Motivation: ... or, how this is related to HMS

- rigorous mathematical definition due to Kontsevich and Kapranov
- $D^b(X) := D^b\mathrm{Coh}(X)$
- $\mathrm{Auteq}\, D^b(X)$
- Galois action on categories
- Symplectic duality

based on: H.Nakajima, D.Pei, ... (2020 - 2021)
Work in progress

Motivation: ... quantum algebra

- braiding and Knizhnik-Zamolodchikov equations

- decorated TQFT

- $X = T^* \mathrm{Gr}_G$ ➡ BPS q-series $\widehat{Z}_b(M_3; q)$

Non-perturbative complex Chern-Simons

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

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- decorated TQFT
- $X = T^* \mathrm{Gr}_G$ ⇨ BPS q-series $\widehat{Z}_b(M_3; q)$

Non-perturbative complex Chern-Simons

Conjecture:

$$D^b(X) \simeq U_q(\mathfrak{g})\text{-mod}^{\cdots} \qquad \text{at generic } q$$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

Coulomb branches of 3d $\mathcal{N} = 4$ theories:

$$X = T^*\mathbb{C}\mathbf{P}^1$$

Lee-Weinberg-Yi

$$X = T^*\mathbb{C}\mathbf{P}^n$$

Taubian-Calabi

$$X = T^*Gr(n, N)$$

quiver varieties

$$X = \mathrm{Hilb}^n(\mathbb{C}^2)$$

$$X = \mathcal{M}_H(G, C)$$

moduli spaces of monopoles

ALE, ALF, ALG, ALG* ...
metrics

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non-compact
hyper-Kahler

I J K

ω_I ω_J ω_K

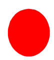
Hyper-Kähler Geometry and Invariants of Three-Manifolds

L. Rozansky

E. Witten

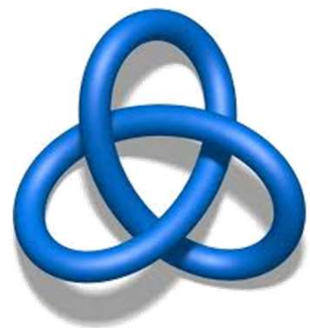
For compact X, we expect that the theory under consideration in this paper will obey the full axioms of a (non-unitary) topological quantum field theory as formalized by Atiyah [21]. (Unitarity would mean that the vector spaces $\mathcal{H}_\Sigma$ associated to two-manifolds Σ have hermitian metrics compatible in a natural way with the rest of the data; that is so in Chern-Simons theory but not in the theory considered here.) Proving this goes beyond the scope of the present paper, though we develop many of the relevant facts in Section 5. We suspect that a full direct proof would be far simpler than the corresponding analysis of Chern-

Modern low-dimensional topology:



"local"
operators

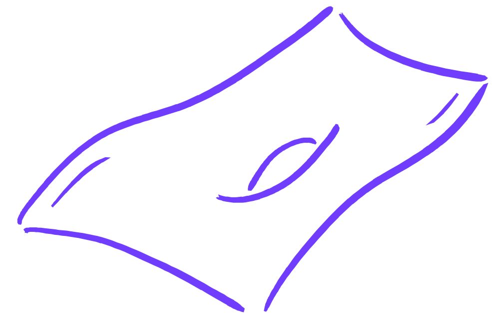
supported
at points



line
operators

webs

supported on
1-manifolds

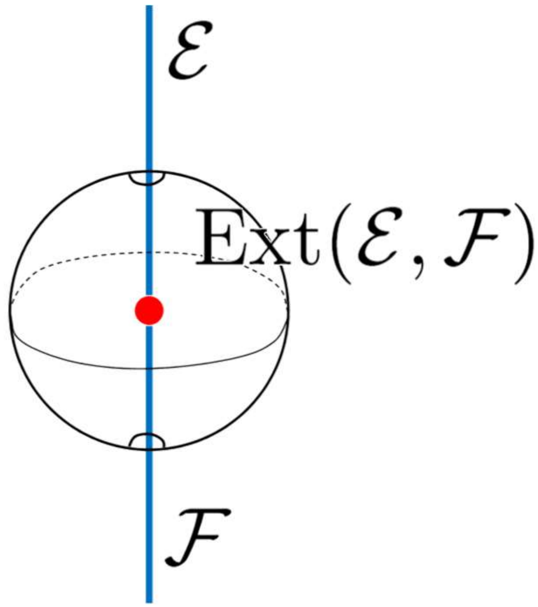


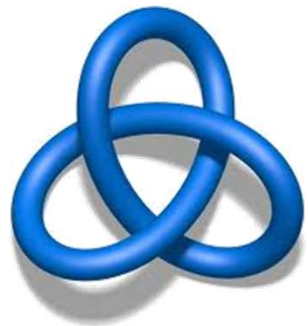
surface
operators

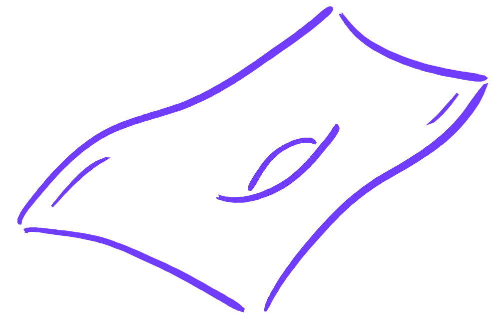
foams

supported on
2-manifolds

Modern low-dimensional topology:







“local” operators

line operators

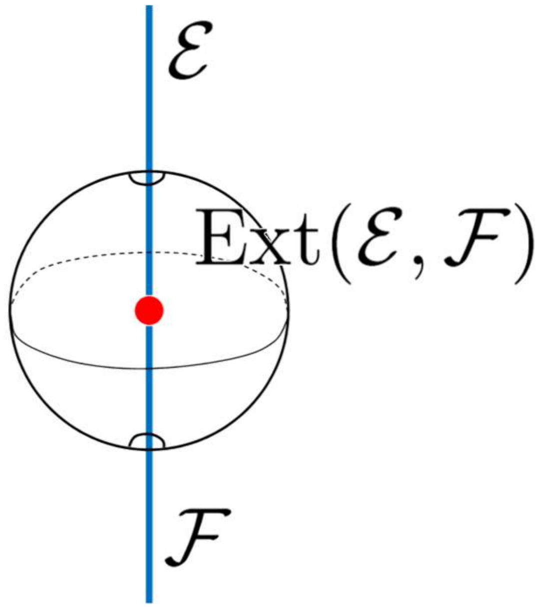
surface operators

Logarithmic (non-semisimple)

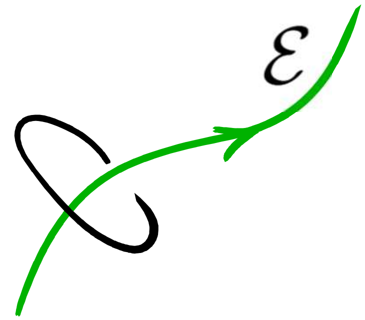
Symmetries
decorated TQFT

Higher groups

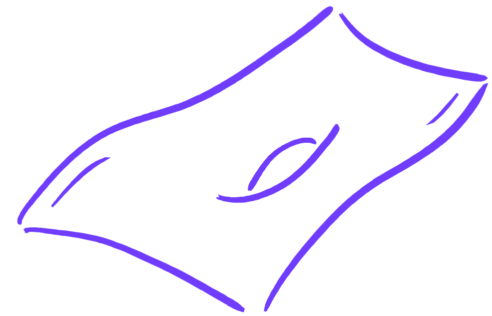
Modern low-dimensional topology:



"local" operators



line operators

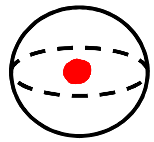
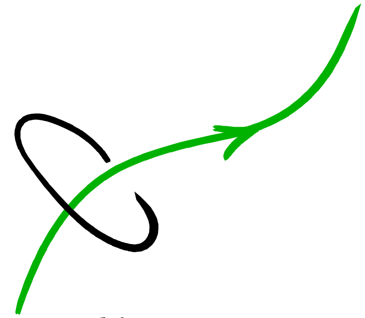
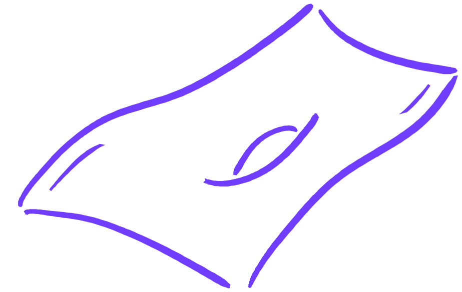


