Higgs-Coulomb correspondence and wall-crossing in abelian gauged linear sigma models

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dedicated to the memory of Professor Bumsig Kim

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Outline

1. Gauged linear sigma models (GLSMs)
2. Higgs branch
   - Landau-Ginzburg (LG) quasimaps
   - LG loop spaces and I-functions
   - central charge $Z([\mathcal{B}])$ of a B-type D-brane $\mathcal{B}$
3. Coulomb branch
   - hemisphere partition function $Z_{D^2}([\mathcal{B}])$
   - (2d) Higgs-Coulomb correspondence $Z_{D^2}([\mathcal{B}]) \longrightarrow Z([\mathcal{B}])$
4. Wall-crossing

1. Gauged linear sigma models (GLSMs)

The input data of a gauged linear sigma model (GLSM) is a 5-tuple $(V, G, \mathbb{C}^*_R, W, \zeta)$

(1) (linear space) $V = \mathrm{Spec}\mathbb{C}[x_1, \ldots, x_m] \simeq \mathbb{C}^m$

(2) (gauge group) $G \subset GL(V) \simeq GL_m(\mathbb{C})$ linear reductive

(3) (R symmetries) $\mathbb{C}^*_R \subset GL(V)$, $\mathbb{C}^*_R \cong \mathbb{C}^*$.

$G, \mathbb{C}^*_R$ commute, $G \cap \mathbb{C}^*_R = \langle J \rangle = \mu_r$

$\mathbb{C}^*_R$ acts on V by weights $c_1, \ldots, c_m \in \mathbb{Z}$, R charges $q_j = \frac{2c_j}{r}$

(4) (superpotential) $W \in \mathbb{C}[x_1, \ldots, x_m]$

- G-invariant: $W(g \cdot x) = W(x) \forall g \in G \Leftrightarrow W \in \mathbb{C}[x_1, \ldots, x_m]^G$
- quasi-homogeneous: $W(t \cdot x) = t^r W(x) \forall t \in \mathbb{C}^*_R$

(5) (stability condition) $\zeta \in \mathrm{Hom}(G, \mathbb{C}^*) \Leftrightarrow G$-linearization on V

assumption: $V^{ss}_G(\zeta) = V^s_G(\zeta)$

$$\mathcal{X}_\zeta = [V^{ss}_G(\zeta)/G] \quad \text{smooth DM stack}$$

$$\downarrow$$

$$\mathbb{C}^*_\omega \curvearrowright X_\zeta = V^{ss}_G(\zeta)/G = V/\!/_\zeta G \quad \text{GIT quotient}$$

$$:= \mathbb{C}^*_R/\langle J \rangle \quad \downarrow \text{projective} \qquad w(t \cdot [x]) = tw([x]), t \in \mathbb{C}^*_\omega, [x] \in X_\zeta$$

$$X_0 = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_m]^G) \xrightarrow{w} \mathbb{C}$$

A GLSM is abelian if the gauge group G is abelian.
In most of this talk, $G = (\mathbb{C}^*)^\kappa$.
We have a short exact sequence of abelian groups (let $n = m - \kappa$)

$$1 \to G \xrightarrow{(D_1,\dots,D_{n+\kappa})} \widetilde{T} \simeq (\mathbb{C}^*)^{n+\kappa} \longrightarrow T \simeq (\mathbb{C}^*)^n \to 1$$

$$\cap \text{maximal torus}$$

$$GL_{n+\kappa}(\mathbb{C})$$

where $D_j \in \mathrm{Hom}(G, \mathbb{C}^*) = \mathbb{L}^\vee \simeq \mathbb{Z}^\kappa$. Then

- $\mathcal{X}_\zeta$ is a smooth toric DM stack (Borisov-Chen-Smith)
- $X_\zeta = V/\!/_\zeta G$ is a semiprojective simplicial toric variety
- $\mathcal{X}_\zeta = [\mu^{-1}(\zeta)/G_\mathbb{R}]$ where $G_\mathbb{R} = U(1)^\kappa \subset G = (\mathbb{C}^*)^\kappa$, and $\mu : V = \mathbb{C}^{n+\kappa} \to \mathrm{Lie}(G_\mathbb{R}) \simeq \mathbb{L}^\vee_\mathbb{R} := \mathbb{L}^\vee \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^\kappa$ is the moment map of Hamiltonian $G_\mathbb{R}$-action on $\mathbb{C}^{n+\kappa}$.
- $\zeta \in \mathbb{L}^\vee_\mathbb{R} \simeq \mathbb{R}^\kappa \supset$ secondary fan

