Twisted Shklyarov pairings and applications

Shaoyun Bai
Simons Center for Geometry and Physics

January 24, 2023

(Based on joint work in progress with Paul Seidel)

Table of Contents

Fixed point Floer cohomology

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.
- $\forall x \in \mathrm{Fix}(\phi)$, $\det(D\phi_x - \mathrm{id}) \neq 0 \Rightarrow$ only finitely many fixed points.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.
- $\forall x \in \mathrm{Fix}(\phi)$, $\det(D\phi_x - \mathrm{id}) \neq 0 \Rightarrow$ only finitely many fixed points.
- $CF^*(\phi) := \bigoplus_{x \in \mathrm{Fix}(\phi)} \mathbb{C}x$, $\mathbb{Z}/2$-graded.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.
- $\forall x \in \mathrm{Fix}(\phi)$, $\det(D\phi_x - \mathrm{id}) \neq 0 \Rightarrow$ only finitely many fixed points.
- $CF^*(\phi) := \bigoplus_{x \in \mathrm{Fix}(\phi)} \mathbb{C}x$, $\mathbb{Z}/2$-graded.
- Choose $J_t, t \in \mathbb{R}$ compatible, $\phi^* J_{t+1} = J_t$.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.
- $\forall x \in \mathrm{Fix}(\phi)$, $\det(D\phi_x - \mathrm{id}) \neq 0 \Rightarrow$ only finitely many fixed points.
- $CF^*(\phi) := \bigoplus_{x \in \mathrm{Fix}(\phi)} \mathbb{C}x$, $\mathbb{Z}/2$-graded.
- Choose $J_t, t \in \mathbb{R}$ compatible, $\phi^* J_{t+1} = J_t$.
- Count $u : \mathbb{R}^2_{s,t} \to M$

$$\phi \circ u(t+1, s) = u(t, s), \quad \partial_s u + J_t(u)\partial_t u = 0$$

to define differential on $CF^*(\phi)$.

Fixed point Floer cohomology

- (M^{2n}, ω), closed, monotone: $[\omega] = c_1(M)$.
- $\phi : M \to M$ symplectic automorphism: $\phi \in \mathrm{Diff}(M)$ and $\phi^*\omega = \omega$.
- $\forall x \in \mathrm{Fix}(\phi)$, $\det(D\phi_x - \mathrm{id}) \neq 0 \Rightarrow$ only finitely many fixed points.
- $CF^*(\phi) := \bigoplus_{x \in \mathrm{Fix}(\phi)} \mathbb{C}x$, $\mathbb{Z}/2$-graded.
- Choose $J_t, t \in \mathbb{R}$ compatible, $\phi^* J_{t+1} = J_t$.
- Count $u : \mathbb{R}^2_{s,t} \to M$

$$\phi \circ u(t+1, s) = u(t, s), \quad \partial_s u + J_t(u)\partial_t u = 0$$

  to define differential on $CF^*(\phi)$.
- Output: *fixed point Floer cohomology* $HF^*(\phi)$.

Some algebraic structures

Some algebraic structures

- Poincaré type pairing

$$HF^*(\phi) \otimes HF^{2n-*}(\phi^{-1}) \to \mathbb{C},$$

nondegenerate, coincides with the Poincaré pairing on $QH^*(M) = H^*(M)$ for $\phi = \text{id}$.

Some algebraic structures

- Poincaré type pairing

$$HF^*(\phi) \otimes HF^{2n-*}(\phi^{-1}) \to \mathbb{C},$$

  nondegenerate, coincides with the Poincaré pairing on $QH^*(M) = H^*(M)$ for $\phi = \mathrm{id}$.

- Conjugation isomorphism: ψ symplectic automorphism, then

$$HF^*(\phi) \xrightarrow{\simeq} HF^*(\psi\phi\psi^{-1}).$$

Some algebraic structures

- Poincaré type pairing

$$HF^*(\phi) \otimes HF^{2n-*}(\phi^{-1}) \to \mathbb{C},$$

nondegenerate, coincides with the Poincaré pairing on $QH^*(M) = H^*(M)$ for $\phi = \mathrm{id}$.

- Conjugation isomorphism: ψ symplectic automorphism, then

$$HF^*(\phi) \xrightarrow{\sim} HF^*(\psi\phi\psi^{-1}).$$

- $\Rightarrow HF^*(\phi^r)$ admits a $\mathbb{Z}/r$-action induced by conjugation with ϕ.

Table of Contents

Twisted Hochschild homology

Twisted Hochschild homology

- $\mathcal{F}$: strictly proper A_∞ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.

Twisted Hochschild homology

- $\mathcal{F}$: strictly proper A_∞ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.
  $\Rightarrow \forall X, Y \in \mathrm{Ob}(\mathcal{F})$, $\mathcal{F}(X, Y)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $\mathbb{C}$.

Twisted Hochschild homology

- $\mathcal{F}$: strictly proper A_∞ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.
  $\Rightarrow \forall X, Y \in \mathrm{Ob}(\mathcal{F})$, $\mathcal{F}(X, Y)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $\mathbb{C}$.
- $\Phi : \mathcal{F} \to \mathcal{F}$ strict A_∞ automorphism.

Twisted Hochschild homology

- $\mathcal{F}$: strictly proper A_∞ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.
  $\Rightarrow \forall X, Y \in \mathrm{Ob}(\mathcal{F})$, $\mathcal{F}(X, Y)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $\mathbb{C}$.
- $\Phi : \mathcal{F} \to \mathcal{F}$ strict A_∞ automorphism.
  $\Rightarrow$ an A_∞ functor with $\Phi^k = 0, \forall k \geq 2$, Φ^1 is an isomorphism of graded vector spaces.