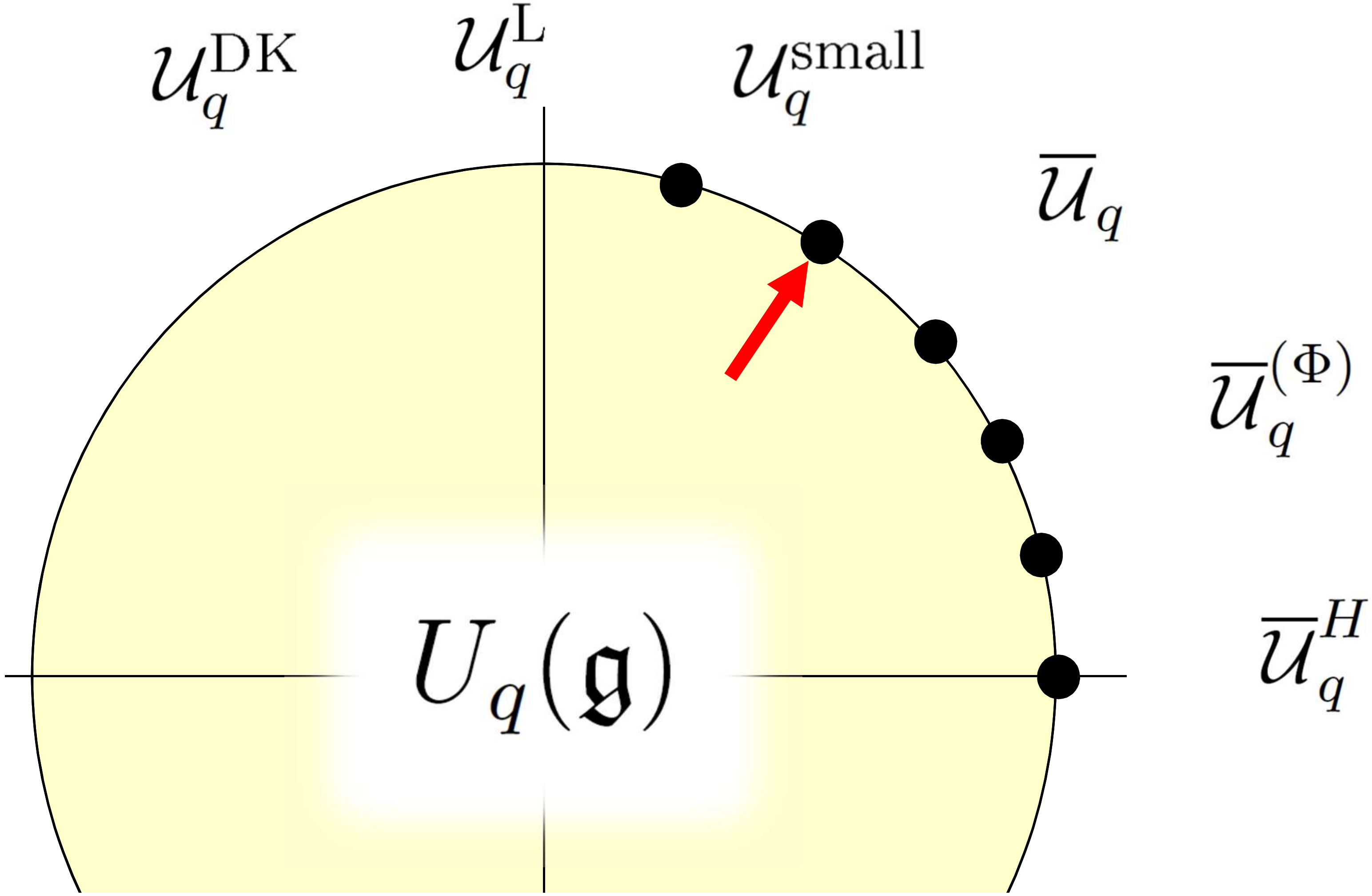
surface operators

<u>Theorem:</u>

MTC → 3d TQFT

Reshetikhin-Turaev construction

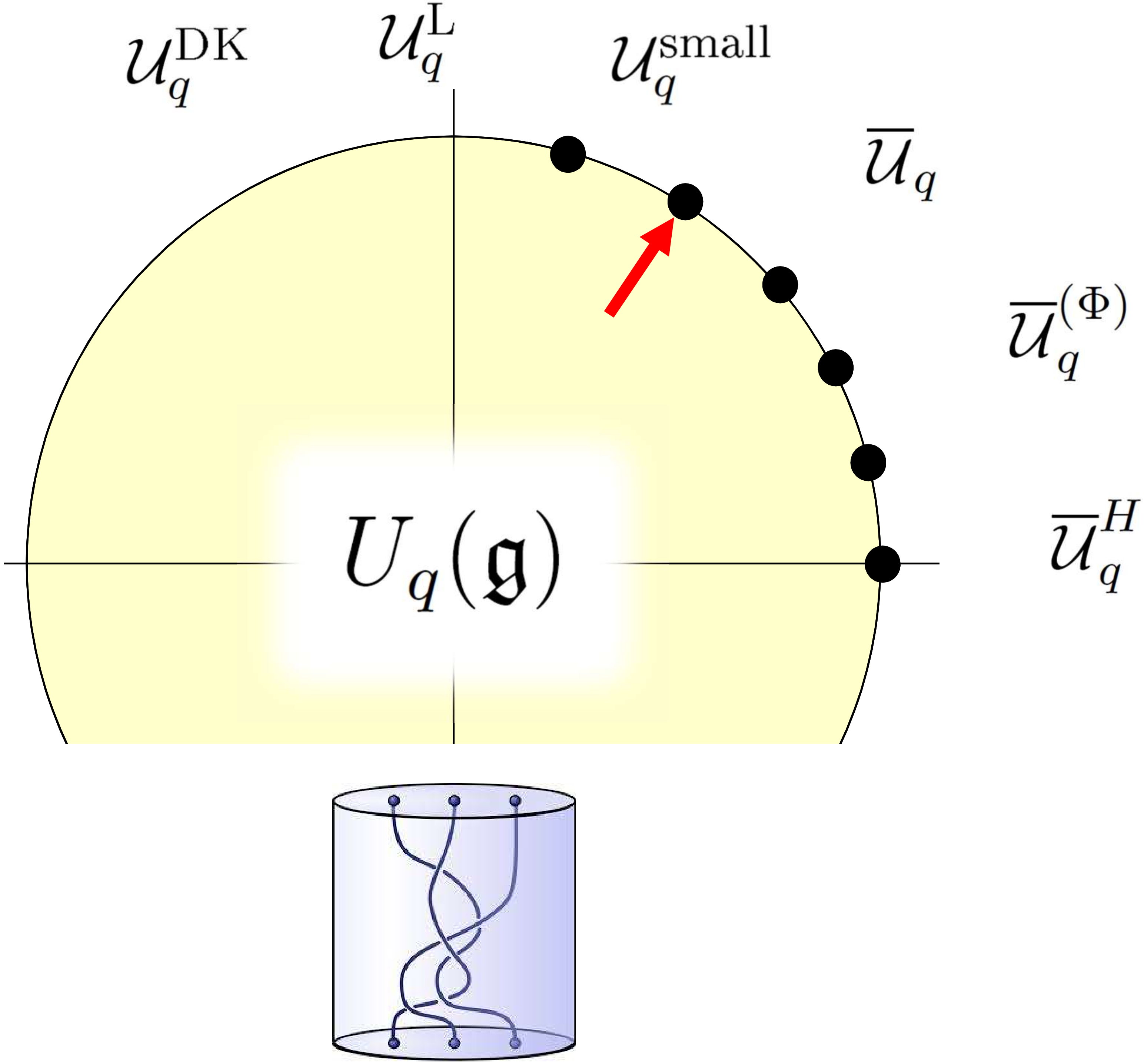
quantum groups $U_q(\mathfrak{g})$	 "local" operators	 line operators	 surface operators
WRT : roots of 1	✗	✓	✗
GPPV : generic q	✓	✓	?



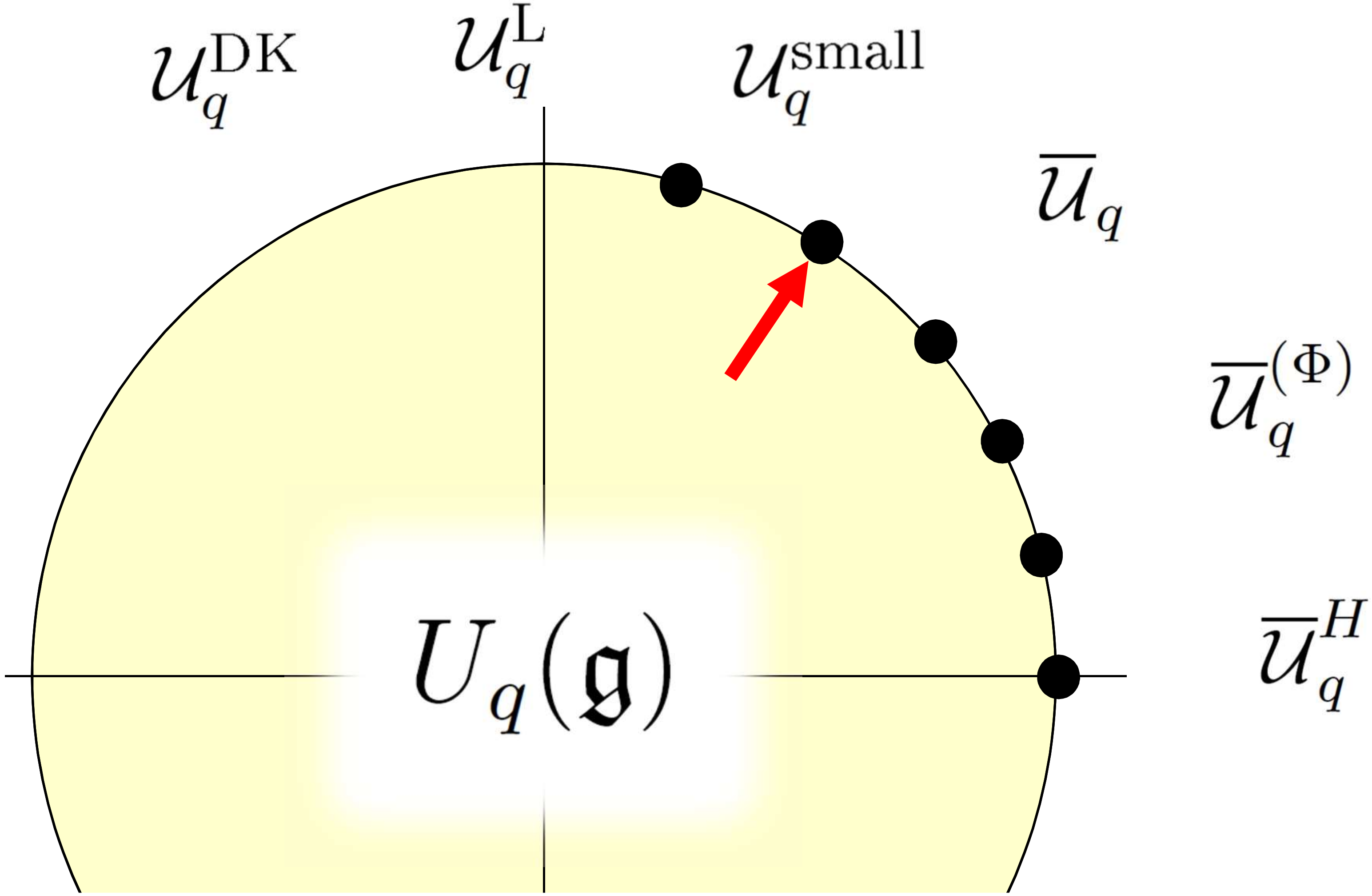
$\mathfrak{g} = \mathfrak{sl}_2$:

$$KEK^{-1} = q^2 E\,, \;\; KFK^{-1} = q^{-2} F$$

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$



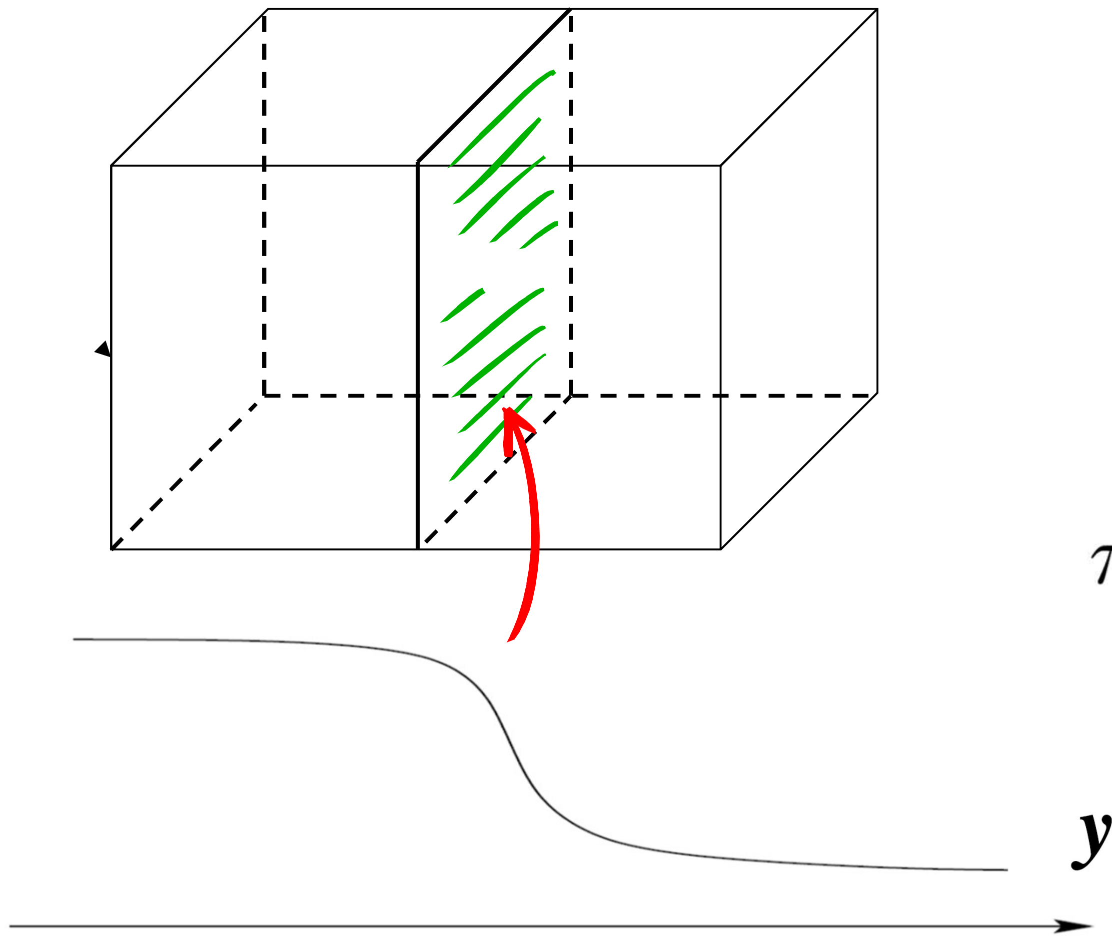
$\mathcal{U}_q^{\mathrm{DK}}$
$\mathcal{U}_q^{\mathrm{L}}$
$\mathcal{U}_q^{\mathrm{small}}$
$\overline{\mathcal{U}}_q$
$\overline{\mathcal{U}}_q^{(\Phi)}$
$\overline{\mathcal{U}}_q^{H}$
$U_q(\mathfrak{g})$



Kazhdan-Lusztig

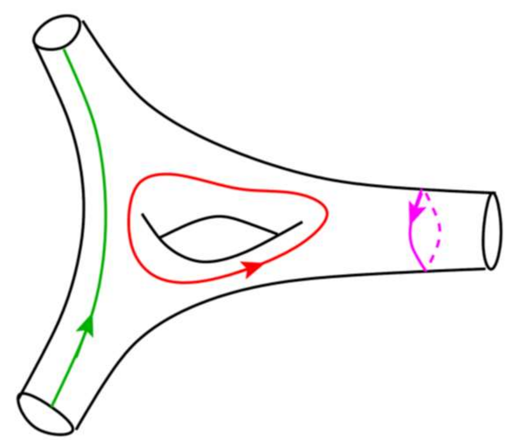
$U_q^{\cdots}(\mathfrak{g})\text{-mod}^{\cdots} \simeq \text{VOA-mod}^{\cdots} \simeq$ Geometric realization ?

Parameter walls / interfaces / surface operators in 3d:



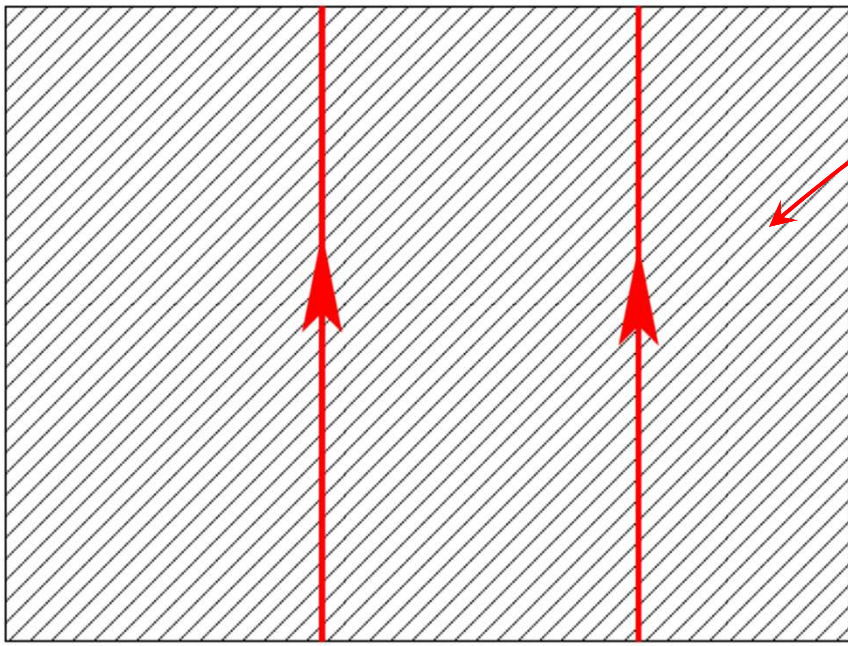
Kahler / Stab

$$\pi_1(\{\text{parameters}\})$$



A.Gadde, S.G., P.Putrov (2013)

cf. (codim-1) lines in 2d A-model / B-model:



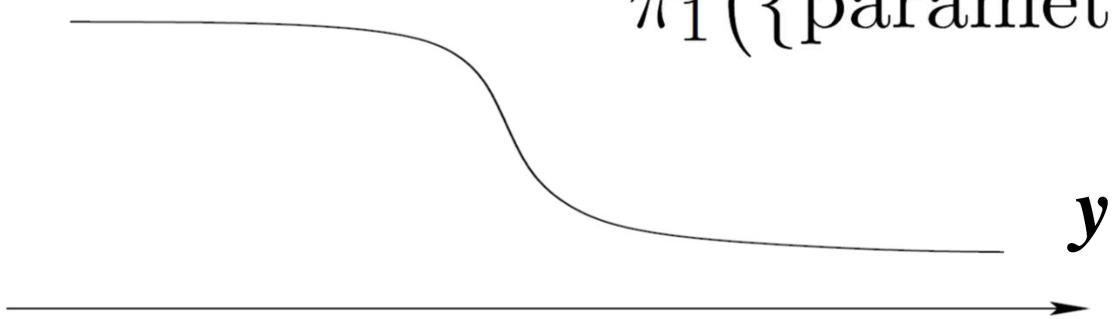
A-model / B-model to

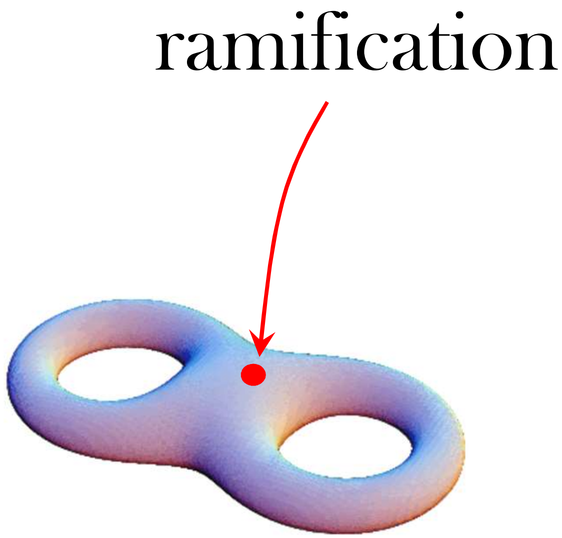
$$Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_2} \cong T^*\mathbb{C}\mathbf{P}^1$$

$$x^2 + y^2 + z^2 = \text{const}$$

S.G, E.Witten ('06)

$$\pi_1(\{\text{parameters}\}) = ?$$





$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma) \cong \mathcal{M}_H(G, \Sigma)$$

$$(\alpha, \beta, \gamma, \eta) \in \left(\mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times \mathbb{T}^\vee\right)/\mathcal{W}$$