(cf. Ernesto Lupercio's talk)

Example: quintic

$V = \mathbb{C}^6 = \mathrm{Spec}\,\mathbb{C}[x_1, \ldots, x_5, p], \quad G = \mathbb{C}^*, \quad \zeta \in \mathbb{R} - \{0\}$

gauge charges $\quad G$ acts by weights $(1, \ldots, 1, -5)$

R charges $\quad \mathbb{C}^*_R$ acts by weights $(0, \ldots, 0, 1)$

$G \cap \mathbb{C}^*_R = \{1\}$

superpotential $\quad W = p(x_1^5 + \cdots + x_5^5) = pW_5(x)$

- $\zeta > 0$: Calabi-Yau (CY)/geometric phase

$$\mathcal{X}_\zeta = \left((\mathbb{C}^5 - \{0\}) \times \mathbb{C}\right)/G = K_{\mathbb{P}^4}$$

$$\mathrm{Crit}(w) = \{W_5(x) = p = 0\} = X_5 \text{ Fermat quintic}$$

$$\subset \{p = 0\} = \mathbb{P}^4$$

GLSM invariants = Gromov-Witten (GW) invariants of X_5

- $\zeta < 0$: Landau-Ginzburg (LG) phase

$$\mathcal{X}_\zeta = \left[\left(\mathbb{C}^5 \times (\mathbb{C} - \{0\})\right)/\mathbb{C}^*\right] = [\mathbb{C}^5/\mu_5]$$

$$\mathrm{Crit}(w)_{\mathrm{red}} = [0/\mu_5] \simeq B\mu_5$$

GLSM invariants = Fan-Jarvis-Ruan-Witten (FJRW) invariants of (W_5, μ_5)

Chiodo-Ruan (2008) LG/CY correspondence for quintic 3-folds:

GW invariants of $X_5 \longleftrightarrow$ FJRW invariants of (W_5, μ_5)

(1) (ϵ-wall-crossing) Givental style mirror theorems

- CY phase (Givental, Lian-Liu-Yau 1996-7):

  $J_+ = \frac{I_+}{I_+^0}$ under the mirror map

- LG phase (Chiodo-Ruan 2008): $J_- = \frac{I_-}{I_-^0}$ under the mirror map

$I_\pm$, $J_\pm$ are functions of 1 variable

take values in a 4-dimensional complex symplectic space

$$H(z)_\pm = zH^0_\pm \oplus H^2_\pm \oplus \frac{1}{z} H^4_\pm \oplus \frac{1}{z^2} H^6_\pm$$

(2) (ζ-wall-crossing) I_+ and I_- are related by analytic continuation and a $\mathbb{C}$-linear symplectic isomorphism $\phi : H(z)_+ \to H(z)_- \in Sp_4(\mathbb{C})$

Question: ϵ-wall-crossing and ζ-wall-crossing for GLSM

Cheong, Ciocan-Fontanine & Kim "Orbifold Quasimap Theory"
ϵ-stable quasimaps to $\mathcal{X}_\zeta = [V^{ss}(\zeta)/G]$, $\epsilon \in \mathbb{Q}_{>0}$.

$$I \xleftarrow{\epsilon\to 0^+} J^\epsilon \xrightarrow{\epsilon\to+\infty} J$$

quasimap wall-crossing (ϵ-wall-crossing)
$\Rightarrow$ Givental style mirror theorems
$\Rightarrow$ mirror theorem for smooth toric DM stacks
(Coates-Corti-Iritani-Tseng)

Y. Zhou: ϵ-wall-crossing in orbifold quasimap theory in all genera in full generality. It is expected that Y. Zhou's proof is generalizable to GLSM $\Rightarrow$ Givental style mirror theorems for all GLSM in all phases (Shoemaker)
Clader-Janda-Ruan, "Higher-genus wall-crossing in the gauged linear sigma model", with an appendix by Y. Zhou:
GLSM for complete intersections in weighted projective spaces

Today: I-functions and ζ-wall-crossing for abelian GLSMs

2. Higgs branch

- Fan-Jarvis-Ruan, "A mathematical theory of the gauged linear sigma model" 2015, 2018, 2020
- Favero-Kim, "General GLSM invariants and their cohomological field theories," 2020
  - Polischuk-Vaintrob: affine LG models
  - Ciocan-Fontanine-Favero-Guéré-Kim-Shoemaker: convex hybrid models