Twisted Hochschild homology

- $\mathcal{F}$: strictly proper A_∞ category over $\mathbb{C}$, $\mathbb{Z}/2$-graded.
$\Rightarrow \forall X, Y \in \mathrm{Ob}(\mathcal{F})$, $\mathcal{F}(X, Y)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $\mathbb{C}$.
- $\Phi : \mathcal{F} \to \mathcal{F}$ strict A_∞ automorphism.
$\Rightarrow$ an A_∞ functor with $\Phi^k = 0, \forall k \geq 2$, Φ^1 is an isomorphism of graded vector spaces.
- Example: $\mathcal{F} =$ monotone Fukaya category of (M, ω),
$\Phi =$ automorphism of $\mathcal{F}$ induced by ϕ, which can be made strict by formally introducing more objects in $\mathcal{F}$.

Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(\mathcal{F}, \Phi) := \bigoplus_{L_0,\dots,L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$.

Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(\mathcal{F}, \Phi) := \bigoplus_{L_0,\dots,L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$.
- Differential: bar differential, only using $\mu^k_{\mathcal{F}}$, with (co)homology denoted by $HH_*(\mathcal{F}, \Phi)$.

Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(\mathcal{F}, \Phi) := \bigoplus_{L_0,\dots,L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$.
- Differential: bar differential, only using $\mu^k_{\mathcal{F}}$, with (co)homology denoted by $HH_*(\mathcal{F}, \Phi)$.

Proposition

There exists a bilinear pairing

$$\langle -, - \rangle_\Phi : HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1}) \to \mathbb{C}.$$

If $\mathcal{F}$ is homologically smooth, then it is nondegenerate. For $r \geq 1$, $HH_*(\mathcal{F}, \Phi^r)$ admits a $\mathbb{Z}/r$-action which is generated by conjugation with Φ.

Twisted Hochschild homology

- Twisted Hochschild chain complex $CC_*(\mathcal{F}, \Phi) := \bigoplus_{L_0,\dots,L_k} \mathcal{F}(L_{k-1}, L_k)[1] \otimes \cdots \otimes \mathcal{F}(L_0, L_1)[1] \otimes \mathcal{F}(\Phi(L_k), L_0)$.
- Differential: bar differential, only using $\mu^k_{\mathcal{F}}$, with (co)homology denoted by $HH_*(\mathcal{F}, \Phi)$.

Proposition

There exists a bilinear pairing

$$\langle -, - \rangle_\Phi : HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1}) \to \mathbb{C}.$$

If $\mathcal{F}$ is homologically smooth, then it is nondegenerate. For $r \geq 1$, $HH_*(\mathcal{F}, \Phi^r)$ admits a $\mathbb{Z}/r$-action which is generated by conjugation with Φ.

- The pairing generalizes the pairing defined by Shklyarov for $\Phi = \mathrm{id}$.

Twisted Shklyarov pairing

Twisted Shklyarov pairing

- The construction of $\langle -, - \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1})$.

Twisted Shklyarov pairing

- The construction of $\langle -, - \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1})$.
- The proof of the nondegeneracy is a generalization of the snake relation in the presence of an automorphism.

Twisted Shklyarov pairing

- The construction of $\langle -, - \rangle_\Phi$ relies on using the diagonal bimodule to construct an element in $HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1})$.
- The proof of the nondegeneracy is a generalization of the snake relation in the presence of an automorphism.
- Here is an explicit formula:

$$\langle -, - \rangle_\Phi : CC_*(\mathcal{F}, \Phi) \otimes CC_{-*}(\mathcal{F}, \Phi^{-1}) \longrightarrow \mathbb{C},$$

$$\langle a_m \otimes \cdots \otimes a_1 \otimes \underline{a}_0, b_n \otimes \cdots \otimes b_1 \otimes \underline{b}_0 \rangle_\Phi$$

$$= \sum_{ijkl} \mathrm{Str}\big(y \mapsto \pm \mu_{\mathcal{F}}^{i-j-k+l+m+2}(a_i, \ldots, \underline{a}_0, \Phi a_m, \ldots, \Phi a_{k+1}, \mu_{\mathcal{F}}^{-i+j+k-l+n+2}(\Phi a_k, \ldots, \Phi a_{i+1}, \Phi y, \Phi b_j, \ldots, \Phi b_1, \Phi \underline{b}_0, b_n, \ldots, b_{l+1}), b_l, \ldots, b_{j+1})\big).$$

Twisted Shklyarov pairing

- For length 1 Hochschild chains, the pairing is reduced to

$$\mathcal{F}(\Phi(L_0), L_0) \otimes \mathcal{F}(\Phi^{-1}(L_1), L_1) \to \mathbb{C}$$
$$\underline{a}_0 \otimes \underline{b}_0 \mapsto \pm \mathrm{Str}(y \mapsto \mu^2(\underline{a}_0, \mu^2(\Phi y, \Phi \underline{b}_0))),$$

where the super-trace is taken over $\mathcal{F}(L_0, L_1)$.

Twisted Shklyarov pairing

- For length 1 Hochschild chains, the pairing is reduced to

$$\mathcal{F}(\Phi(L_0), L_0) \otimes \mathcal{F}(\Phi^{-1}(L_1), L_1) \to \mathbb{C}$$

$$\underline{a}_0 \otimes \underline{b}_0 \mapsto \pm \mathrm{Str}(y \mapsto \mu^2(\underline{a}_0, \mu^2(\Phi y, \Phi \underline{b}_0))),$$

where the super-trace is taken over $\mathcal{F}(L_0, L_1)$.

