Quivers and Curves in Higher Dimension

Hülya Argüz

University of Georgia

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Plan of the talk

- **Algebra**: Quiver Donaldson–Thomas (DT) Invariants
  - The attractor flow tree formula (calculating quiver DT invariants via tropical geometry)
- **Geometry**: Counts of log curves in toric varieties
  - From quivers to toric varieties
  - Log Gromov–Witten (GW) invariants of toric varieties
  - Calculating log GW invariants tropically

- Quiver DT invariants $\leftrightarrow$ log GW invariants of toric varieties
- Quiver DT invariants $\leftrightarrow$ log GW invariants of cluster varieties
A **quiver** is a finite oriented graph $Q = (Q_0, Q_1, s, t)$.

- $Q_0$: set of vertices.
- $Q_1$: set of arrows.
- $s: Q_1 \to Q_0$ maps an arrow to its *source*.
- $t: Q_1 \to Q_0$ maps an arrow to its *target*.

$Q_0 = \{1, 2, 3\}$
A *representation of a quiver* is an assignement of
- a vector space $V_v$, for each vertex $v \in Q_0$, and
- a linear transformation $f_{ij} \in \text{Hom}_\mathbb{C}(V_{s(e)}, V_{t(e)})$ for each edge $e \in Q_1$.

*Dimension* of a quiver representation is a vector

$$\gamma = (\gamma_i)_{i \in Q_0} \in N^+,$$

where $N := \mathbb{Z}^{Q_0}$ and $N^+ = \mathbb{N}^{Q_0} \setminus \{0\}$, encoding dimensions of the vector spaces assigned to vertices.
There is a natural notion of morphisms/isomorphisms between two quiver representations \((f_{ij})\) and \((g_{ij})\):

- automorphisms \(h_i: \mathbb{C}^{\gamma_i} \rightarrow \mathbb{C}^{\gamma_i}\) such that \(g_{ij} = f_{ij} \circ h_i\).

**Definition (King’s notion of stability)**

- \(V\): quiver representation of dimension \(\gamma \in \mathbb{N}^+\).
- \(M := \text{Hom}(N, \mathbb{Z})\) and \(M_\mathbb{R} = \text{Hom}(N, \mathbb{R}) = M \otimes \mathbb{R}\)
- \(\theta \in \gamma^\perp := \{\theta \in M_\mathbb{R}, \theta(\gamma) = 0\} \subset M_\mathbb{R}\): stability parameter.
  - \(V\): \(\theta\)-stable if \(\forall \{0\} \subsetneq V' \subsetneq V\) we have \(\theta(\dim(V')) < 0\).
  - \(V\): \(\theta\)-semi-stable if \(\forall V' \subsetneq V\) we have \(\theta(\dim(V')) \leq 0\).

- \(\mathcal{M}_\theta^{\gamma}\): Moduli space of \(\theta\) semi-stable quiver representations of \(Q\) dimension \(\gamma\).
"In nice cases" (when $M_\gamma^\theta$: smooth) we define quiver DT invariants as the topological Euler characteristics:

$$DT_\gamma^\theta := e(M_\gamma^\theta) = \sum_k (-1)^k \dim H^k(M_\gamma^\theta, \mathbb{C}).$$

Piecewise constant dependence on $\theta \in \gamma^\perp$: wall-crossing, universal wall-crossing formula (Kontsevich-Soibelman).
Example

- $Q$: $n$-Kronecker quiver
- $V$: representation with $\gamma := \dim(V) = (1, 1) \in \mathbb{N}$
- $\theta = (\theta_1, -\theta_1) \in \gamma^\perp \subset M_{\mathbb{R}}$

\begin{align*}
\xi_1 &\quad \xi_2 &\quad \xi_n \\
\quad &\quad &\quad V
\end{align*}