A-model / B-model of $Y = \overline{\mathcal{O}}_{\mathbb{C}} = \mathbb{C}^2/\mathbb{Z}_2$

$$x^2 + y^2 + z^2 = 0$$

Model	Complex Modulus	Kahler Modulus
I	$\beta + i\gamma$	$\alpha + i\eta$
J	$\gamma + i\alpha$	$\beta + i\eta$
K	$\alpha + i\beta$	$\gamma + i\eta$

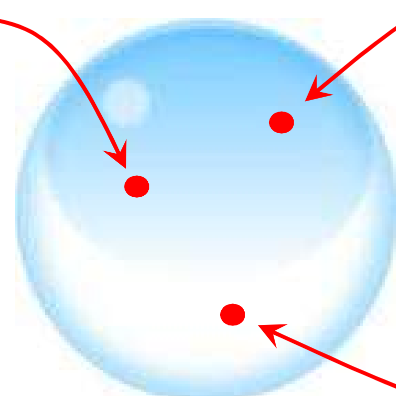
In homology / K-theory:

M.Kapranov, E.Vasserot
P.Seidel, R.Thomas
A.Ishii, H.Uehara
T.Bridgeland

$$M_{\mathcal{G}} = M_{\mathcal{K}} M_{\mathcal{L}}$$

$$M_{\mathcal{K}}^2 = 1 \quad , \quad M_{\mathcal{G}}^2 = 1$$

$$M_{\mathcal{K}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_{\mathcal{G}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$$

$$M_{\mathcal{L}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$$

In homology / K-theory:

N.Chriss, V.Ginzburg
M.Kapranov, E.Vasserot
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In the derived category:

S.G, E.Witten

R.Bezrukavnikov

$$\Phi_{\mathcal{G}}^2 = 1 \quad , \quad \Phi_{\mathcal{K}}^2 \neq 1$$

affine Weyl group $\mathcal{W}_{\mathrm{aff}}$ *acts on* $K(\mathcal{M}_H)$
affine Hecke algebra H_{aff} *acts on* $K^{\mathbb{C}^*}(\mathcal{M}_H)$
affine braid group B_{aff} *acts on* $D^b(\mathcal{M}_H)$

In homology / K-theory:

N.Chriss, V.Ginzburg

$$(\alpha, \beta, \gamma) \in \big(\mathfrak{t} \times \mathfrak{t} \times \mathfrak{t}\big)/\mathcal{W}_{\text{aff}}$$

$$\pi_1\big(\{(\alpha, \beta, \gamma)\}^{\text{reg}}\big) = \mathcal{W}_{\text{aff}}$$

S.G, E.Witten

In the derived category:

$$\pi_1\big(\{(\beta, \eta)\}^{\text{reg}}\big) = B_{\text{aff}}$$

R.Bezrukavnikov

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affine braid group B_{aff} *acts on* $D^b(\mathcal{M}_H)$

affine Weyl group: $q = 1$

N.Chriss, V.Ginzburg

affine Hecke algebra:

$$(T+1)(T-q) = 0$$

$$TX^{-1} - XT = (1-q)X$$

$$XX^{-1} = X^{-1}X = 1$$

affine braid group: omit quadratic relation on T

affine Weyl group $\mathcal{W}_{\mathrm{aff}}$ *acts on* $K(\mathcal{M}_H)$
affine Hecke algebra H_{aff} *acts on* $K^{\mathbb{C}^*}(\mathcal{M}_H)$
affine braid group B_{aff} *acts on* $D^b(\mathcal{M}_H)$

quantized K-theoretic Coulomb branches:

$K^{G(\mathcal{O})\times\mathbb{C}^*}(\mathrm{Gr}_G)$ spherical nil-DAHA

$\vdots$

M.Finkelberg, A.Tsymbaliuk
R.Bezrukavnikov, M.Finkelberg, I.Mirković
A.Braverman, M.Finkelberg, H.Nakajima
:

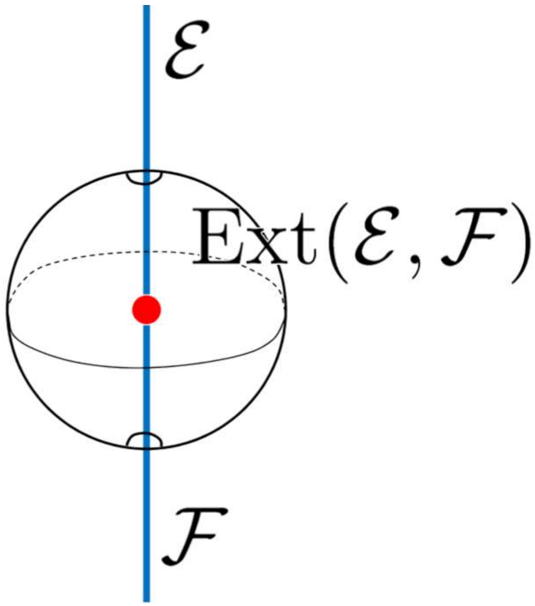
$\mathcal{T} =$ space of BFN triples (cf. $T^*\mathrm{Gr}_G$)

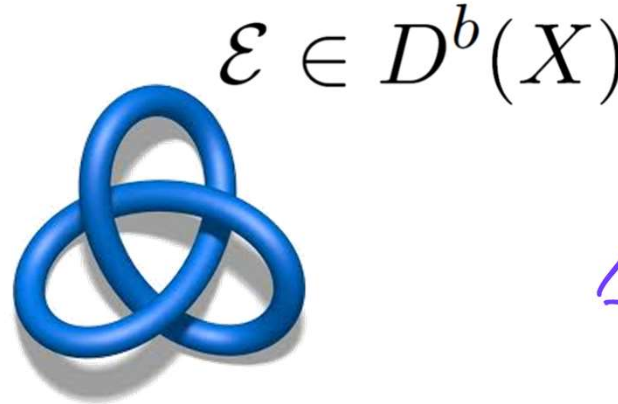
The "ravioli space" $\mathcal{R}$ is a sub-variety of $\mathcal{T}$

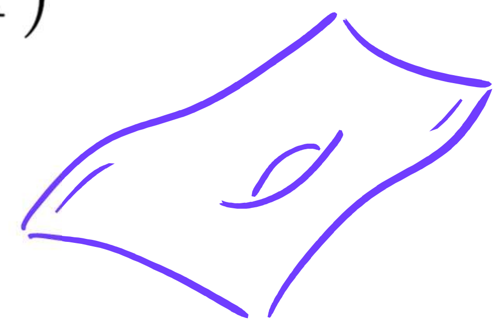
$$K^{G(O)}(\mathrm{St}_G) \cong \mathbb{C}[\mathbb{T}_{\mathbb{C}} \times {}^L\mathbb{T}_{\mathbb{C}}]^W$$

cf. geometric Satake correspondence:

$$\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \cong \mathrm{Rep}\, G^\vee$$







"local"
operators

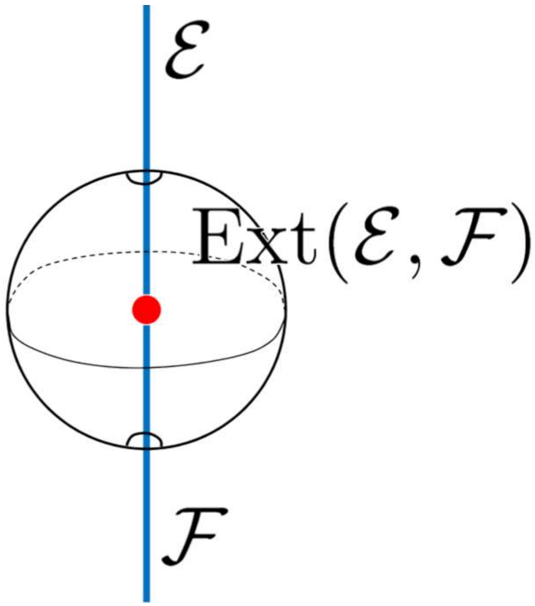
line
operators

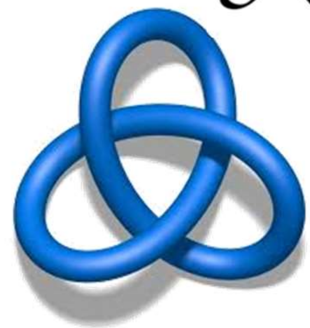
surface
operators

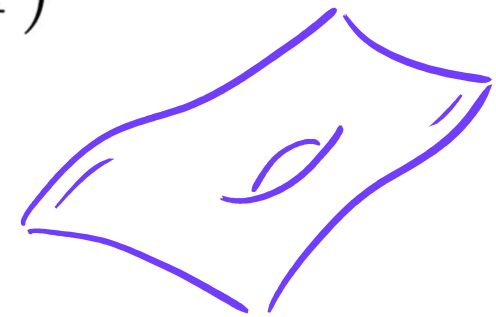
Symmetries
decorated TQFT

Logarithmic
(non-semisimple)

Higher groups







"local" operators

line operators

surface operators

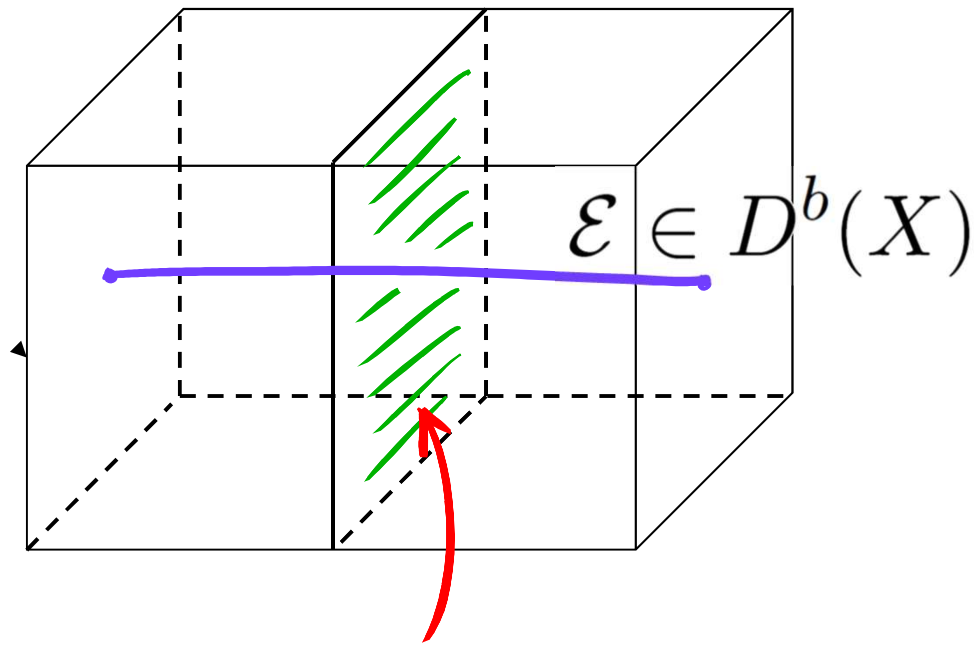
$$\mathrm{Auteq}\, D^b(X) \;\cong\; \pi_1(\{\text{parameters}\})$$