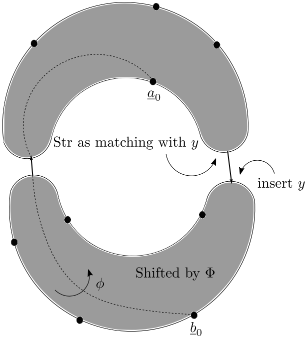


Table of Contents

Twisted open-closed string map

Twisted open-closed string map

Proposition

For $\mathcal{F} = \mathcal{F}(M, \omega)$, and $\phi : (M, \omega) \to (M, \omega)$, there exists a twisted open-closed string map $OC(\phi) : HH_*(\mathcal{F}, \Phi) \to HF^{*+n}(\phi)$ making the following diagram commute.

Twisted open-closed string map

Proposition

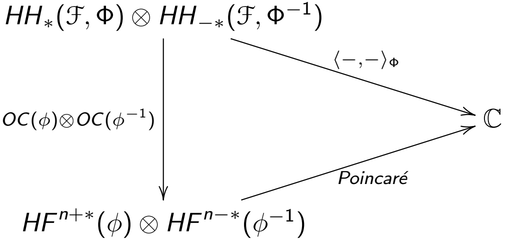
For $\mathcal{F} = \mathcal{F}(M, \omega)$, and $\phi : (M, \omega) \to (M, \omega)$, there exists a twisted open-closed string map $OC(\phi) : HH_*(\mathcal{F}, \Phi) \to HF^{*+n}(\phi)$ making the following diagram commute.

$$
\begin{array}{ccc}
HH_*(\mathcal{F}, \Phi) \otimes HH_{-*}(\mathcal{F}, \Phi^{-1}) & & \\
\Big\downarrow {\scriptstyle OC(\phi) \otimes OC(\phi^{-1})} & \searrow^{\langle -, - \rangle_\Phi} & \\
 & & \mathbb{C} \\
HF^{n+*}(\phi) \otimes HF^{n-*}(\phi^{-1}) & \nearrow_{\text{Poincaré}} &
\end{array}
$$

Twisted open-closed string map

Proposition

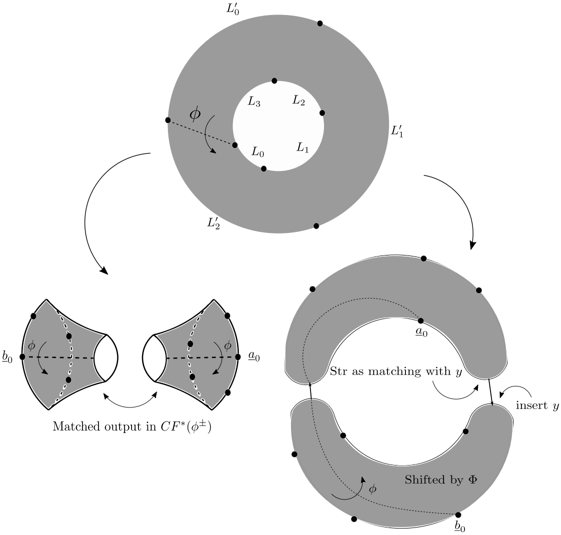
For $\mathcal{F} = \mathcal{F}(M, \omega)$, and $\phi : (M, \omega) \to (M, \omega)$, there exists a twisted open-closed string map $OC(\phi) : HH_*(\mathcal{F}, \Phi) \to HF^{*+n}(\phi)$ making the following diagram commute.



Corollary

If $\mathcal{F}$ is homologically smooth, $OC(\phi)$ is injective.

A Cardy relation



Some further remarks on $OC(\phi)$

Some further remarks on $OC(\phi)$

- $OC(\phi^r) : HH_*(\mathcal{F}, \Phi^r) \to HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.

Some further remarks on $OC(\phi)$

- $OC(\phi^r) : HH_*(\mathcal{F}, \Phi^r) \to HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.
- For (M, ω) monotone, $\mathcal{F}(M, \omega)$ is decomposed into smaller pieces according to the value of the disc potential. All the above constructions "respect" such a decomposition.

Some further remarks on $OC(\phi)$

- $OC(\phi^r) : HH_*(\mathcal{F}, \Phi^r) \to HF^{n+*}(\phi^r)$ is $\mathbb{Z}/r$-equivariant.
- For (M, ω) monotone, $\mathcal{F}(M, \omega)$ is decomposed into smaller pieces according to the value of the disc potential. All the above constructions "respect" such a decomposition.
- An interesting question: replace Φ by Lagrangian correspondences.

Table of Contents

Lefschetz fibrations

Lefschetz fibrations

- Let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, with fiber (F, ω, θ) a Liouville domain.

Lefschetz fibrations

- Let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, with fiber (F, ω, θ) a Liouville domain.
- $\mu : F \to F$ global monodromy: it's compactly supported, and $\exists G_\phi : F \to F$ such that $\phi^*\theta - \theta = dG_\phi$.

Lefschetz fibrations

- Let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, with fiber (F, ω, θ) a Liouville domain.
- $\mu : F \to F$ global monodromy: it's compactly supported, and $\exists G_\phi : F \to F$ such that $\phi^*\theta - \theta = dG_\phi$.
- The Hamiltonian

$$H^{\text{rot}} : (\mathbb{C}, \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}) \to \mathbb{R}$$
$$z \mapsto \pi|z|^2$$

defines a Hamiltonian $H^{\text{rot}} \circ \pi : E \to \mathbb{R}$.

