Twisted Shklyarov pairings and applications

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(Based on joint work in progress with Paul Seidel)
Main construction in a toy model
- Fixed point Floer cohomology
- Twisted Shklyarov pairing
- A Cardy relation

Lefschetz fibrations and noncommutative divisor
- Hamiltonian Floer cohomology of the global monodromy
- Noncommutative anti-canonical divisor

Other applications
- Collapsing critical values
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- Count $u : \mathbb{R}^2_{s,t} \to M$
  \[
  \phi \circ u(t + 1, s) = u(t, s), \quad \partial_s u + J_t(u)\partial_t u = 0
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  to define differential on $CF^*(\phi)$.
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to define differential on \(\text{CF}^*(\phi)\).
- Output: fixed point Floer cohomology \(HF^*(\phi)\).
Some algebraic structures
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- Poincaré type pairing

\[ HF^*(\phi) \otimes HF^{2n-*}(\phi^{-1}) \to \mathbb{C}, \]

nondegenerate, coincides with the Poincaré pairing on \( QH^*(M) = H^*(M) \) for \( \phi = \text{id} \).
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• Conjugation isomorphism: \( \psi \) symplectic automorphism, then

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- \( \Rightarrow HF^*(\phi^r) \) admits a \( \mathbb{Z}/r \)-action induced by conjugation with \( \phi \).
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Twisted Hochschild homology

• $\mathcal{F}$: strictly proper $A_\infty$ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.
  $\Rightarrow \forall X, Y \in \text{Ob}(\mathcal{F}), \mathcal{F}(X, Y)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $\mathbb{C}$. 
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- \( \Phi : \mathcal{F} \to \mathcal{F} \) strict \( A_\infty \) automorphism.
  \( \Rightarrow \) an \( A_\infty \) functor with \( \Phi^k = 0, \forall k \geq 2 \), \( \Phi^1 \) is an isomorphism of graded vector spaces.

- Example: \( \mathcal{F} = \) monotone Fukaya category of \( (M, \omega) \), \( \Phi = \) automorphism of \( \mathcal{F} \) induced by \( \phi \), which can be made strict by formally introducing more objects in \( \mathcal{F} \).
Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(\mathcal{F}, \Phi) := \bigoplus_{L_0, \ldots, L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$. 

Differential: bar differential, only using $\mu_k \mathcal{F}$, with (co)homology denoted by $HH^*($\mathcal{F}, \Phi$).

Proposition
There exists a bilinear pairing $\langle - , - \rangle_{\Phi} : HH^*(\mathcal{F}, \Phi) \otimes HH_{-\ast}(\mathcal{F}, \Phi) \rightarrow \mathbb{C}$.

If $\mathcal{F}$ is homologically smooth, then it is nondegenerate. For $r \geq 1$, $HH^*(\mathcal{F}, \Phi)^r$ admits a $\mathbb{Z}/r$-action which is generated by conjugation with $\Phi$. 

The pairing generalizes the pairing defined by Shklyarov for $\Phi = \text{id}$. 
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There exists a bilinear pairing $\langle -,- \rangle_{\Phi} : HH_* (\mathcal{F}, \Phi) \otimes HH_{*-1} (\mathcal{F}, \Phi) \to \mathbb{C}$.
If $\mathcal{F}$ is homologically smooth, then it is nondegenerate. For $r \geq 1$, $HH_* (\mathcal{F}, \Phi_r)$ admits a $\mathbb{Z}/r$-action which is generated by conjugation with $\Phi$. 
- The pairing generalizes the pairing defined by Shklyarov for $\Phi = \text{id}$. 


Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(F, \Phi) := \bigoplus_{L_0, \ldots, L_k} F(L_{k-1}, L_k)[1] \otimes \cdots \otimes F(L_0, L_1)[1] \otimes F(\Phi(L_k), L_0)$.

- Differential: bar differential, only using $\mu^k_F$, with (co)homology denoted by $HH_*(F, \Phi)$.

Proposition

There exists a bilinear pairing

$$\langle -, - \rangle_\Phi : HH_*(F, \Phi) \otimes HH_{-*}(F, \Phi^{-1}) \to \mathbb{C}.$$  

If $F$ is homologically smooth, then it is nondegenerate. For $r \geq 1$, $HH_*(F, \Phi^r)$ admits a $\mathbb{Z}/r$-action which is generated by conjugation with $\Phi$.  

Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_\ast(\mathcal{F}, \Phi) := \bigoplus_{L_0, \ldots, L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$.

- Differential: bar differential, only using $\mu_k^\mathcal{F}$, with (co)homology denoted by $HH_\ast(\mathcal{F}, \Phi)$.

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Twisted Shklyarov pairing
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- The construction of $\langle -,- \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_\ast(\mathcal{F}, \Phi) \otimes HH_{-\ast}(\mathcal{F}, \Phi^{-1})$. 
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- The construction of $\langle - , - \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_* (\mathcal{F}, \Phi) \otimes HH_{-*} (\mathcal{F}, \Phi^{-1})$.

- The proof of the nondegeneracy is a generalization of the snake relation in the presence of an automorphism.
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• The construction of $\langle -,- \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1})$.

• The proof of the nondegeneracy is a generalization of the snake relation in the presence of an automorphism.

• Here is an explicit formula:

$$\langle -,- \rangle_\Phi : CC_*(\mathcal{F}, \Phi) \otimes CC_{-*}(\mathcal{F}, \Phi^{-1}) \longrightarrow \mathbb{C},$$

$$\langle a_m \otimes \cdots \otimes a_1 \otimes a_0, b_n \otimes \cdots \otimes b_1 \otimes b_0 \rangle_\Phi$$

$$= \sum_{ijkl} \text{Str}(y \mapsto \pm \mu_{ij}^{i-j-k+l+m+2}(a_i, \ldots, a_0, \Phi a_m, \ldots, \Phi a_{k+1},$$

$$\mu_{ij}^{i+j+k-l+n+2}(\Phi a_k, \ldots, \Phi a_{i+1}, \Phi y, \Phi b_j, \ldots, \Phi b_1, \Phi b_0,$$

$$b_n, \ldots, b_{l+1}), b_l, \ldots, b_{j+1})).$$
Twisted Shklyarov pairing

• For length 1 Hochschild chains, the pairing is reduced to

\[ \mathcal{F}(\Phi(L_0), L_0) \otimes \mathcal{F}(\Phi^{-1}(L_1), L_1) \to \mathbb{C} \]

\[ a_0 \otimes b_0 \mapsto \pm \text{Str}(y \mapsto \mu^2(a_0, \mu^2(\Phi y, \Phi b_0))), \]

where the super-trace is taken over \( \mathcal{F}(L_0, L_1) \).
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**Proposition**

For $\mathcal{F} = \mathcal{F}(M, \omega)$, and $\phi : (M, \omega) \to (M, \omega)$, there exists a twisted open-closed string map $OC(\phi) : HH_*(\mathcal{F}, \Phi) \to HF^{*+n}(\phi)$ making the following diagram commute.
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\begin{array}{ccc}
HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1}) & \xrightarrow{\langle -,- \rangle_{\Phi}} & HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1})\\
\downarrow & & \downarrow Poincaré\\
OC(\phi) \otimes OC(\phi^{-1}) & \xrightarrow{OC(\phi)} & \mathbb{C}\\
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\begin{array}{ccc}
HH_* (\mathcal{F}, \Phi) \otimes HH_{-*} (\mathcal{F}, \Phi^{-1}) & \xrightarrow{\langle -, - \rangle_\phi} & C \\
OC(\phi) \otimes OC(\phi^{-1}) & \xrightarrow{\text{Poincaré}} & HF^{n+*} (\phi) \otimes HF^{n-*} (\phi^{-1})
\end{array}
\]

Corollary

If $\mathcal{F}$ is homologically smooth, $OC(\phi)$ is injective.
A Cardy relation

Matched output in $CF^\star(\phi^\perp)$

Shifted by $\Phi$

Str as matching with $y$

insert $y$

$\phi$

$b_0$

$\bar{a}_0$

$L_0$

$L_1$

$L_2$

$L_3$

$L_0'$

$L_1'$
Some further remarks on $OC(\phi)$
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- $OC(\phi^r): HH_*(\mathcal{F}, \Phi^r) \to HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.
Some further remarks on $OC(\phi)$