Example: $X = K3$

$$\mathrm{Pic}\, X \rtimes \mathrm{Aut}\, X \times \mathbb{Z}[1]$$

H.Uehara
:

Modern low-dimensional topology:



$\operatorname{Auteq} D^b(X)$

<u>Example:</u> $X = K3$

$$\operatorname{Pic} X \rtimes \operatorname{Aut} X \times \mathbb{Z}[1]$$

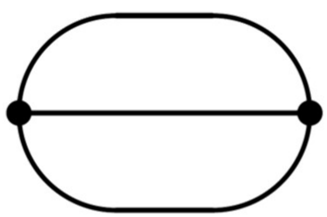
H.Uehara
:

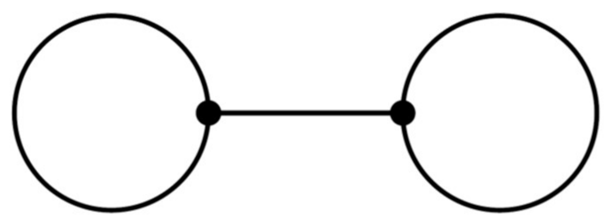
Rozansky-Witten vs Chern-Simons

$$\chi\wedge D\chi+\chi\wedge\chi\wedge\chi\eta \qquad\qquad A\wedge DA+A\wedge A\wedge A$$

RW field	on M_3	on X	#(zero modes)
η	0-form	$d\bar{z}$	$2\dim_{\mathbb{H}} X$
χ	1-form	dz	$2b_1\dim_{\mathbb{H}} X$

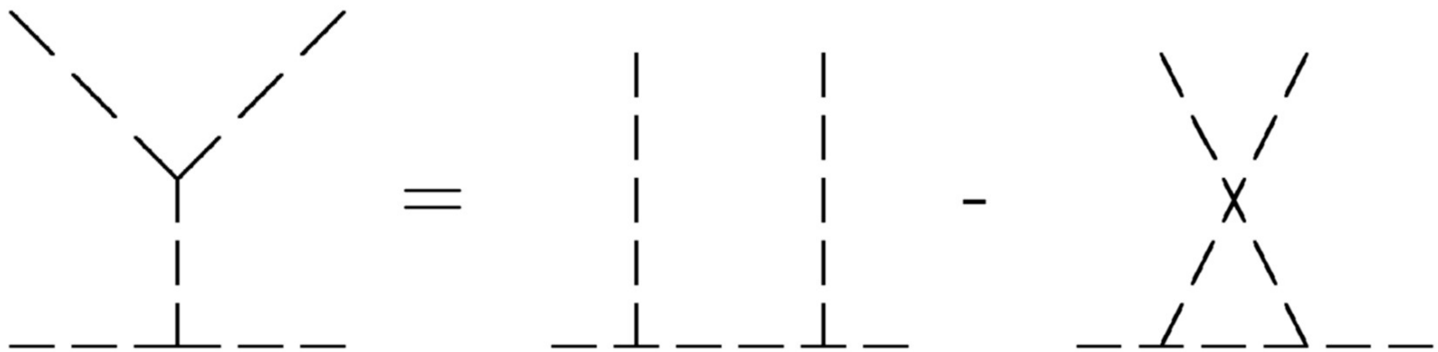
- trivalent graph with 2n vertices = n+1 loops in CS = finite type of order 3(n-1)





- For X of real dim 4n: only trivalent graphs with 2n vertices

The IHX relation for the weight follows from Jacobi identity in WRT, from Bianchi identity in RW (ff+ff+ff=0 vs DR=0 on the Riemann tensor)



Finite type invariants of 3-manifolds introduced by T.Ohtsuki (1996)
For knots: Vassiliev, Gusarov, …

The number of independent degree-n Vassiliev inv'ts:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
dim V_n	1	1	2	3	6	10	19	33	60	104	184	316	548

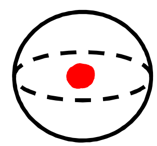
Rozansky-Witten vs Chern-Simons

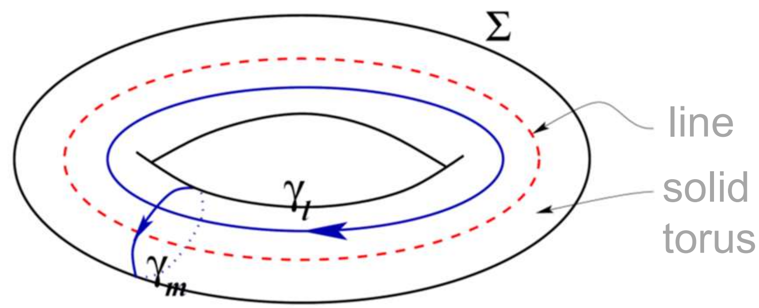
$$\chi \wedge D\chi + \chi\wedge\chi\wedge\chi\eta \qquad\qquad A\wedge DA + A\wedge A\wedge A$$

RW field	on M_3	on X	#(zero modes)
η	0-form	$d\overline{z}$	$2\dim_{\mathbb{H}} X$
χ	1-form	dz	$2b_1 \dim_{\mathbb{H}} X$

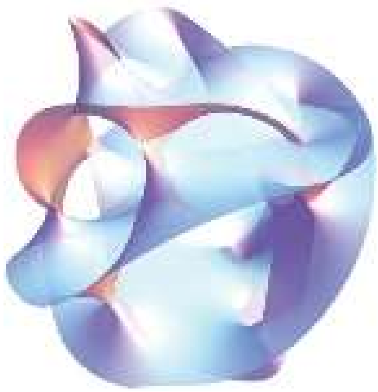
Q-cohomology:

$$\mathcal{H}(\Sigma_g) = \begin{cases} \bigoplus_{l=0}^{2n} H^{0,l}(X)\,, & g=0 \\ \bigoplus_{l,m=0}^{2n} H^{l,m}(X)\,, & g=1 \\ \vdots & \end{cases}$$





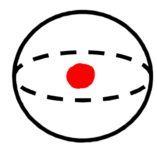
Example: $X = K3$

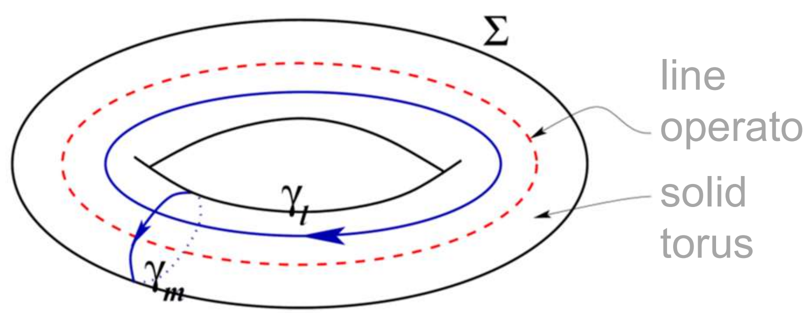


$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Q-cohomology:

$$\mathcal{H}(\Sigma_g) = \begin{cases} \bigoplus_{l=0}^{2n} H^{0,l}(X)\,, & g=0 \\ \bigoplus_{l,m=0}^{2n} H^{l,m}(X)\,, & g=1 \\ \vdots & \end{cases}$$





tri-holomorphic
(on arbitrary 3-manifolds) ⇒ Spin^c - decorated TQFT

$$U(1)_x$$

$$X = \text{Coulomb branch, ...}$$

$$U(1)_t$$

holomorphic: on Seifert 3-manifolds

$$Z_{\mathrm{RW}[X]}(M_3; x) = \sum_{b \in \mathrm{Spin}^c(M_3)} x^b Z_{\mathrm{RW}[X]}(M_3; b)$$

There is a canonical map:

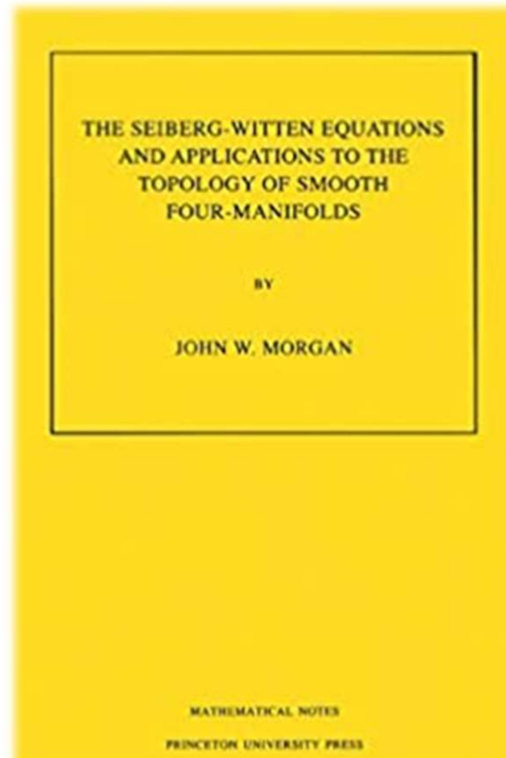
$$\sigma : \mathrm{Spin}(M_3) \times H_1(M_3, \mathbb{Z}) \longrightarrow \mathrm{Spin}^c(M_3)$$

induced by

$$B\mathrm{Spin} \times BU(1) \to B\mathrm{Spin}^c$$

which, in turn, is part of the fiber sequence for the classifying spaces.