Let $\underline{\mathfrak{X}} = (V, G, \mathbb{C}^*_R, W, \zeta)$ be a general GLSM.
G and $\mathbb{C}^*_R$ generate $\Gamma \subset GL(V)$

$$1 \to G \to \Gamma \xrightarrow{\chi} \Gamma/G = \mathbb{C}^*_R/\langle J \rangle = \mathbb{C}^*_\omega \to 1.$$

Landau-Ginzburg (LG) quasimaps

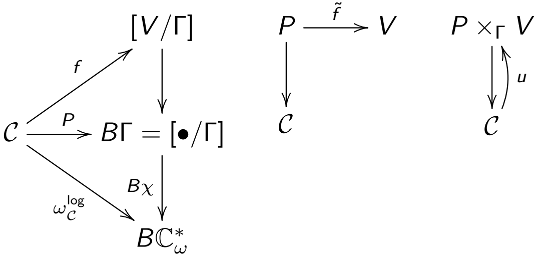
A genus-g, ℓ-pointed, degree $\beta \in H_2(BG;\mathbb{Q})$ ϵ-stable LG quasimap to $\underline{\mathfrak{X}}$ is a 4-tuple $\mathrm{Q} = ((\mathcal{C}, \mathfrak{z}_1, \dots, \mathfrak{z}_\ell), P, \rho, u))$ where

(1) $(\mathcal{C}, \mathfrak{z}_1, \dots, \mathfrak{z}_\ell)$ is a genus-g, ℓ-pointed nodal orbicurve

(2) $P \to \mathcal{C}$ is a principal Γ-bundle

(3) $\rho : P \times_\chi \mathbb{C} \xrightarrow{\cong} \omega_{\mathcal{C}}^{\log} = \omega_{\mathcal{C}}(\mathfrak{z}_1 + \cdots + \mathfrak{z}_\ell)$

(4) $u \in H^0(\mathcal{C}, P \times_\Gamma V)$



(5) The base locus $B(\mathrm{Q}) := u^{-1}(P \times_\Gamma V_G^{un}(\zeta))$ is purely 0-dim'l and disjoint from marked points and nodes in $\mathcal{C}$

$+$ ϵ-stability condition

Moduli spaces

$LG^{\epsilon}_{g,\ell}(\underline{\mathfrak{X}},\beta)$:= moduli of genus-g, ℓ-pointed, degree β ϵ-stable LG quasimaps to $\underline{\mathfrak{X}}$. We have evaluation maps

$$\mathrm{ev}_i : LG^{\epsilon}_{g,\ell}(\underline{\mathfrak{X}},\beta) \longrightarrow I\mathcal{X}_\zeta = \bigsqcup_{v\in B_\zeta} \mathcal{X}_{\zeta,v}, \quad i=1,\ldots,\ell$$

Fix g,ℓ,β, and $\vec{v}=(v_1,\ldots,v_\ell)$ where $v_i\in B_\zeta$, let

$$\mathbf{X} = LG^{\epsilon}_{g,\vec{v}}(\underline{\mathfrak{X}},\beta) := \bigcap_{i=1}^{\ell} \mathrm{ev}_i^{-1}(\mathcal{X}_{\zeta,v_i}).$$

Then $\mathrm{ev}_i : \mathbf{X} \to \mathcal{X}_{\zeta,v_i}$.
Let $\mathcal{Z}_\zeta \subset \mathcal{X}_\zeta$, $\mathbf{Z}\subset\mathbf{X}$ be the closed substack defined by $\mathrm{Crit}(W)\subset V$. By Fan-Jarvis-Ruan, $\mathbf{X}$ is a (usually non-proper) separated DM stack of finite type, equipped with a perfect obstruction theory of virtual dimension r. If $\mathcal{Z}_\zeta$ is proper then $\mathbf{Z}$ is a proper DM stack.

Favero-Kim's construction

There exists a *smooth* DM stack $\mathbf{U}$ of finite type, a vector bundle bundle $B_{\mathbf{U}}$ over $\mathbf{U}$ such that $\mathbf{X}$ is embedded in $\mathbf{U}$ as the zero locus of $\beta_{\mathbf{U}} \in \Gamma(\mathbf{U}, B_{\mathbf{U}})$:

$$\iota_{\mathbf{X}} : \mathbf{X} = Z(\beta_{\mathbf{U}}) \hookrightarrow \mathbf{U}, \quad [\mathbf{X}]^{\mathrm{vir}}_{\mathrm{BF}} = c_b(B_{\mathbf{U}}, \beta_{\mathbf{U}}) \cap [\mathbf{U}] \in A_r(\mathbf{X}; \mathbb{Q}).$$

where $b = \mathrm{rank} B_{\mathbf{U}}$, and that the evaluation map $\mathrm{ev}_i : \mathbf{X} \to \mathcal{X}_{\zeta, v_i}$ extends to $\mathrm{ev}_i^{\mathbf{U}} : \mathbf{U} \to \mathcal{X}_{\zeta, v_i}$ which is a smooth map between smooth DM stacks.