- $\theta_1 > 0$ and $(\xi_1, \ldots, \xi_n) \neq 0 \implies V$ is $\theta$ semi-stable, $M_\gamma^\theta \cong \mathbb{C}P^{n-1}$
- $\theta_1 < 0 \implies M_\gamma^\theta = \emptyset$. 
Quivers with potentials

- **Path algebra $\mathbb{C}Q$:** $\mathbb{C}$-linear combinations of paths in $Q$ with concatenation product.

  \[
  \mathbb{C} Q = \mathbb{C} v \oplus \mathbb{C} e \oplus \mathbb{C} w
  \]

  \[
  v^2 = v, \quad w^2 = w \quad ev = we = e
  \]

- **Potential $W \in \mathbb{C}Q$:** Formal linear combination of oriented cycles.

  - **Acyclic Quiver**
    
    \[
    W = 0
    \]

  - **Not allowed!**
    
    \[
    W = 2abc + 5(abc)^2
    \]

- **We assume quivers do not have oriented two-cycles.**
For \((Q, W = \sum \lambda_c c)\) define the **trace function**

\[
\text{Tr}(c)_{\gamma}^\theta : M_{\gamma}^\theta \to \mathbb{C}
\]

\[
V = ((V_i)_{i \in Q_0}, (f_{\alpha})_{\alpha \in Q_1}) \mapsto \text{Tr}(f_{\alpha_n} \circ \ldots \circ f_{\alpha_1})
\]

\[
\text{Tr}(W)_{\gamma}^\theta = \sum_c \lambda_c \text{Tr}(c)_{\gamma}^\theta
\]

- \(C_{\gamma}^\theta\): Critical locus of \(\text{Tr}(W)_{\gamma}^\theta \subset M_{\gamma}^\theta\).
- “In nice cases” (\(M_{\gamma}^\theta\) smooth and \(\text{Tr}(W)_{\gamma}^\theta\) Morse-Bott)

\[
\Omega_{\gamma}^\theta := e(C_{\gamma}^\theta) = \sum_k (-1)^k \dim H^k(C_{\gamma}^\theta, \mathbb{C})
\]
The (general) definition of DT invariants

**Definition**

For \((Q, W)\): quiver with potential, \(\gamma \in N^+\), and \(\theta \in \gamma^\perp \subset M_{\mathbb{R}}\), the Donaldson–Thomas (DT) invariant \(\Omega_\gamma^\theta \in \mathbb{Z}\) for \(((Q, W), \gamma, \theta)\) is defined by

\[
\Omega_\gamma^\theta = e(C_\gamma^\theta, \phi_{\text{Tr}(W)_\gamma^\theta} IC_{M_\gamma^\theta})
\]

- \(IC_{M_\gamma^\theta}\): intersection cohomology sheaf on \(M_\gamma^\theta\)
  - \(IC_{M_\gamma^\theta}\) is a perverse sheaf (\(M_\gamma^\theta\) smooth \(\Rightarrow IC_{M_\gamma^\theta}\) is the constant sheaf with stalk \(\mathbb{Q}\))
- \(\phi_{\text{Tr}(W)_\gamma^\theta}: \text{vanishing cycle functor}\) for the function \(\text{Tr}(W)_\gamma^\theta\)
  - \(\phi_{\text{Tr}(W)_\gamma^\theta} IC_{M_\gamma^\theta}\): sheaf on the critical locus \(C_\gamma^\theta \subset M_\gamma^\theta\)
- We will work with rational DT invariants given by

\[
\overline{\Omega}_\gamma^\theta := \sum_{\gamma = k\gamma', k \in \mathbb{Z}_{\geq 1}, \gamma' \in N^+} \frac{(-1)^{k-1}}{k^2} \Omega_{\gamma'}^\theta
\]

- See Kontsevich–Soibelman, Joyce–Song, Reineke, Davison–Meinhardt
Ex: $\Omega^\theta_\gamma$ can generally be very complicated

- The 3-Kronecker quiver appears in $\mathcal{N} = 2, 4d$ SU(3) super Yang-Mills theory\(^1\)

**Figure:** Values of $\Omega^\theta_\gamma$ for the 3-Kronecker quiver

Why are refined DT invariants of quivers interesting?