$$b \in \mathrm{Spin}^c(M_3) \cong H_1(M_3; \mathbb{Z})$$



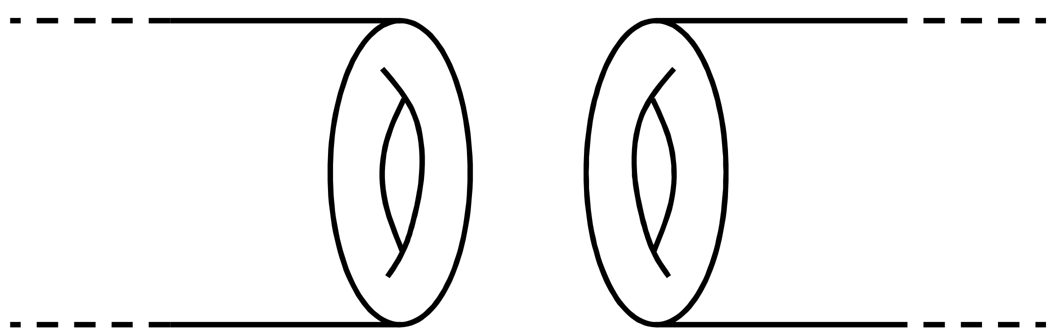
d-dimensional TQFT

<u>Def:</u> $H^{n+1}(\,\cdot\,; G)$-decorated TQFT

G abelian

(n+1)-form $A \in H^{n+1}(M_d; G)$

connection on a flat n-gerbe



F.Costantino, S.G., P.Putrov

d=3: $$\omega \in H^1(M_3;G) \cong \mathrm{Hom}(H_1(M_3;\mathbb{Z}),G) \cong \mathrm{Hom}(H^2(M_3;\mathbb{Z}),G)$$

$M_3 = \Sigma \times S^1$:

$$\omega \in H^1(\Sigma \times S^1;G)$$
$$\cong \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}),G) \oplus \mathrm{Hom}(H_0(\Sigma;\mathbb{Z}),G)$$
$$\cong H^1(\Sigma;G) \oplus \mathrm{Hom}(H_0(\Sigma;\mathbb{Z}),G)$$

$\mathcal{H}(\Sigma)$	structure
graded by	$H^2(\Sigma;\widehat{G})$
decorated by	$H^1(\Sigma;G)$

← $\widehat{G} = \mathrm{Hom}(G,U(1))$

Pontryagin dual

F.Costantino, S.G., P.Putrov

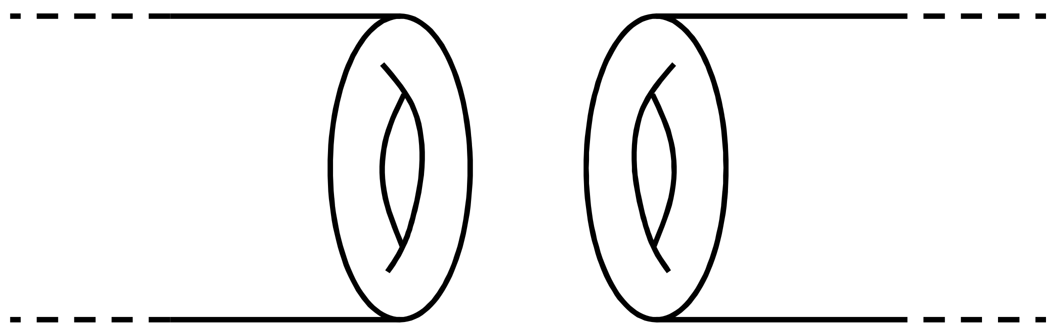
3d $H^2(\,\cdot\,,\mathbb{Z})$-decorated TQFTs

$$G = \mathbb{Z}$$

$$b \in H^2(\Sigma \times S^1, G) \cong H^2(\Sigma,\mathbb{Z}) \oplus \underbrace{H_1(\Sigma,\mathbb{Z})}$$

$\mathcal{H}(\Sigma)$ graded by

$$\mathrm{Hom}(H_1(\Sigma,\mathbb{Z}),U(1)) \cong H^1(\Sigma,U(1))$$



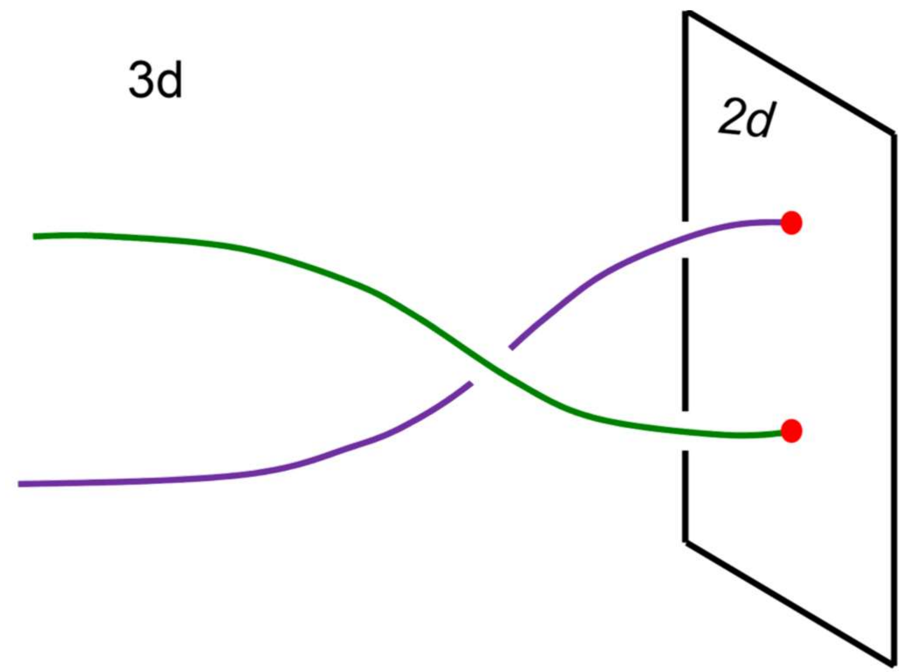
M.Jagadale
F.Costantino, S.G., P.Putrov

Decorated TQFT $\longleftrightarrow$ 2d VOA / CFT

G-crossed MTCs, $\longleftrightarrow$ twisted sectors
"enriched" SPT phases,
bordered, sutured, ...

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

$$\mathcal{C}_{g_1} \boxtimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2}$$



M.Jagadale
F.Costantino, S.G., P.Putrov

Hyper-Kähler Geometry and Invariants of Three-Manifolds

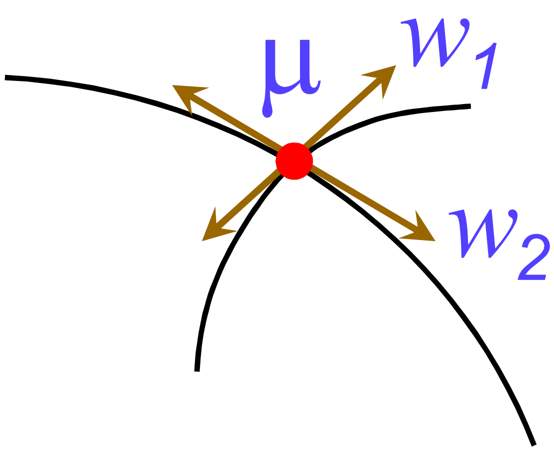
L. Rozansky

E. Witten

answers. In general, the analysis by cutting and summing over physical states is likely to be quite subtle if X is not compact, roughly because there is a continuum of almost Q-invariant states starting at zero energy. In the presence of such a continuum, formal arguments claiming to show a reduction to the Q-cohomology are hazardous at best. But if X is compact, the spectrum is discrete, and one will get a quite straightforward formalism involving a sum over finitely many physical states.

(cf. eq. (5.8)). If X is non-compact, the continuous spectrum starting at zero energy obstructs a reduction to a description with a finite-dimensional space of physical states. We therefore consider only compact X, such as $X = \mathrm{K3}$, to obtain the surgery formulas.

$U(1)_t \circlearrowright X =$ Coulomb branch, ...



$$T_{\lambda\lambda} = t^{\mu(\lambda)}$$

$$(S_{0\lambda})^2 = \frac{K_X^{1/2}}{\text{K-theory Euler class}(T_\lambda X)}$$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

Example:

wild

$\mathcal{M}_H\Big($  $,\ \text{ramification}\Big)$

2 fixed points		
$\mu =$	0	$1/5$
$w_1 =$	$2/5$	$6/5$
$w_2 =$	$3/5$	$-1/5$

$\Rightarrow$

Fibonacci MTC

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{-\frac{\pi i}{15}} & 0 \\ 0 & e^{\frac{11\pi i}{15}} \end{pmatrix}$$

M.Dedushenko, S.G., H.Nakajima, D.Pei, K.Ye

3d reduction of 4d Argyres-Doulas theory

$$U(1)_t \circlearrowright \; X = \text{Coulomb branch, ...}$$

2 fixed points

$$\mu = \quad 0 \quad 1/5$$

$$w_1 = \quad 2/5 \quad 6/5$$

$$w_2 = \quad 3/5 \quad -1/5$$

$\Rightarrow$

Fibonacci MTC

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{-\frac{\pi i}{15}} & 0 \\ 0 & e^{\frac{11\pi i}{15}} \end{pmatrix}$$

M.Dedushenko, S.G., H.Nakajima, D.Pei, K.Ye

Geometry

of

Galois action?