$\mathbf{X}$, $\mathbf{U}$, $B_{\mathbf{U}}$, $\beta_{\mathbf{U}}$ depend on $(V, G, \mathbb{C}^*_R, \zeta)$ but not on W.

Let $w_{\zeta,v} : \mathcal{X}_{\zeta,v} \to \mathbb{C}$ be the restriction of the superpotential $w_\zeta : \mathcal{X}_\zeta \to \mathbb{C}$, and define

$$w_{\mathbf{U}} := \sum_{i=1}^{\ell} (\mathrm{ev}_i^{\mathbf{U}})^* w_{\zeta, v_i} : \mathbf{U} \to \mathbb{C}.$$

Then $\iota_X^* w_{\mathbf{U}} = 0$. Favero-Kim constructed

$$[\mathbf{U}]^{\mathrm{vir}}_w \in \mathbb{H}^{even}_{\mathbf{Z}} \left(\mathbf{U}, (\Omega^\bullet_{\mathbf{U}}, -dw_{\mathbf{U}})\right).$$

LG loop spaces

In orbifold quasimap theory, the I-function is obtained by $\mathbb{C}^*$ localization on stacky loop spaces (orbifold version of Givental's toric map spaces). We will introduce LG loop spaces which are analogues of stacky loop spaces.

The domain is $(\mathbb{P}[a,1], \infty = [1,0])$ where $a \in \mathbb{Z}_{>0}$, $(g,\ell) = (0,1)$.

M. Shoemaker "Towards a mirror theorem for GLSMs" $(g,\ell) = (0,2)$.

Given $p = 0, 1$, $a \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}$, define

$$H^p\left(\mathbb{P}^1, \mathcal{O}(m/a)\right) := H^p\left(\mathbb{P}[a,1], \mathcal{O}_{\mathbb{P}[a,1]}(m)\right).$$

Given an effective class $\beta \in \mathbb{K}^\zeta \subset \mathbb{L}_\mathbb{Q} := \mathbb{L} \otimes_\mathbb{Z} \mathbb{Q}$, define

$$V_\beta = \bigoplus_{i=1}^{n+\kappa} H^0\left(\mathbb{P}^1, \mathcal{O}(\langle D_i, \beta\rangle - \frac{q_i}{2})\right), \quad W_\beta = \bigoplus_{i=1}^{n+\kappa} H^1\left(\mathbb{P}^1, \mathcal{O}(\langle D_i, \beta\rangle - \frac{q_i}{2})\right)$$

degree β LG loop space $\mathcal{X}_{\beta,\zeta} = [V_\beta^{ss}(\zeta)/G]$

degree β obstruction bundle $Ob_\beta = [\left(V_\beta^{ss}(\zeta) \times W_\beta\right)/G]$

$\mathbb{C}_q^*$ rotates $\mathbb{P}^1$, $\widetilde{T}$ and $\mathbb{C}_q^*$ act linearly on V_β, W_β.
$\widetilde{T}\times\mathbb{C}_q^*$ acts on the smooth toric DM stack $\mathcal{X}_{\beta,\zeta}$.
Ob_β is a $\widetilde{T}\times\mathbb{C}_q^*$-equivariant vector bundle over $\mathcal{X}_{\beta,\zeta}$.
Let $\mathcal{X}^\circ_{\beta,\zeta}=[V^\circ_\beta/G]\subset\mathcal{X}_{\beta,\zeta}=[V^{ss}_\beta(\zeta)/G]$ is the open substack such that the evaluation at ∞ is defined:

$$\mathrm{ev}_\infty:\mathcal{X}^\circ_{\beta,\zeta}\longrightarrow\mathcal{X}_{\zeta,v(\beta)},\quad \iota_{\beta\to v(\beta)}:\mathcal{F}_{\beta,\zeta}:=(\mathcal{X}^\circ_{\beta,\zeta})^{\mathbb{C}_q^*}\hookrightarrow\mathcal{X}_{\zeta,v(\beta)}.$$

$$N_\beta^{\mathrm{vir}}=\iota^*_{\beta\to v(\beta)}\widetilde{N}_\beta^{\mathrm{vir}}\quad \text{where } \widetilde{N}_\beta^{\mathrm{vir}}\in K_{\mathbb{C}_q^*\times\widetilde{T}}(\mathcal{X}_{\zeta,v(\beta)}).$$