- Coherent sheaves in CY3’s
- Special Lagrangians in CY3’s

Geometric DT theory

DT invariants of quiver representations

Supersymmetric quantum mechanics

- Supersymmetric ground states
- BPS particles/black holes
DT invariants from a “simple” set of invariants

Is there a primitive set of DT invariants from which we could determine all DT invariants?

- Yes! We calculate quiver DT invariants using wall structures and flow trees, from simpler (attractor) DT invariants.

Let \( \{s_1, \ldots, s_{|Q_0|}\} \) be a basis for \( N \). Define a skew symmetric form \( \langle -, - \rangle \) on \( N \) by

\[
\langle s_i, s_j \rangle := a_{ij} - a_{ji}.
\]

where \( a_{ij} \) is the number of arrows from \( i \) to \( j \).

Fix \( \gamma \in N \). The chamber containing \( \langle \gamma, - \rangle \in \gamma^\perp \in M_\mathbb{R} \) is an attractor chamber for \( \gamma \) (generally not \( \gamma \)-generic).

**Definition (Alexandrov–Pioline, Mozgovoy–Pioline, Kontsevich–Soibelman)**

Let \( \theta \in \gamma^\perp \subset M_\mathbb{R} \) be a small perturbation of \( \langle \gamma, - \rangle \) which is \( \gamma \)-generic. Define the **attractor DT invariants** by \( \Omega^\star_{\gamma} := \Omega_{\gamma}^\theta \).

\( \Omega^\star_{\gamma} \) do not depend on the stability parameter \( \theta \), and are generally much simpler to compute.
The attractor DT invariants

**Theorem (Bridgeland)**


If $Q$ is acyclic then

$$\Omega^*_\gamma = \begin{cases} 
1 & \text{if } \gamma = (0, \ldots, 0, 1, 0, \ldots, 0) \\
0 & \text{otherwise}
\end{cases}$$

**Conjecture (Beaujard–Manschot–Pioline, Mozgovoy–Pioline)**

Proven for the local P2 by Bousseau–Descombes–Le Floch–Pioline, arxiv:2210.10712

- $K_S$: local del Pezzo (canonical bundle over a del Pezzo surface $S$)
- $Q$: quiver, with the additional data of a potential functions $W$ s.t. $D^b \text{Rep}(Q, W) \cong D^b \text{Coh}(K_S)$, then $\Omega^*_\gamma = 0$, unless either $\gamma = (0, \ldots, 0, 1, 0, \ldots, 0) \implies \Omega^*_\gamma = 1$ or $\gamma$ is a multiple of a class of a point, in which case $\Omega^*_\gamma$ equals the Euler characteristic of $S$. 
(\(Q, W\)): quiver with potential

\[\gamma \in \mathbb{N}^+, \text{ and } \theta \in \gamma^\perp, \gamma\text{-generic.}\]

Iterative application of the Kontsevich-Soibelman wall-crossing formula:

\[
\overline{\Omega}^\theta_\gamma = \sum_{\gamma = \gamma_1 + \cdots + \gamma_r} \frac{1}{|\text{Aut}((\gamma_i)_i)|} F_r^\theta(\gamma_1, \ldots, \gamma_r) \prod_{i=1}^r \overline{\Omega}^*_{\gamma_i}.
\]

where

- \(|\text{Aut}((\gamma_i)_i)|\) is the order of the group of permutation symmetries of the decomposition \(\gamma = \gamma_1 + \cdots + \gamma_r\), and