Lefschetz fibrations

- Let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, with fiber (F, ω, θ) a Liouville domain.
- $\mu : F \to F$ global monodromy: it's compactly supported, and $\exists G_\phi : F \to F$ such that $\phi^*\theta - \theta = dG_\phi$.
- The Hamiltonian

$$H^{\mathrm{rot}} : (\mathbb{C}, \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}) \to \mathbb{R}$$
$$z \mapsto \pi|z|^2$$

  defines a Hamiltonian $H^{\mathrm{rot}} \circ \pi : E \to \mathbb{R}$.
- The associated Hamiltonian diffeomorphism is equal to identity viewed from $\mathbb{C}$, and restricts to μ on each fiber over the complement of a compact subset.

Hamiltonian Floer cohomology

Question

How to define the Hamiltonian Floer cohomology of $H^{rot} \circ \pi$*?*

Hamiltonian Floer cohomology

Question

How to define the Hamiltonian Floer cohomology of $H^{\mathrm{rot}} \circ \pi$?

- Traditionally: perturb $H^{\mathrm{rot}} \circ \pi$ using the lift of $\epsilon|z|^2$ for $0 < |\epsilon| < 1$.

Hamiltonian Floer cohomology

Question

How to define the Hamiltonian Floer cohomology of $H^{\mathrm{rot}} \circ \pi$?

- Traditionally: perturb $H^{\mathrm{rot}} \circ \pi$ using the lift of $\epsilon |z|^2$ for $0 < |\epsilon| < 1$.
- Alternatively: use the lift of $z \mapsto A \cdot \mathrm{Re}(z)$ for some $A \neq 0$.

Hamiltonian Floer cohomology

Question

How to define the Hamiltonian Floer cohomology of $H^{\text{rot}} \circ \pi$?

- Traditionally: perturb $H^{\text{rot}} \circ \pi$ using the lift of $\epsilon|z|^2$ for $0 < |\epsilon| < 1$.
- Alternatively: use the lift of $z \mapsto A \cdot \text{Re}(z)$ for some $A \neq 0$.
- Define Hamiltonian Floer cohomology:

$$HF^*(E, 1) := HF^*(H^{\text{rot}} \circ \pi \text{ perturbed by } A \cdot \text{Re}(z))$$

$$HF^*(E, 1 \pm \frac{1}{2}) := HF^*(H^{\text{rot}} \circ \pi \text{ perturbed by } \pm \frac{1}{2} \cdot |z|^2).$$

Hamiltonian Floer cohomology

Question

How to define the Hamiltonian Floer cohomology of $H^{\mathrm{rot}} \circ \pi$?

- Traditionally: perturb $H^{\mathrm{rot}} \circ \pi$ using the lift of $\epsilon|z|^2$ for $0 < |\epsilon| < 1$.
- Alternatively: use the lift of $z \mapsto A \cdot \mathrm{Re}(z)$ for some $A \neq 0$.
- Define Hamiltonian Floer cohomology:

$$HF^*(E,1) := HF^*(H^{\mathrm{rot}} \circ \pi \text{ perturbed by } A \cdot \mathrm{Re}(z))$$
$$HF^*(E, 1 \pm \frac{1}{2}) := HF^*(H^{\mathrm{rot}} \circ \pi \text{ perturbed by } \pm \frac{1}{2} \cdot |z|^2).$$

Proposition

There exist long exact sequences

$$\cdots \to HF^*(E, \frac{1}{2}) \to HF^*(E,1) \to HF^{*-1}(\mu) \to \cdots$$
$$\cdots \to HF^*(E,1) \to HF^*(E, 1 + \frac{1}{2}) \to HF^*(\mu) \to \cdots .$$

Iterating the monodromy

Iterating the monodromy

- More generally, we can consider $rH^{\mathrm{rot}} \circ \pi$ for $r \in \mathbb{Z}$, and Floer cohomology groups $HF^*(E, r)$ and $HF^*(E, r \pm \frac{1}{2})$.

Iterating the monodromy

- More generally, we can consider $rH^{\text{rot}} \circ \pi$ for $r \in \mathbb{Z}$, and Floer cohomology groups $HF^*(E, r)$ and $HF^*(E, r \pm \frac{1}{2})$.
- They fit into long exact sequences

$$\cdots \to HF^*(E, r - \frac{1}{2}) \to HF^*(E, r) \to HF^{*-1}(\mu^r) \to \cdots$$

$$\cdots \to HF^*(E, r) \to HF^*(E, r + \frac{1}{2}) \to HF^*(\mu^r) \to \cdots .$$

Iterating the monodromy

- More generally, we can consider $rH^{\mathrm{rot}} \circ \pi$ for $r \in \mathbb{Z}$, and Floer cohomology groups $HF^*(E, r)$ and $HF^*(E, r \pm \frac{1}{2})$.
- They fit into long exact sequences

$$\cdots \to HF^*(E, r - \frac{1}{2}) \to HF^*(E, r) \to HF^{*-1}(\mu^r) \to \cdots$$

$$\cdots \to HF^*(E, r) \to HF^*(E, r + \frac{1}{2}) \to HF^*(\mu^r) \to \cdots .$$