Fix β and let $v=v(\beta)$. $\mathbf{X}=\mathcal{F}_{\beta,\zeta}$ is the zero locus of a *regular* section $\beta_{\mathbf{U}}$ of a $\widetilde{T}$-equivariant vector bundle $B_{\mathbf{U}}$ on $\mathbf{U}=\mathcal{X}_{\zeta,v}$.

$$-w_{\mathbf{U}}=-w_{\zeta,v(\beta)}=\langle\alpha_{\mathbf{U}},\beta_{\mathbf{U}}\rangle$$

for some $\alpha_{\mathbf{U}}\in\Gamma(\mathbf{U},B^\vee_{\mathbf{U}})$. Then $\mathbb{K}_\beta=\{\alpha_{\mathbf{U}},\beta_{\mathbf{U}}\}$ is a Kozual matrix factorization of $(\mathcal{X}_{\zeta,v},-w_{\zeta,v})$. If $w_{\zeta,v}=0$ (which is true when $\mathcal{X}_{\zeta,v}$ is compact, i.e. when v is "narrow") then $\alpha_{\mathbf{U}}=0$ and $\mathbb{K}_\beta=\iota_{\beta\to v*}\mathcal{O}_{\mathcal{F}_{\beta,\zeta}}$ is the Kozual complex.

I-functions

For each effective class $\beta \in \mathbb{K}^\zeta$, define

$$F_\beta^w := \mathrm{tdch}_{\mathcal{Z}_{\zeta,v}}^{\mathcal{X}_{\zeta,v}}(\mathbb{K}_\beta) \in \mathbb{H}_{\mathcal{Z}_{\zeta,v}}^{even}\left(\mathcal{X}_{\zeta,v}, (\Omega^\bullet_{\mathcal{X}_{\zeta,v}}, -dw_{\zeta,v})\right)$$

$$F_\beta^{\widetilde{T}} = e_{\widetilde{T}}(B_{\mathbf{U}}) \in H^*_{\widetilde{T}}(\mathcal{X}_{\zeta,v}).$$

The GLSM I-function of $(V, G, \mathbb{C}^*_R, W, \zeta)$ is

$$I_w(y,z) = \sum_{v \in B_\zeta} I_{w,v}, \quad I_{w,v} = (\dots) \sum_{\substack{\beta \in \mathbb{K}^\zeta \\ v(\beta)=v}} y^\beta \frac{1}{e_{\mathbb{C}^*_q}(\widetilde{N}_\beta^{\mathrm{vir}})} F_\beta^w$$

The $\widetilde{T}$-equivariant I-function of $(V, G, \mathbb{C}^*_R, 0, \zeta)$ is

$$I_{\widetilde{T}}(y,z) = \sum_{v \in B_\zeta} I_{\widetilde{T},v}, \quad I_{\widetilde{T},v} = (\dots) \sum_{\substack{\beta \in \mathbb{K}^\zeta \\ v(\beta)=v}} y^\beta \frac{1}{e_{\mathbb{C}^*_q \times \widetilde{T}}(\widetilde{N}_\beta^{\mathrm{vir}})} F_\beta^{\widetilde{T}}$$

Central charges

Given $\mathcal{B} \in MF(\mathcal{X}_\zeta, w)$, $[\mathcal{B}] \in K(MF(\mathcal{X}_\zeta, w))$, define

GLSM central charge $\quad Z_w([\mathcal{B}]) = \langle I_w, \hat{\Gamma}_w \mathrm{ch}_w([\mathcal{B}]) \rangle$

where $\hat{\Gamma}_w \mathrm{ch}_w \in \bigoplus_{v \in B_\zeta} H_{w,v} \otimes_{\mathbb{C}} \mathbb{C}(\!(z^{-1})\!)$,

$H_{w,v} = \mathbb{H}^* \left(\mathcal{X}_{\zeta,v}, (\Omega^\bullet_{\mathcal{X}_{\zeta,v}}, dw_{\zeta,v}) \right) \cong H^*(\mathcal{X}_{\zeta,v}, w^\infty_{\zeta,v}; \mathbb{C})$.

$\hat{\Gamma}_w$ analogue of Iritani's Γ-class

$\mathrm{ch}_w([\mathcal{B}])$ defined by Choa-Kim-Sreedhar.