- The coefficients \(F_r^\theta(\gamma_1, \ldots, \gamma_r)\) are sums of contributions from attractor trees with leaves decorated by \(\gamma_1, \ldots, \gamma_r\) and with root at \(\theta\):

\[
F_r^\theta(\gamma_1, \ldots, \gamma_r) = \sum_{T \in T_r^\theta(\gamma_1, \ldots, \gamma_r)} F_{r, T}^\theta(\gamma_1, \ldots, \gamma_r)
\]

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M. Kontsevich, Y. Soibelman: “Wall-crossing structures in Donaldson–Thomas invariants, integrable systems and mirror symmetry”. In Homological mirror symmetry and tropical geometry (pp. 197-308), Springer, 2014
Attractor trees are in particular tropical trees – they satisfy the “tropical balancing condition” (weighted directions around edges add up to zero)
The flow tree formula

**Theorem (Flow tree formula (A-Bousseau))**

- \((Q, W)\): quiver with potential
- \(\gamma \in N^+, \) and \(\theta \in \gamma^\perp, \) \(\gamma\)-generic.

\[
\Omega_{\gamma}^\theta = \sum_{\gamma=\gamma_1+\cdots+\gamma_r} \frac{1}{|\text{Aut}((\gamma_i)_i)|} F^\theta_r(\gamma_1, \ldots, \gamma_r) \prod_{i=1}^r \Omega^{*}_{\gamma_i}.
\]

where

- \(F^\theta_r(\gamma_1, \ldots, \gamma_r) \in \mathbb{Q}\) are described concretely in terms of **binary** trees.

Binary attractor trees are perturbations of the (generally non-binary) attractor trees with roots at \(\theta\).

- Conjectured by Alexandrov-Pioline.
- Proof uses wall structures.
- A variant of this formula is proven by Mozgovoy using operads.
The coefficients $F_{r}^{\theta}(\gamma_{1}, \ldots, \gamma_{r})$

- For $(Q, W)$, let $\gamma = \gamma_{1} + \cdots + \gamma_{r} \in \mathbb{N}^{+}$. (repetitions allowed!)
- Simplifying assumption for now: $\{\gamma_{1}, \ldots, \gamma_{r}\}$ is a basis for $\mathcal{N}$.

$$F_{r}^{\theta}(\gamma_{1}, \ldots, \gamma_{r}) := \sum_{T_{r}} \prod_{v \in V_{T_{r}}^{\circ}} -\tilde{\epsilon}_{T_{r}, v}^{\tilde{\theta}} \langle e_{v'}, e_{v''} \rangle.$$ 

- $T_{r}$: rooted binary trees with $r$ leaves (decorated by $\{\gamma_{1}, \ldots, \gamma_{r}\}$),
- $V_{T_{r}}^{\circ}$: set of interior vertices of $T_{r}$,
- $e_{v} \in \mathcal{N}$ is the sum of $\gamma_{i}$’s attached to leaves descendant from $v$ for any $v \in V_{T_{r}}^{\circ}$,
- $\tilde{\theta}$ is a small generic perturbation of $\theta$ in $M_{\mathbb{R}}$
- $\tilde{\epsilon}_{T_{r}, v}^{\tilde{\theta}} \in \{-1, 0, 1\}$ is a sign defined via “flows” (these signs control the realizability of $T_{r}$ as a binary attractor tree with root at $\tilde{\theta}$).
The general case

- Generally, for $\gamma = \gamma_1 + \ldots + \gamma_r$, if $\{\gamma_1, \ldots, \gamma_r\}$ is not a basis, we introduce a bigger lattice

$$\mathcal{N} := \bigoplus_{i=1}^{r} \mathbb{Z}e_i$$

and consider the map

- $p: \mathcal{N} \to N$ defined by $e_i \mapsto \gamma_i$

- Define a skew-symmetric form $\eta$ on $\mathcal{N}$ by $\eta(e_i, e_j) := \langle \gamma_i, \gamma_j \rangle$.