Proposition

There exists a nondegenerate pairing

$$\langle -, - \rangle_r : HF^*(E, r) \otimes HF^{2n-*}(E, -r) \to \mathbb{C}.$$

Iterating the monodromy

- More generally, we can consider $rH^{\mathrm{rot}} \circ \pi$ for $r \in \mathbb{Z}$, and Floer cohomology groups $HF^*(E, r)$ and $HF^*(E, r \pm \frac{1}{2})$.
- They fit into long exact sequences

$$\cdots \to HF^*(E, r - \frac{1}{2}) \to HF^*(E, r) \to HF^{*-1}(\mu^r) \to \cdots$$

$$\cdots \to HF^*(E, r) \to HF^*(E, r + \frac{1}{2}) \to HF^*(\mu^r) \to \cdots .$$

Proposition

There exists a nondegenerate pairing

$$\langle -, - \rangle_r : HF^*(E, r) \otimes HF^{2n-*}(E, -r) \to \mathbb{C}.$$

For each $r \in \mathbb{Z}_{\geq 1}$, $HF^*(E, r)$ admits a $\mathbb{Z}/r$-action, so that $HF^*(E, r) \to HF^{*-1}(\mu^r)$ is $\mathbb{Z}/r$-equivariant.

Fukaya–Seidel category and Serre functor

Fukaya–Seidel category and Serre functor

- Let $\mathcal{A}$ be the Fukaya–Seidel category (over $\mathbb{C}$) of $\pi : E \to \mathbb{C}$ (with objects *all* Lefschetz thimbles).

Fukaya–Seidel category and Serre functor

- Let $\mathcal{A}$ be the Fukaya–Seidel category (over $\mathbb{C}$) of $\pi : E \to \mathbb{C}$ (with objects *all* Lefschetz thimbles).

Proposition (folklore, attributed to Kontsevich–Seidel)

The Hamiltonian diffeomorphism defined by $H^{rot} \circ \pi$ (perturbed by $A \cdot Re(z)$) induces the inverse Serre functor $\mathcal{S}^{-1}$ on $\mathcal{A}$.

Fukaya–Seidel category and Serre functor

- Let $\mathcal{A}$ be the Fukaya–Seidel category (over $\mathbb{C}$) of $\pi : E \to \mathbb{C}$ (with objects *all* Lefschetz thimbles).

Proposition (folklore, attributed to Kontsevich–Seidel)

The Hamiltonian diffeomorphism defined by $H^{rot} \circ \pi$ *(perturbed by* $A \cdot Re(z)$*) induces the inverse Serre functor* $\mathcal{S}^{-1}$ *on* $\mathcal{A}$.

- Recall: $\mathcal{A}(\mathcal{S}X, Y) \cong \mathcal{A}(Y, X)^\vee$, $\mathcal{S}$ represents the *linear dual bimodule* $\mathcal{A}^\vee$.

Fukaya–Seidel category and Serre functor

- Let $\mathcal{A}$ be the Fukaya–Seidel category (over $\mathbb{C}$) of $\pi : E \to \mathbb{C}$ (with objects *all* Lefschetz thimbles).

Proposition (folklore, attributed to Kontsevich–Seidel)

The Hamiltonian diffeomorphism defined by $H^{rot} \circ \pi$ (perturbed by $A \cdot Re(z)$) induces the inverse Serre functor $\mathcal{S}^{-1}$ on $\mathcal{A}$.

- Recall: $\mathcal{A}(\mathcal{S}X, Y) \cong \mathcal{A}(Y, X)^\vee$, $\mathcal{S}$ represents the *linear dual bimodule* $\mathcal{A}^\vee$.
- Here $\mathcal{A}^\vee(X, Y) = \mathrm{Hom}(\mathcal{A}(X, Y), \mathbb{C})$, and

$$\langle \mu_{\mathcal{A}^\vee}(a_s, \ldots, a_1, \pi, b_r, \ldots, b_1), p \rangle = \pm \langle \pi, \mu^{r+1+s}(b_r, \ldots, b_1, p, a_s, \ldots, a_1) \rangle.$$

Twisted open-closed maps for Lefschetz fibrations

Twisted open-closed maps for Lefschetz fibrations

Theorem (B–Seidel)

For each $r \in \mathbb{Z}$, there exists a $\mathbb{Z}/r$-equivariant twisted open-closed string map

$$OC(r) : HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \to HF^*(E, -r)[n(1+r)]$$

such that the following diagram commutes

Twisted open-closed maps for Lefschetz fibrations

Theorem (B–Seidel)

For each $r \in \mathbb{Z}$, there exists a $\mathbb{Z}/r$-equivariant twisted open-closed string map

$$OC(r) : HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \to HF^*(E, -r)[n(1+r)]$$

such that the following diagram commutes

$$\begin{array}{ccc}
HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \otimes HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes -r}) & \xrightarrow{\text{Shklyarov pairing}} & \mathbb{C}. \\
\downarrow{\scriptstyle OC(r)\otimes OC(-r)} & \nearrow{\scriptstyle \langle -,- \rangle_r} & \\
HF^*(E, -r)[n(1+r)] \otimes HF^*(E, r)[n(1-r)] & &
\end{array}$$

Twisted open-closed maps for Lefschetz fibrations

- The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with J-holomorphic curves (cf. Seidel's Lefschetz IV and IV 1/2).

Twisted open-closed maps for Lefschetz fibrations

- The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with J-holomorphic curves (cf. Seidel's Lefschetz IV and IV 1/2).

Proposition

$\mathcal{A}$ is homologically smooth.