Given $\mathcal{B} \in Coh_{\widetilde{T}}(\mathcal{X}_\zeta)$, $[\mathcal{B}] \in K_{\widetilde{T}}(\mathcal{X}_\zeta)$, define

$\widetilde{T}$-equivariant central charge

$$Z_{\widetilde{T}}([\mathcal{B}]) = \langle I_{\widetilde{T}}, \hat{\Gamma}_{\widetilde{T}} \mathrm{ch}_{\widetilde{T}}([\mathcal{B}]) \rangle = \sum_{I \in \mathcal{A}^{\mathrm{min}}_\zeta} Z^I_{\widetilde{T}}([\mathcal{B}])$$

where $\hat{\Gamma}_{\widetilde{T}} \mathrm{ch}_{\widetilde{T}}([\mathcal{B}]) \in \bigoplus_{v \in B_\zeta} H^*_{\widetilde{T}}(\mathcal{X}_{\zeta,v}) \otimes_{R_{\widetilde{T}}} R_{\widetilde{T}}(\!(z^{-1})\!)$,

$R_{\widetilde{T}} = H^*_{\widetilde{T}}(\bullet) = \mathbb{C}[\lambda_1, \ldots, \lambda_{n+\kappa}]$.

3. Coulomb branch

Hori-Romo 2013 (Sugishita-Terashima, Honda-Okuda)

Consider an abelian GLSM $(V, G, \mathbb{C}^*_R, W, \zeta)$ where
$G \simeq (\mathbb{C}^*)^\kappa \subset SL(V)$ (Calabi-Yau)
$\theta = \zeta + 2\pi\sqrt{-1}B \in \mathbb{L}^\vee_\mathbb{C}$ complexified/stringy Kähler class
ζ = (extended) Kähler class, B = B-field
$\alpha = (\alpha_1, \ldots, \alpha_{n+\kappa}) \in \mathbb{R}^{n+\kappa}$, $\delta \in \mathbb{L}_\mathbb{R}$, $\langle D_i, \delta\rangle + \alpha_i > 0$ for
$1 \leq i \leq n + \kappa$
Given $[\mathcal{B}] \in K_{\widetilde{T}}([V/G])$, define the
(α-perturbed) hemisphere partition function

$$Z_{D^2}([\mathcal{B}]) = \frac{1}{(2\pi\sqrt{-1})^\kappa}\int_{\delta+\sqrt{-1}\mathbb{L}_\mathbb{R}} d\sigma \Gamma(\sigma)\mathrm{ch}[\mathcal{B}](\sigma)e^{\langle\theta,\sigma\rangle}$$

where $\Gamma(\sigma) = \prod_{i=1}^{n+\kappa}\Gamma(\langle D_i, \sigma\rangle + \alpha_i)$, and

$$\mathrm{ch}[\mathcal{B}](\sigma) = \sum_{t\in\mathbb{L}^\vee} c_t e^{2\pi\sqrt{-1}\langle t,\sigma\rangle} \text{ if } [\mathcal{B}] = \sum_{t\in\mathbb{L}^\vee} c_t\mathcal{L}_t \in K_{\widetilde{T}}([V/G]), c_t \in R_{\widetilde{T}}.$$

- $Z_{D^2}([\mathcal{B}])$ is a multidimensional inverse Mellin transform of $\Gamma(\sigma)\mathrm{ch}[\mathcal{B}](\sigma)$
- (R-wall-crossing) $\begin{cases} \alpha_i \to 0: & \text{without superpotential} \\ \alpha_i \to q_i/2: & \text{with superpotential} \end{cases}$

Proposition

There is an open subset $U \subset \mathbb{L}_{\mathbb{R}}^{\vee}$ such that

$$Z_{D^2}(\mathcal{L}_t) = \frac{1}{(2\pi\sqrt{-1})^{\kappa}} \int_{\delta+\sqrt{-1}\mathbb{L}_{\mathbb{R}}} d\sigma \Gamma(\sigma) e^{\langle \theta + 2\pi\sqrt{-1}t, \sigma \rangle}$$

is analytic in θ on $\{\theta = \zeta + 2\pi\sqrt{-1}B \mid \zeta \in \mathbb{L}_{\mathbb{R}}^{\vee}, B + t \in U\}$. More precisely,

$$|\langle B + t, \sigma \rangle| < \frac{1}{4} \sum_{i=1}^{n+\kappa} |\langle D_i, \sigma \rangle| \text{ for all } \sigma \in L_{\mathbb{R}} \setminus \{0\}.$$

Theorem 1 (Aleshkin-L)

Let C be a phase of the GLSM, and let $\zeta_0 \in C$.

