k-1)!(k-1)!}(\mu-k)^{\mu-k}k^{k}.$$

On the other hand, it turns out that the double of this number is the number of all root bases for $D_{\mu,k}$.
(follows from results by Kluitmann and Nakamura–Shiraishi–T)

Thus, $M_{D_{\mu,k}}$ passes a compatibility test.

Thank you very much!