Twisted open-closed maps for Lefschetz fibrations

- The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with J-holomorphic curves (cf. Seidel's Lefschetz IV and IV 1/2).

Proposition

$\mathcal{A}$ *is homologically smooth.*

- A distinguished basis defines a directed category, which has "automatic" smoothness. The point is that *all* Lefschetz thimbles are generated by the ones from a distinguished basis.

Twisted open-closed maps for Lefschetz fibrations

- The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with J-holomorphic curves (cf. Seidel's Lefschetz IV and IV 1/2).

Proposition

$\mathcal{A}$ is homologically smooth.

- A distinguished basis defines a directed category, which has "automatic" smoothness. The point is that *all* Lefschetz thimbles are generated by the ones from a distinguished basis.

Corollary

$\forall r \in \mathbb{Z}$, $OC(r) : HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \to HF^*(E, -r)[n(1+r)]$ is injective.

Twisted open-closed maps for Lefschetz fibrations

- The proof conceptually follows from the argument in the monotone case, but requires a different technical framework to deal with J-holomorphic curves (cf. Seidel's Lefschetz IV and IV 1/2).

Proposition

$\mathcal{A}$ *is homologically smooth.*

- A distinguished basis defines a directed category, which has "automatic" smoothness. The point is that *all* Lefschetz thimbles are generated by the ones from a distinguished basis.

Corollary

$\forall r \in \mathbb{Z}$, $OC(r) : HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes r}) \to HF^*(E, -r)[n(1+r)]$ *is injective.*

- A conjecture of Seidel expects it to be an isomorphism.

Table of Contents

Noncommutative anti-canonical divisor

Noncommutative anti-canonical divisor

Question

How to construct an A_∞ category $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory, and $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1-n]$ as A_∞ bimodules?

Noncommutative anti-canonical divisor

Question

How to construct an A_∞ category $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory, and $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1-n]$ as A_∞ bimodules?

- At the level of morphisms, $\mathcal{B}(X, Y) = \mathcal{A}(X, Y) \oplus \mathcal{A}^\vee(X, Y)[1-n]$. The "0-th" order information is contained in $\mu^k_{\mathcal{A}}$ and $\mu^{r|1|s}_{\mathcal{A}^\vee}$.

Noncommutative anti-canonical divisor

Question

How to construct an A_∞ category $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory, and $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1-n]$ as A_∞ bimodules?

- At the level of morphisms, $\mathcal{B}(X, Y) = \mathcal{A}(X, Y) \oplus \mathcal{A}^\vee(X, Y)[1-n]$. The "0-th" order information is contained in $\mu^k_{\mathcal{A}}$ and $\mu^{r|1|s}_{\mathcal{A}^\vee}$.
- The "first" order information is a bimodule homomorphism

$$\theta : T(\mathcal{A}[1]) \otimes \mathcal{A}^\vee[-n] \otimes T(\mathcal{A}[1]) \to \mathcal{A},$$

which defines a class in $H^0(\hom(\mathcal{A}^\vee[-n], \mathcal{A})) \cong HH_*(\mathcal{A}, \mathcal{S}^{-1})$.

Noncommutative anti-canonical divisor

Question

How to construct an A_∞ category $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory, and $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1-n]$ as A_∞ bimodules?

- At the level of morphisms, $\mathcal{B}(X,Y) = \mathcal{A}(X,Y) \oplus \mathcal{A}^\vee(X,Y)[1-n]$. The "0-th" order information is contained in $\mu^k_{\mathcal{A}}$ and $\mu^{r|1|s}_{\mathcal{A}^\vee}$.
- The "first" order information is a bimodule homomorphism

$$\theta : T(\mathcal{A}[1]) \otimes \mathcal{A}^\vee[-n] \otimes T(\mathcal{A}[1]) \to \mathcal{A},$$

which defines a class in $H^0(\hom(\mathcal{A}^\vee[-n], \mathcal{A})) \cong HH_*(\mathcal{A}, \mathcal{S}^{-1})$.
- The "higher" information is encoded in

$$H^*(\hom((\mathcal{A}^\vee)^{\otimes r}, \mathcal{A}))^{\mathbb{Z}/(r+1)} \cong HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}.$$

Noncommutative anti-canonical divisor

Proposition

If $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}$ is supported on non-negative degrees, an A_∞ structure on $\mathcal{A} \oplus \mathcal{A}^\vee[1-n]$ is uniquely determined by the first-order information $\theta \in HH_*(\mathcal{A}, \mathcal{S}^{-1})$.

Noncommutative anti-canonical divisor

Proposition

If $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}$ is supported on non-negative degrees, an A_∞ structure on $\mathcal{A} \oplus \mathcal{A}^\vee[1-n]$ is uniquely determined by the first-order information $\theta \in HH_*(\mathcal{A}, \mathcal{S}^{-1})$.

- The element θ can be viewed as a "section of the anti-canonical bundle."

Noncommutative anti-canonical divisor

Proposition

If $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}$ *is supported on non-negative degrees, an* A_∞ *structure on* $\mathcal{A} \oplus \mathcal{A}^\vee[1-n]$ *is uniquely determined by the first-order information* $\theta \in HH_*(\mathcal{A}, \mathcal{S}^{-1})$.

- The element θ can be viewed as a "section of the anti-canonical bundle."