$\Rightarrow C = \bigcap_{I \in \mathcal{A}^{\min}_{\zeta_0}} \angle_I \subset \mathbb{L}^\vee_{\mathbb{R}}$ where $\angle_I = \{\sum_{i \in I} a_i D_i \mid a_i \in (0, +\infty)\}$.

Then there is an open subset $U_C = \bigcap_{I \in \mathcal{A}^{\min}_{\zeta_0}} U_I \subset \mathbb{L}^\vee_{\mathbb{R}}$ where

$$U_I = \{\sum_{i \in I} a_i D_i \mid a_i \in (N_i, +\infty)\} \ (N_i \gg 0) \ = \text{shifted } \angle_I$$

such that if $\zeta \in U_C$ then $Z_{D^2}(\mathcal{L}_t) = \sum_{I \in \mathcal{A}^{\min}_{\zeta_0}} Z^I(\mathcal{L}_t)$, where

$$Z^I(\mathcal{L}_t) = \frac{1}{|G_I|} \sum_{m \in (\mathbb{Z}_{\geq 0})^I} \prod_{\bar{i} \in \bar{I}} \Gamma\left(\langle D_i, \sigma_m \rangle + \alpha_i\right) \prod_{i \in I} \frac{(-1)^{m_i}}{m_i!} e^{\langle \theta + 2\pi\sqrt{-1}t, \sigma_m \rangle}$$

$$\sigma_m = -\sum_{i \in I} (m_i + \alpha_i) D_i^{*I} \text{ where } \{D_i^{*I} : i \in I\} \text{ is a basis of } \mathbb{L}_{\mathbb{Q}}$$

dual to the basis $\{D_i : i \in I\}$ of $\mathbb{L}^\vee_{\mathbb{Q}}$. The infinite series $Z^I(\mathcal{L}_t)$ converges absolutely and uniformly on $\{\theta = \zeta + 2\pi\sqrt{-1}B : \zeta \in U_I, B \in \mathbb{L}^\vee_{\mathbb{R}}\}$.

Moreover, we have the following Higgs-Coulomb correspondence

$$Z_{D^2}([\mathcal{B}])\Big|_{\theta=-\sum_{a=1}^{\kappa}(\log y_a)\xi_a,\ \alpha_i=\lambda_j+\frac{q_i}{2}} = Z_{\widetilde{T}}([\mathcal{B}])\Big|_{z=1}.$$

where $\{\xi_1, \ldots, \xi_\kappa\}$ is an integral basis of $\mathbb{L}^\vee$ and $1 \le i \le n+\kappa$.
cf. Knapp-Romo-Scheidegger 2020

Proof by careful manipulation of κ-dimensional cycles and convergence checks of integrals $\int$ and series $\sum$.

$$Z_{D^2}(\mathcal{L}_t) = \int_{\mathbb{R}^\kappa}(\cdots) = \sum_{\mathcal{A}_1}\sum_{m\in\mathbb{Z}_{\ge 0}}\int_{S^1\times\mathbb{R}^{\kappa-1}}(\cdots) = \cdots$$

$$= \sum_{\mathcal{A}_\ell}\sum_{m\in(\mathbb{Z}_{\ge 0})^\ell}\int_{(S^1)^\ell\times\mathbb{R}^{\kappa-\ell}}(\cdots) = \cdots = \sum_{\mathcal{A}_\kappa}\sum_{m\in(\mathbb{Z}_{\ge 0})^\kappa}\underbrace{\int_{(S^1)^\kappa}(\cdots)}_{\kappa\text{-dimensional residue}}$$

- $\mathcal{A}_1, \ldots, \mathcal{A}_\kappa = \mathcal{A}_{\zeta_0}^{\min}$ are finite sets.
- Up to translation, $\mathbb{R}^{\kappa-\ell} \subset \sqrt{-1}\mathbb{L}_\mathbb{R}$.
- Use the Calabi-Yau condition.

4. Wall-Crossing

abelian GLSMs without superpotentials:

- Borisov-Horja "Mellin-Barnes integrals as Fourier-Mukai transforms"
- Coates-Iritani-Jiang "The Crepant Transformation Conjecture for Toric Complete Intersections."