- $OC(\phi^r) : HH_* (\mathcal{F}, \Phi^r) \rightarrow HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.
- For $(M, \omega)$ monotone, $\mathcal{F}(M, \omega)$ is decomposed into smaller pieces according to the value of the disc potential. All the above constructions “respect” such a decomposition.
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- $OC(\phi^r) : HH_* (\mathcal{F}, \Phi^r) \to HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.
- For $(M, \omega)$ monotone, $\mathcal{F}(M, \omega)$ is decomposed into smaller pieces according to the value of the disc potential. All the above constructions “respect” such a decomposition.
- An interesting question: replace $\Phi$ by Lagrangian correspondences.
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- Let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, with fiber $(F, \omega, \theta)$ a Liouville domain.
Lefschetz fibrations

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• $\mu : F \to F$ global monodromy: it’s compactly supported, and $\exists G_\phi : F \to F$ such that $\phi^* \theta - \theta = dG_\phi$. 
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• The Hamiltonian

$$H^{\text{rot}} : (\mathbb{C}, \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}) \to \mathbb{R}$$

$$z \mapsto \pi |z|^2$$

defines a Hamiltonian $H^{\text{rot}} \circ \pi : E \to \mathbb{R}$. 
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• The associated Hamiltonian diffeomorphism is equal to identity viewed from $\mathbb{C}$, and restricts to $\mu$ on each fiber over the complement of a compact subset.
Hamiltonian Floer cohomology

Question

*How to define the Hamiltonian Floer cohomology of $H^{\text{rot}} \circ \pi$?*
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Question

How to define the Hamiltonian Floer cohomology of $H^{\text{rot}} \circ \pi$?

- Traditionally: perturb $H^{\text{rot}} \circ \pi$ using the lift of $\epsilon|z|^2$ for $0 < |\epsilon| < 1$.
- Alternatively: use the lift of $z \mapsto A \cdot \text{Re}(z)$ for some $A \neq 0$. 
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- Alternatively: use the lift of $z \mapsto A \cdot \text{Re}(z)$ for some $A \neq 0$.
- Define Hamiltonian Floer cohomology:

  $$HF^*(E, 1) := HF^*(H^{\text{rot}} \circ \pi \text{ perturbed by } A \cdot \text{Re}(z))$$

  $$HF^*(E, 1 \pm \frac{1}{2}) := HF^*(H^{\text{rot}} \circ \pi \text{ perturbed by } \pm \frac{1}{2} \cdot |z|^2).$$
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Proposition

There exist long exact sequences

$$\cdots \rightarrow HF^*(E, \frac{1}{2}) \rightarrow HF^*(E, 1) \rightarrow HF^{*-1}(\mu) \rightarrow \cdots$$

$$\cdots \rightarrow HF^*(E, 1) \rightarrow HF^*(E, 1 + \frac{1}{2}) \rightarrow HF^*(\mu) \rightarrow \cdots.$$
Iterating the monodromy
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- More generally, we can consider $rH^{\text{rot}} \circ \pi$ for $r \in \mathbb{Z}$, and Floer cohomology groups $HF^*(E, r)$ and $HF^*(E, r \pm \frac{1}{2})$. 

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- They fit into long exact sequences

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There exists a nondegenerate pairing

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Proposition

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\]

For each \( r \in \mathbb{Z}_{\geq 1} \), \( HF^*(E, r) \) admits a \( \mathbb{Z}/r \)-action, so that \( HF^*(E, r) \rightarrow HF^{*-1}(\mu^r) \) is \( \mathbb{Z}/r \)-equivariant.
Fukaya–Seidel category and Serre functor
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Fukaya–Seidel category and Serre functor

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**Proposition (folklore, attributed to Kontsevich–Seidel)**

*The Hamiltonian diffeomorphism defined by $H^{\text{rot}} \circ \pi$ (perturbed by $A \cdot \text{Re}(z)$) induces the inverse Serre functor $S^{-1}$ on $\mathcal{A}$.**
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The Hamiltonian diffeomorphism defined by $H^{\text{rot}} \circ \pi$ (perturbed by $A \cdot \text{Re}(z)$) induces the inverse Serre functor $S^{-1}$ on $\mathcal{A}$.