Lemma

If $\pi : E \to \mathbb{C}$ *is constructed by removing the fiber over* ∞ *of an anti-canonical Lefschetz pencil, then* $HF^*(E, r)$ *is supported on non-negative degrees.*

Noncommutative anti-canonical divisor

Proposition

If $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}$ is supported on non-negative degrees, an A_∞ structure on $\mathcal{A} \oplus \mathcal{A}^\vee[1-n]$ is uniquely determined by the first-order information $\theta \in HH_*(\mathcal{A}, \mathcal{S}^{-1})$.

- The element θ can be viewed as a "section of the anti-canonical bundle."

Lemma

If $\pi : E \to \mathbb{C}$ is constructed by removing the fiber over ∞ of an anti-canonical Lefschetz pencil, then $HF^*(E, r)$ is supported on non-negative degrees.

- Because $OC(r)$ is injective, we conclude that $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})$ is supported on non-negative degrees. The $\mathbb{Z}/r$-equivariance of $OC(r)$ implies that $HH_*(\mathcal{A}, \mathcal{S}^{-(r+1)})^{\mathbb{Z}/(r+1)}$ satisfies the same property.

Mirror symmetry implications

Mirror symmetry implications

- For $\pi : E \to \mathbb{C}$ anti-canonical, Fukaya category of the fiber $\mathcal{B}$ and the restriction functor $\mathcal{A} \to \mathcal{B}$ define a noncommutative anti-canonical divisor (cf. Seidel's Lefschetz VI).

Mirror symmetry implications

- For $\pi : E \to \mathbb{C}$ anti-canonical, Fukaya category of the fiber $\mathcal{B}$ and the restriction functor $\mathcal{A} \to \mathcal{B}$ define a noncommutative anti-canonical divisor (cf. Seidel's Lefschetz VI).
- The above discussion shows that to reconstruct $\mathcal{B}$ from $\mathcal{A}$, we just need a natural transformation

$$\mathcal{S} \to \mathrm{id},$$

which is in fact realized as the identity element in $HF^*(E, 1)$ under the open-closed map.

Mirror symmetry implications

- For $\pi : E \to \mathbb{C}$ anti-canonical, Fukaya category of the fiber $\mathcal{B}$ and the restriction functor $\mathcal{A} \to \mathcal{B}$ define a noncommutative anti-canonical divisor (cf. Seidel's Lefschetz VI).
- The above discussion shows that to reconstruct $\mathcal{B}$ from $\mathcal{A}$, we just need a natural transformation

$$\mathcal{S} \to \mathrm{id},$$

  which is in fact realized as the identity element in $HF^*(E, 1)$ under the open-closed map.
- The same actually holds for the deformation of the pair $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$ induced by adding back the base locus.

Mirror symmetry implications

- For $\pi : E \to \mathbb{C}$ anti-canonical, Fukaya category of the fiber $\mathcal{B}$ and the restriction functor $\mathcal{A} \to \mathcal{B}$ define a noncommutative anti-canonical divisor (cf. Seidel's Lefschetz VI).
- The above discussion shows that to reconstruct $\mathcal{B}$ from $\mathcal{A}$, we just need a natural transformation

$$\mathcal{S} \to \mathrm{id},$$

  which is in fact realized as the identity element in $HF^*(E, 1)$ under the open-closed map.
- The same actually holds for the deformation of the pair $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$ induced by adding back the base locus.
- This is a step towards showing that the compact Fukaya category of the Calabi–Yau hypersurface $\overline{\mathcal{B}}$ is actually defined over a polynomial ring after applying a mirror map.

Table of Contents

Isolated singularities

Isolated singularities

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that 0 is an isolated singularity.

Isolated singularities

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that 0 is an isolated singularity.
- The monodromy surrounding 0, $\mu : F \to F$, can be modified to be an exact symplectic automorphism of the Milnor fiber.

Isolated singularities

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that 0 is an isolated singularity.
- The monodromy surrounding 0, $\mu : F \to F$, can be modified to be an exact symplectic automorphism of the Milnor fiber.

Theorem (McLean)

Let m be the multiplicity of f and 0. Then for any $r < m$, the fixed point Floer cohomology $HF^*(\mu^r) = 0$.

Isolated singularities

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that 0 is an isolated singularity.
- The monodromy surrounding 0, $\mu : F \to F$, can be modified to be an exact symplectic automorphism of the Milnor fiber.

Theorem (McLean)

Let m be the multiplicity of f and 0. Then for any $r < m$, the fixed point Floer cohomology $HF^*(\mu^r) = 0$.

- Using the long exact sequence

$$\cdots \to HF^*(E, \frac{1}{2}) \cong HF^*(\mathbb{C}^n) \to HF^*(E, 1) \to HF^{*-1}(\mu) \to \cdots ,$$

we know $\mathrm{rank} HF^*(E, 1) \leq 1$.

Isolated singularities

- Let $f : \mathbb{C}^n \to \mathbb{C}$ be a (germ of) holomorphic function defined near $0 \in \mathbb{C}^n$, such that 0 is an isolated singularity.
- The monodromy surrounding 0, $\mu : F \to F$, can be modified to be an exact symplectic automorphism of the Milnor fiber.

Theorem (McLean)

Let m be the multiplicity of f and 0. Then for any $r < m$, the fixed point Floer cohomology $HF^*(\mu^r) = 0$.

- Using the long exact sequence

$$\cdots \to HF^*(E, \frac{1}{2}) \cong HF^*(\mathbb{C}^n) \to HF^*(E, 1) \to HF^{*-1}(\mu) \to \cdots,$$

  we know $\mathrm{rank} HF^*(E, 1) \leq 1$.
- After Morsifying f, the associated Fukaya–Seidel category is nontrivial $\Rightarrow HH^*(\mathcal{A}, \mathcal{A}) \cong HH_{*+n}(\mathcal{A}, \mathcal{S}^{-1}) \neq 0$.