MTC

$\mu = \quad 0 \quad 1/5$

$w_1 = \quad 2/5 \quad 6/5$

$w_2 = \quad 3/5 \quad -1/5$

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

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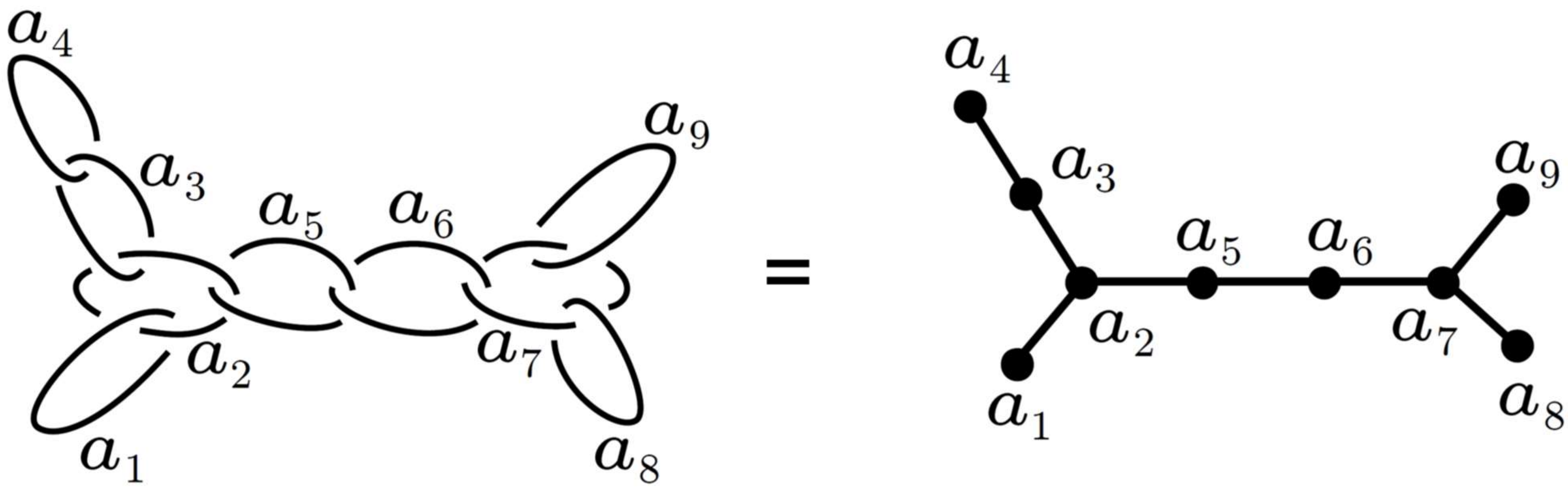
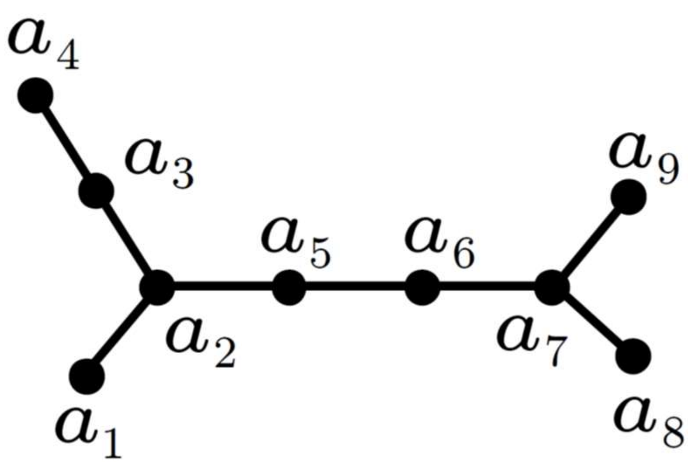
M.Dedushenko, S.G., H.Nakajima, D.Pei, K.Ye



$$\sum_{\lambda_v} \prod_{\text{vertices}} S_{0\lambda_v}^{2-\deg(v)} T_{\lambda_v\lambda_v}^{a_v} \prod_{\text{edges}} S_{\lambda_v \lambda_v'}$$

$U(1)_x \circlearrowright X =$ Coulomb branch, ...

tri-holomorphic



$b \in \mathrm{Spin}^c(M_3)$

$$Z_{\mathrm{RW}[X]}(M_3) = \sum_{m_i \in \mathbb{Z}} \int \prod_{i \in \mathrm{vertices}} \frac{dx_i}{2\pi i x_i} x_i^{\sum_j Q^{ij} m_j} g(x_i)^{\deg(i)-2} x_i^{b_i}$$

$$\frac{1}{g^2(x)} = Z_{\mathrm{RW}[X]}\left(S^1 \times S^2\right) = \text{Hilbert series of } X$$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

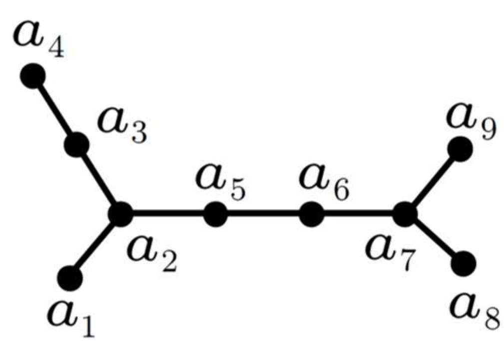
LATTICE COHOMOLOGY AND q-SERIES INVARIANTS OF 3-MANIFOLDS

ROSTISLAV AKHMECHET, PETER K. JOHNSON, AND VYACHESLAV KRUSHKAL

Definition 4.1. Fix a commutative ring $\mathcal{R}$. A family of functions $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n\geq 0}$ is *admissible* if

(A1) $F_2(0) = 1$ and $F_2(r) = 0$ for all $r \neq 0$.
(A2) For all $n \geq 1$ and $r \in \mathbb{Z}$,

$$F_n(r+1) - F_n(r-1) = F_{n-1}(r).$$



Theorem 5.10. *For any admissible family of functions F, the weighted graded root is an invariant of the 3-manifold $Y(\Gamma)$ equipped with the* spin^c *structure $[k]$.*

$$U(1)_x \times U(1)_t \circlearrowright X = T^*\mathbb{C}\mathbf{P}^1$$

tri-holomorphic ($U(1)_x$) holomorphic ($U(1)_t$)

$$TX|_{p_1} = x + t/x \quad \Rightarrow \quad (S_{00})^2 = \frac{1}{(1-x)(1-t/x)}$$

$$TX|_{p_2} = x^{-1} + tx \quad \Rightarrow \quad (S_{01})^2 = \frac{1}{(1-x^{-1})(1-tx)}$$

$$U(1)_x \times U(1)_t \circlearrowright \; X = T^*\mathbb{C}\mathbf{P}^1$$

tri-holomorphic ($U(1)_x$) holomorphic ($U(1)_t$)

$$\sum_\lambda (S_{0\lambda})^2 = Z_{\mathrm{RW}[X]}\left(S^1 \times S^2\right) = \int_X \widehat{A}(TX)$$

$$= \frac{1+t}{(1-tx)(1-t/x)}$$

Hilbert series of X

$$U(1)_x \times U(1)_t \circlearrowright X = T^*\mathbb{C}\mathbf{P}^1$$

tri-holomorphic ($U(1)_x$) holomorphic ($U(1)_t$)

$$\sum_\lambda (S_{0\lambda})^2 = Z_{\text{RW}[X]}\left(S^1 \times S^2\right) = \int_X \widehat{A}(TX)$$

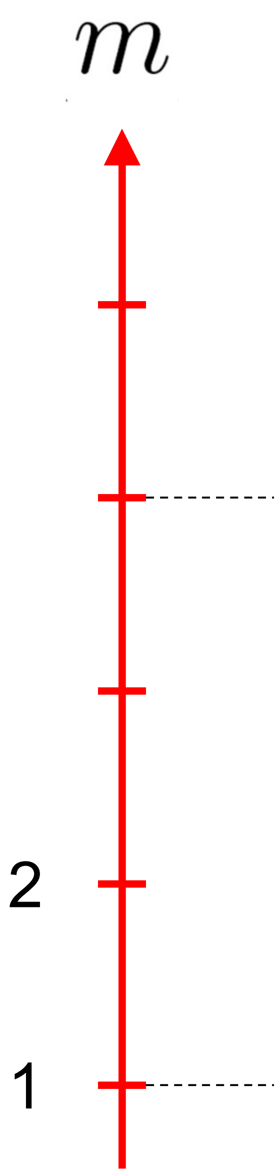
$$= \frac{1+t}{(1-tx)(1-t/x)}$$

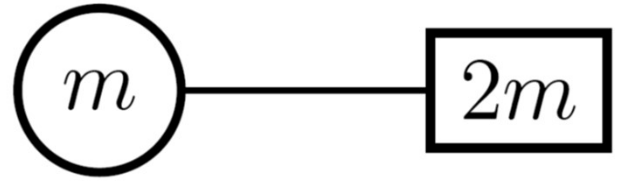
$$Z_{\text{RW}[X]}(M_3; x) = \sum_{b \in \text{Spin}^c(M_3)} x^b Z_{\text{RW}[X]}(M_3; b)$$

Geometry | Algebra / TQFT

$X \longleftrightarrow Z_{\mathrm{RW}[X]}(M_3)$

$$Z_{\mathrm{RW}[X]}\left(L(k,1)\right) = \int_X e^{k\omega} \wedge \widehat{A}(TX) = \sum_{\lambda} (S_{0\lambda})^2\, T^k_{\lambda\lambda}$$





$$X_m = \mathbb{H}^{2m^2} /// U(m)$$

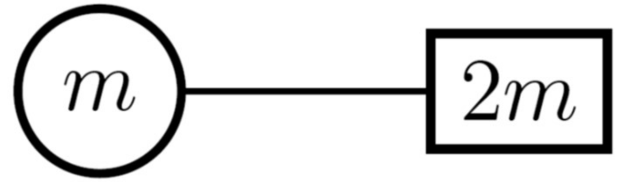
$$\dim_{\mathbb{H}} X_m = m^2$$

$$X_1 = T^* \mathbb{C}\mathbf{P}^1$$

$$X_m = \mathbb{H}^{2m^2} /// U(m)$$

$$\circlearrowright$$

$$U(1)_x \times U(1)_t$$



$$\sum_{\lambda} (S_{0\lambda})^2 = \text{ Hilbert series of } X_m$$

$$= \prod_{j=1}^{m} \frac{1 - t^{2m+1-j}}{(1 - t^j)(1 - x t^{m+1-j})(1 - x^{-1} t^{m+1-j})}$$

$m \to \infty$:

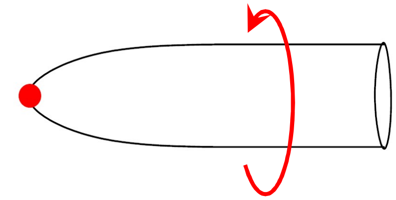
$$\frac{1}{(tx;t)_\infty (t;t)_\infty (tx^{-1};t)_\infty}$$

$$\sum_\lambda (S_{0\lambda})^2 = Z_{\mathrm{RW}[X]}\left(S^1 \times S^2\right) = \int_X \widehat{A}(TX)$$

cotangent bundle:

$$\frac{(x;q)_\infty (q;q)_\infty (x^{-1};q)_\infty}{(qtx;q)_\infty (qt;q)_\infty (qtx^{-1};q)_\infty}$$

$X = \mathcal{M}_H(G, \Sigma)$ for $\Sigma = D^2$



$=$ cotangent bundle to Affine Grassmannian

$$\mathrm{Gr}_G = \frac{G_{\mathbb{C}}(\text{functions holo on } \mathbb{C}^*)}{G_{\mathbb{C}}(\text{functions holo on } \mathbb{C})}$$

$$x \in \mathrm{Hom}\Big(\mathrm{Spin}^c(M_3), \mathbb{C}^*\Big)$$

$$\widehat{Z}(S^1 \times S^2) = \frac{(x;q)_\infty (q;q)_\infty (x^{-1};q)_\infty}{(qtx;q)_\infty (qt;q)_\infty (qtx^{-1};q)_\infty}$$

loop rotation

holomorphic

Annals of Mathematics, **168** (2008), 175–220

The strong Macdonald conjecture and Hodge theory on the loop Grassmannian

By Susanna Fishel, Ian Grojnowski, and Constantin Teleman

11.12 Theorem. *For simply laced G, the vacuum vector $\omega \in \mathbf{H}_0$ gives an isomorphism*

$$(*) \qquad \omega\otimes : \{\mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]} \xrightarrow{\sim} \{\mathbf{H}_0 \otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]}.$$

Consequently, with $q = z^{-1}$,

$$\sum_{p\geq 0} t^p \dim_q \mathrm{Gr}_p \mathbf{H}_0^G = \prod_{k=1}^{\ell} \prod_{n>m_k} (1 - t^{m_k+1} q^n)^{-1}.$$

Thanks for listening.

Questions?

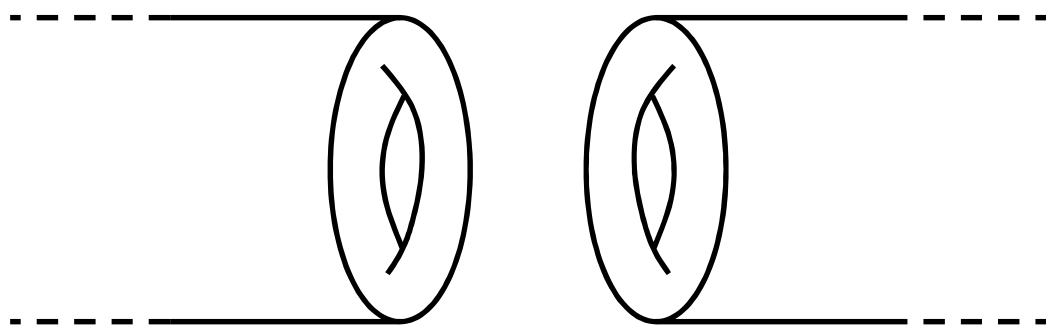
Example: $G = U(1)$ or $\mathbb{C}^*$ 0-form symmetry

$U(1)_k$ Chern-Simons TQFT

$$Z(M_3, \omega) = \int DA \exp\left(\frac{ik}{2\pi} \int_{M_3} AdA + 2\pi i \omega(c_1) \right)$$

$\omega \in H^1(M;G)$

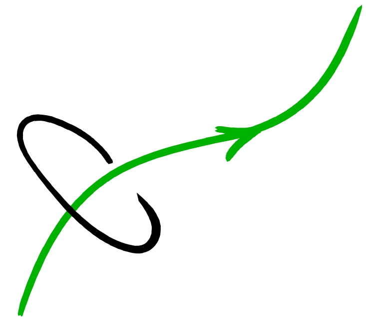
$\mathcal{H}(\Sigma)$	structure
graded by	$H^2(\Sigma;\widehat{G})$
decorated by	$H^1(\Sigma;G)$



Example: $G = U(1)$ or $\mathbb{C}^*$ 0-form symmetry

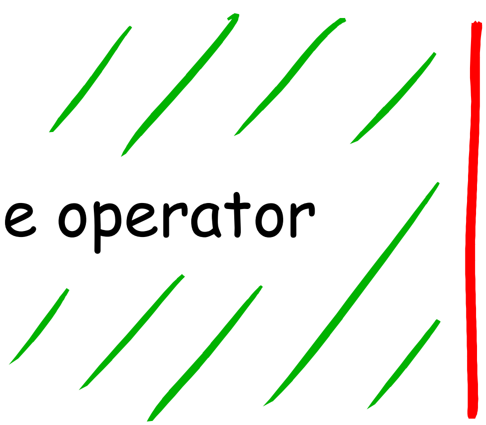
$U(1)_k$ Chern-Simons TQFT

$$\mathcal{H}_{\mathrm{TQFT}}(T^2) = \text{ line operators } = K^0(\mathcal{C})$$



$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

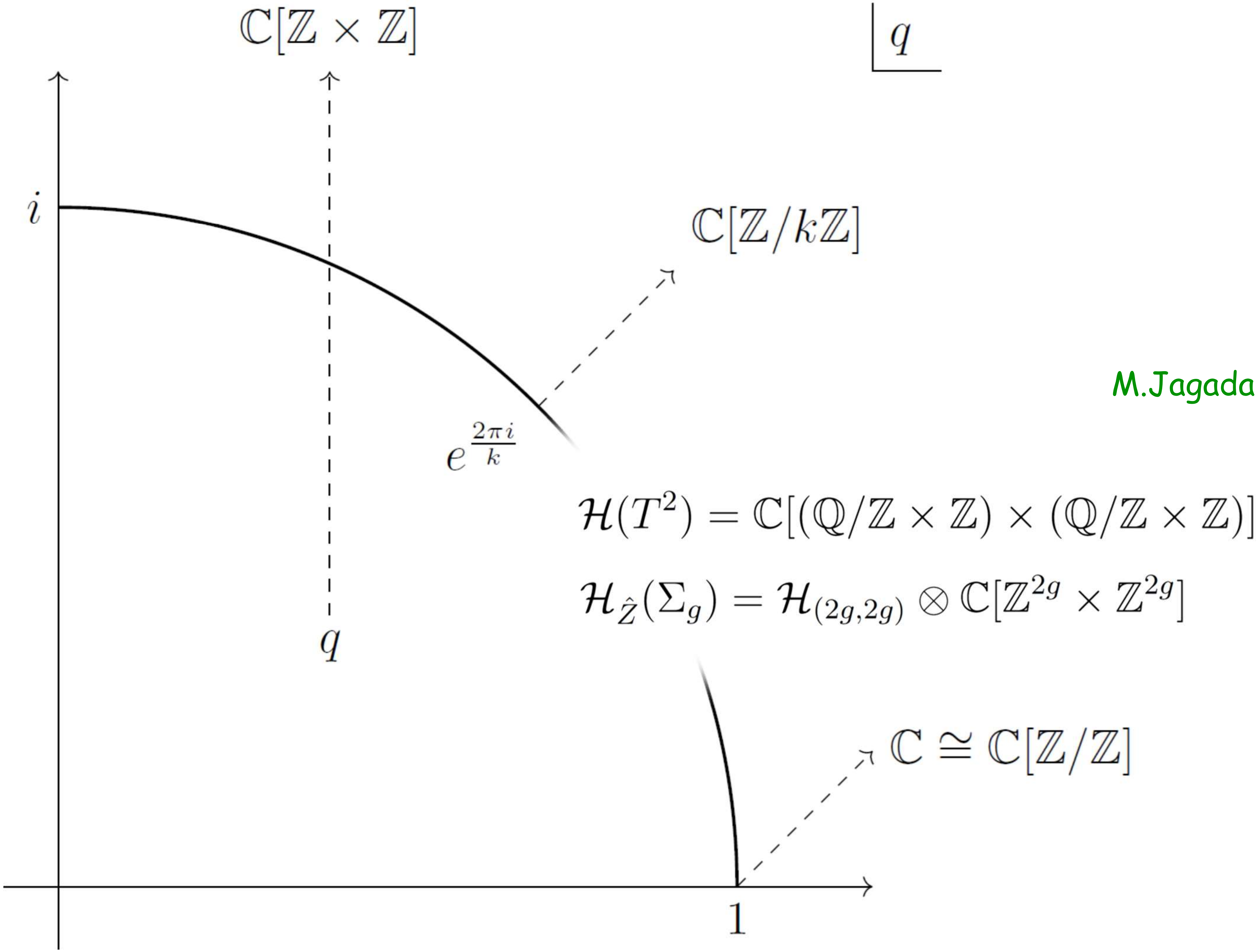
has k simple objects $\begin{cases} \mu \in \mathbb{C}/k\mathbb{Z} \\ \mu \equiv g \bmod 1 \end{cases}$



surface operator

line operator

$$W_\mu(\gamma) = \exp\left(i\mu \int_\gamma A\right)$$


$\mathbb{C}[\mathbb{Z} \times \mathbb{Z}]$
q
i
$\mathbb{C}[\mathbb{Z}/k\mathbb{Z}]$
$e^{\frac{2\pi i}{k}}$
$\mathcal{H}(T^2) = \mathbb{C}[(\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Q}/\mathbb{Z} \times \mathbb{Z})]$
$\mathcal{H}_{\hat{Z}}(\Sigma_g) = \mathcal{H}_{(2g,2g)} \otimes \mathbb{C}[\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}]$
q
$\mathbb{C} \cong \mathbb{C}[\mathbb{Z}/\mathbb{Z}]$
1

M.Jagadale