- Recall: $\mathcal{A}(SX, Y) \cong \mathcal{A}(Y, X)^\vee$, $S$ represents the linear dual bimodule $\mathcal{A}^\vee$. 
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- Recall: $\mathcal{A}(S X, Y) \cong \mathcal{A}(Y, X)^\vee$, $S$ represents the linear dual bimodule $\mathcal{A}^\vee$.

- Here $\mathcal{A}^\vee(X, Y) = \text{Hom}(\mathcal{A}(X, Y), \mathbb{C})$, and

  $\langle \mu_{\mathcal{A}^\vee}(a_s, \ldots, a_1, \pi, b_r, \ldots, b_1), p \rangle = \pm \langle \pi, \mu^{r+1+s}(b_r, \ldots, b_1, p, a_s, \ldots, a_1) \rangle$. 
Twisted open-closed maps for Lefschetz fibrations
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Theorem (B–Seidel)

For each $r \in \mathbb{Z}$, there exists a $\mathbb{Z}/r$-equivariant twisted open-closed string map

$$OC(r) : HH_*(A, (A^\vee)^\otimes r) \to HF^*(E, -r)[n(1 + r)]$$

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\begin{array}{ccc}
HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \otimes HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes -r}) & \xrightarrow{\text{Shklyarov pairing}} & \mathbb{C} \\
OC(r) \otimes OC(-r) & \xrightarrow{\langle -, - \rangle_r} & HF^*(E, -r)[n(1 + r)] \otimes HF^*(E, r)[n(1 - r)]
\end{array}
$$
Twisted open-closed maps for Lefschetz fibrations

• The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with $J$-holomorphic curves (cf. Seidel’s Lefschetz IV and IV 1/2).
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Proposition

$\mathcal{A}$ is homologically smooth.
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Proposition

$A$ is homologically smooth.

- A distinguished basis defines a directed category, which has “automatic” smoothness. The point is that all Lefschetz thimbles are generated by the ones from a distinguished basis.
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$\forall r \in \mathbb{Z}, \ OC(r) : HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \to HF^*(E, -r)[n(1 + r)]$ is injective.
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- A conjecture of Seidel expects it to be an isomorphism.
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Question

How to construct an $A_\infty$ category $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory, and $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1 - n]$ as $A_\infty$ bimodules?
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- At the level of morphisms, $\mathcal{B}(X, Y) = \mathcal{A}(X, Y) \oplus \mathcal{A}^\vee(X, Y)[1 - n]$. The “0-th” order information is contained in $\mu^k_\mathcal{A}$ and $\mu^{|1-r|}_\mathcal{A}$. 
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- The “first” order information is a bimodule homomorphism

$$\theta : T(\mathcal{A}[1]) \otimes \mathcal{A}^\vee[-n] \otimes T(\mathcal{A}[1]) \to \mathcal{A},$$

which defines a class in $H^0(\text{hom}(\mathcal{A}^\vee[-n], \mathcal{A})) \cong HH_*(\mathcal{A}, S^{-1})$. 
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Question

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- The “higher” information is encoded in

$$H^*(\text{hom}((A^\vee)^r, A))^\mathbb{Z}/(r+1) \cong HH_*(A, S^{-(r+1)})^\mathbb{Z}/(r+1).$$
Noncommutative anti-canonical divisor

Proposition

If $HH_\ast(A, S^{-(r+1)})\mathbb{Z}/(r+1)$ is supported on non-negative degrees, an $A_\infty$ structure on $A \oplus A^\vee [1-n]$ is uniquely determined by the first-order information $\theta \in HH_\ast(A, S^{-1})$. 
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Lemma

If $\pi: E \to \mathbb{C}$ is constructed by removing the fiber over $\infty$ of an anti-canonical Lefschetz pencil, then $HF^\bullet(E, r)$ is supported on non-negative degrees.
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**Lemma**

If \( \pi : E \to \mathbb{C} \) is constructed by removing the fiber over \( \infty \) of an anti-canonical Lefschetz pencil, then \( HF_\ast(E, r) \) is supported on non-negative degrees.