Collapsing critical values

Collapsing critical values

- Using the injectivity of $OC(-1)$, we see that $HF^*(E, 1)$ is exactly 1-dimensional.

Collapsing critical values

- Using the injectivity of $OC(-1)$, we see that $HF^*(E, 1)$ is exactly 1-dimensional.
- This simple computation already gives some interesting applications.

Collapsing critical values

- Using the injectivity of $OC(-1)$, we see that $HF^*(E,1)$ is exactly 1-dimensional.
- This simple computation already gives some interesting applications.

Example

The holomorphic map

$$(\mathbb{C}^*)^n \to \mathbb{C}$$
$$(z_1, \ldots, z_n) \mapsto z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}$$

cannot be deformed to a regular function with isolated singularities but with fewer critical values.

Collapsing critical values

- Recall that $HH^*(\mathcal{A}, \mathcal{A})$ has a ring structure.

Collapsing critical values

- Recall that $HH^*(\mathcal{A}, \mathcal{A})$ has a ring structure.
- We can define the cup-length of $HH^*(\mathcal{A}, \mathcal{A})$ (denoted by $\mathrm{cl}(\mathcal{A})$) to be the maximal $r \in \mathbb{Z}_{\geq 0}$ such that $\exists a_1, \ldots, a_r \in HH^*(\mathcal{A}, \mathcal{A})$ nilpotent and
$$a_1 \cup \cdots \cup a_r \neq 0.$$

Collapsing critical values

- Recall that $HH^*(\mathcal{A}, \mathcal{A})$ has a ring structure.
- We can define the cup-length of $HH^*(\mathcal{A}, \mathcal{A})$ (denoted by $\mathrm{cl}(\mathcal{A})$) to be the maximal $r \in \mathbb{Z}_{\geq 0}$ such that $\exists a_1, \ldots, a_r \in HH^*(\mathcal{A}, \mathcal{A})$ nilpotent and
$$a_1 \cup \cdots \cup a_r \neq 0.$$
- Homological mirror symmetry tells us
$$\mathcal{F}\big((\mathbb{C}^*)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\big) \cong D^b\mathrm{Coh}(\mathbb{CP}^n).$$

Collapsing critical values

- Recall that $HH^*(\mathcal{A}, \mathcal{A})$ has a ring structure.
- We can define the cup-length of $HH^*(\mathcal{A}, \mathcal{A})$ (denoted by $\mathrm{cl}(\mathcal{A})$) to be the maximal $r \in \mathbb{Z}_{\geq 0}$ such that $\exists a_1, \ldots, a_r \in HH^*(\mathcal{A}, \mathcal{A})$ nilpotent and

$$a_1 \cup \cdots \cup a_r \neq 0.$$

- Homological mirror symmetry tells us

$$\mathcal{F}\big((\mathbb{C}^*)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\big) \cong D^b\mathrm{Coh}(\mathbb{CP}^n).$$

- For $\mathcal{A} = D^b\mathrm{Coh}(\mathbb{CP}^n)$, we have $\mathrm{cl}(\mathcal{A}) \geq n$ by looking at n linearly independent holomorphic vector field on $\mathbb{CP}^n$ generated by the torus action.

Collapsing critical values

- If $\mathcal{A} = \mathcal{F}(\pi)$ for $\pi : E \to \mathbb{C}$ being a Morsification of an isolated singularity, we know $\mathrm{cl}(\mathcal{A}) = 0$.

Collapsing critical values

- If $\mathcal{A} = \mathcal{F}(\pi)$ for $\pi : E \to \mathbb{C}$ being a Morsification of an isolated singularity, we know $\mathrm{cl}(\mathcal{A}) = 0$.

Lemma

Suppose $\mathcal{A}$ admits a semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle,$$

then $cl(\mathcal{A}) \leq \sum cl(\mathcal{A}_i) + m - 1$.

Collapsing critical values

- If $\mathcal{A} = \mathcal{F}(\pi)$ for $\pi : E \to \mathbb{C}$ being a Morsification of an isolated singularity, we know $\mathrm{cl}(\mathcal{A}) = 0$.

Lemma

Suppose $\mathcal{A}$ admits a semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle,$$

then $cl(\mathcal{A}) \leq \sum cl(\mathcal{A}_i) + m - 1$.

- If we could collapse the critical values of $z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}$, we would obtain a semi-orthogonal decomposition of

$$\mathcal{F}\left((\mathbb{C}^*)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\right) \cong D^b\mathrm{Coh}(\mathbb{CP}^n)$$

of length $\leq n - 1$.

Collapsing critical values

- If $\mathcal{A} = \mathcal{F}(\pi)$ for $\pi : E \to \mathbb{C}$ being a Morsification of an isolated singularity, we know $\mathrm{cl}(\mathcal{A}) = 0$.

Lemma

Suppose $\mathcal{A}$ admits a semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle,$$

then $cl(\mathcal{A}) \leq \sum cl(\mathcal{A}_i) + m - 1$.

- If we could collapse the critical values of $z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}$, we would obtain a semi-orthogonal decomposition of

$$\mathcal{F}\big((\mathbb{C}^*)^n, z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\big) \cong D^b\mathrm{Coh}(\mathbb{CP}^n)$$

  of length $\leq n - 1$.
- This would imply its cup-length is $\leq n - 1 \Rightarrow$ contradiction!

Thanks for your attention!