Let C_+, C_- be two adjacent chambers in $\mathbb{L}^\vee_\mathbb{R}$ = space of stability conditions. Then $\bar{C}_\pm$ are κ-dimensional cones in the secondary fan, and the $(\kappa-1)$-dimensional cone $\bar{C}_+ \cap \bar{C}_-$ is contained in the hyperplane $(h^\perp)_\mathbb{R} := \{\zeta \in \mathbb{L}^\vee_\mathbb{R} \mid \langle \zeta, h\rangle = 0\}$ for some primitive $h \in \mathbb{L}$.

Let $\zeta_\pm \in C_\pm$, $\mathcal{X}_\pm := \mathcal{X}_{\zeta_\pm}$. Then

$$C_\pm = \bigcap_{I \in \mathcal{A}^{\min}_{\zeta_\pm}} \angle_I, \quad \mathcal{A}^{\min}_{\zeta_\pm} = \mathcal{A}^{\text{ess}}_\pm \cup \underbrace{\mathcal{A}^{\text{noness}}}_{\mathcal{A}^{\min}_{\zeta_+} \cap \mathcal{A}^{\min}_{\zeta_-}}$$

$$\{1,\ldots,n+\kappa\} = I_+ \cup I_- \cup I_0, \text{ where } \begin{matrix} I_+ \\ I_- \\ I_0 \end{matrix} = \{i \mid \langle D_i, h\rangle \begin{matrix} > \\ < \\ = \end{matrix} 0\}$$

$$\mathcal{A}^{\text{ess}}_\pm = \{\{i\} \cup J \mid i \in I_\pm, J \in \mathcal{A}_0\}, \quad J \in \mathcal{A}_0 \Rightarrow J \subset I_0, |J| = \kappa - 1.$$

Theorem 2 (Aleshkin-L)
In the setting above, if $t \in \mathbb{L}^\vee$ satisfies the Grade Restriction Rule

$$|\langle B+t, h\rangle| < \frac{1}{4}\sum_{i=1}^{n+\kappa} |\langle D_i, h\rangle| = \frac{1}{2}\eta$$

where $\eta = \sum_{i\in I_+}\langle D_i, h\rangle = \sum_{i\in I_-}\langle D_i, -h\rangle$. Then there exists an open subset $U \subset U_{C_\pm}$ such that for $\zeta \in U$

$$Z_{D^2}(\mathcal{L}_t)_\pm = \sum_{J\in\mathcal{A}_0} Z_J^{\mathrm{ess}}(\mathcal{L}_t) + \sum_{I\in\mathcal{A}^{\mathrm{noness}}} Z_I(\mathcal{L}_t)$$

- $Z_J^{\mathrm{ess}}(\mathcal{L}_t)$ is an explicit series of integrals over $(S^1)^{\kappa-1}\times\mathbb{R}$.
- $Z_I(\mathcal{L}_t)$ converges uniformly and absolutely for $\zeta \in U_I \supset U_{C_\pm}$.

The Grade Restriction Rule (GRR) $\langle B+t,h\rangle \in (-\frac{\eta}{2},\frac{\eta}{2})$ defines equivalences

$$\mathrm{GR}: \begin{array}{ccc} D^b(\mathcal{X}_+) & \longrightarrow & D^b(\mathcal{X}_-) \\ D^b_T(\mathcal{X}_+) & \longrightarrow & D^b_T(\mathcal{X}_-) \\ D^b_{\widetilde{T}}(\mathcal{X}_+) & \longrightarrow & D^b_{\widetilde{T}}(\mathcal{X}_-) \\ D(MF(\mathcal{X}_+,w)) & \longrightarrow & D(MF(\mathcal{X}_-,w)) \end{array}$$

- Herbst-Hori-Page GRR
- Kawamata $\mathrm{FM}: D^b(\mathcal{X}_+) \xrightarrow{\simeq} D^b(\mathcal{X}_-)$ (Fourier-Mukai)
- Coates-Iritani-Jiang-Segal $\mathrm{GR}=\mathrm{FM}: D^b_T(\mathcal{X}_+) \xrightarrow{\simeq} D^b_T(\mathcal{X}_-)$ (Grade Restriction Rule = Fourier-Mukai)
  Ballard-Favero-Katzarkov, Halpern-Leistner
- Baranovsky-Pecharich, ...

Theorem 2 $\Rightarrow Z_{D^2}([\mathcal{B}])_+$ and $Z_{D^2}(\mathrm{GR}[\mathcal{B}])_-$ are related by analytic continuation. $\mathrm{GR} \rightarrow$ symplectic transform

Future work: discrepant transformation, general G.