- Because \( OC(r) \) is injective, we conclude that \( HH_\ast(\mathcal{A}, S^{-(r+1)}) \) is supported on non-negative degrees. The \( \mathbb{Z}/r \)-equivariance of \( OC(r) \) implies that \( HH_\ast(\mathcal{A}, S^{-(r+1)})^{\mathbb{Z}/(r+1)} \) satisfies the same property.
Mirror symmetry implications

- For $\pi: E \to C$ anti-canonical, Fukaya category of the fiber $B$ and the restriction functor $A \to B$ define a noncommutative anti-canonical divisor (cf. Seidel's Lefschetz VI).
- The above discussion shows that to reconstruct $B$ from $A$, we just need a natural transformation $S \to \text{id}$, which is in fact realized as the identity element in $HF^*(E,1)$ under the open-closed map.
- The same actually holds for the deformation of the pair $A \to B$ induced by adding back the base locus.
- This is a step towards showing that the compact Fukaya category of the Calabi–Yau hypersurface $B$ is actually defined over a polynomial ring after applying a mirror map.
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- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that $0$ is an isolated singularity.
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Theorem (McLean)

Let $m$ be the multiplicity of $f$ and 0. Then for any $r < m$, the fixed point Floer cohomology $HF^\ast(\mu^r) = 0$. 
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**Theorem (McLean)**

Let $m$ be the multiplicity of $f$ and 0. Then for any $r < m$, the fixed point Floer cohomology $HF^*(\mu^r) = 0$.

- Using the long exact sequence

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\cdots \to HF^*(E, \frac{1}{2}) \cong HF^*(\mathbb{C}^n) \to HF^*(E, 1) \to HF^{*-1}(\mu) \to \cdots,
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we know $\text{rank}HF^*(E, 1) \leq 1$. 
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- After Morsifying \( f \), the associated Fukaya–Seidel category is nontrivial \( \Rightarrow HH^*(\mathcal{A}, \mathcal{A}) \cong HH^{*-n}(\mathcal{A}, S^{-1}) \neq 0 \).
Collapsing critical values
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• Using the injectivity of $O_C(-1)$, we see that $HF^*(E, 1)$ is exactly 1-dimensional.
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- Using the injectivity of $OC(-1)$, we see that $HF^*(E, 1)$ is exactly 1-dimensional.
- This simple computation already gives some interesting applications.
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**Example**

The holomorphic map

$$(\mathbb{C}^*)^n \to \mathbb{C}$$

$$(z_1, \ldots, z_n) \mapsto z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}$$

cannot be deformed to a regular function with isolated singularities but with fewer critical values.
Collapsing critical values

- Recall that $HH^*({\mathcal{A}}, {\mathcal{A}})$ has a ring structure.
Collapsing critical values

- Recall that $HH^*(\mathcal{A}, \mathcal{A})$ has a ring structure.
- We can define the cup-length of $HH^*(\mathcal{A}, \mathcal{A})$ (denoted by $cl(\mathcal{A})$) to be the maximal $r \in \mathbb{Z}_{\geq 0}$ such that $\exists a_1, \ldots, a_r \in HH^*(\mathcal{A}, \mathcal{A})$ nilpotent and 
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  $$a_1 \cup \cdots \cup a_r \neq 0.$$  
- Homological mirror symmetry tells us
  $$\mathcal F\left(\left(\mathbb{C}^*\right)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\right) \cong D^b\text{Coh}(\mathbb{CP}^n).$$
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$$\mathcal{F}((\mathbb{C}^*)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}) \cong D^b \text{Coh}(\mathbb{C}\mathbb{P}^n).$$

• For $\mathcal{A} = D^b \text{Coh}(\mathbb{C}\mathbb{P}^n)$, we have $\text{cl}(\mathcal{A}) \geq n$ by looking at $n$ linearly independent holomorphic vector field on $\mathbb{C}\mathbb{P}^n$ generated by the torus action.
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- If $\mathcal{A} = \mathcal{F}(\pi)$ for $\pi : E \to \mathbb{C}$ being a Morsification of an isolated singularity, we know $\text{cl}(\mathcal{A}) = 0$. 
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Lemma

Suppose $\mathcal{A}$ admits a semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle,$$

then $\text{cl}(\mathcal{A}) \leq \sum \text{cl}(\mathcal{A}_i) + m - 1$. 
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of length \( \leq n - 1 \).
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of length $\leq n - 1$.

• This would imply its cup-length is $\leq n - 1 \Rightarrow$ contradiction!
Thanks for your attention!