- In this bigger space we can work with perturbations of attractor trees into binary trees.

$$\begin{align*}
e_1 &= (0, 0, 1) \\
e_1 &= (0, 1, 0) \\
e_1 &= (1, 0, 0) \\
g_1 &= (2, 1) = (1, 0) + (1, 0) + (0, 1) \\
g_3 &= (0, 1) \\
g_1 &= g_2 = (1, 0) \end{align*}$$
Example: for $Q$ the $n$-Kronecker quiver

- $Q$ the $n$-Kronecker quiver
- Let $\theta = (\theta_1, -\theta_1)$ and $\gamma = (1, 1)$, so that $\gamma_1 = (1, 0)$, $\gamma_2 = (0, 1)$. In this case, can actually take $\tilde{\theta} = \theta$.

$$F_1^\theta(\gamma_1, \gamma_2) = 1$$
$$F_2^\theta(\gamma_1, \gamma_2) = -\epsilon_{T, \nu}^\theta n$$

- We have $\theta_1 < 0 \implies \epsilon_{T, \nu}^\theta = 0$ and $\theta_1 > 0 \implies \epsilon_{T, \nu}^\theta = -1$

$$\Omega_\gamma^\theta = F_1^\theta(\gamma)\Omega_{\gamma_1}^* + F_2^\theta(\gamma_1, \gamma_2)\Omega_{\gamma_1}^* \Omega_{\gamma_2}^*$$
$$= 1 \cdot 0 - (-1)n$$
$$= n$$
A correspondence between

$$F_r^\theta (\gamma_1, \ldots, \gamma_r)$$

and counts of rational curves in a toric variety $X_\Sigma$. 
Let $Q$ be a quiver with $I$ vertices and $\gamma \in N$. We set

- $\Sigma$: a fan in $M_{\mathbb{R}}$ of a smooth projective toric variety $X_\Sigma$ containing the rays $\mathbb{R}_{\geq 0} \langle \gamma_i, - \rangle$ for all $i \in I$.
- $H_i \subset D_i$ hypersurfaces defined by $\{z^{\gamma_i} = \text{constant}\}$

- Count genus 0 stable maps $(C, \{p_1, \ldots, p_{r+1}\}) \to X_\Sigma$ satisfying
  - $p_i \mapsto H_i$ for all $1 \leq i \leq r$
  - The contact order of the image of $p_i$ with $D_i$ is the divisibility of $\langle \gamma_i, - \rangle$
• Jun Li: The case $D \subset X$ is smooth. Expand the target;

\[
X \mapsto X[1] = X \amalg_D \mathbb{P}(\mathcal{N}_D|_X \oplus \mathcal{O}_D)
\]

\[
\mapsto X[2] = X[1] \amalg_D \mathbb{P}(\mathcal{N}_D|_X \oplus \mathcal{O}_D)
\]

\[
\mapsto X[3] = \ldots
\]

• Gross–Siebert/Abramovich–Chen: The case $D \subset X$ is log smooth. Record contact orders using “log structures”
A **log structure** on $X$ is a sheaf of monoids $\mathcal{M}_X$ together with a map $\alpha : \mathcal{M} \to (\mathcal{O}_X, \cdot)$ inducing an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. A **log scheme** $(X, \mathcal{M}_X)$ is a scheme with a log structure.

The **ghost sheaf** of a log scheme $(X, \mathcal{M}_X)$ is the sheaf of monoids

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha(\mathcal{O}_X^\times).$$

The **tropicalization** $\Sigma(X)$ of a log scheme $(X, \mathcal{M}_X)$ is the cone complex

$$\coprod_{\eta}(\overline{\mathcal{M}}_X, \eta)_\mathbb{R}^\vee := \text{Hom}(\overline{\mathcal{M}}_X, \eta, \mathbb{R}_{\geq 0}) / \sim$$

indexed by the generic points $\eta$ of the log strata of $X$. 

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Example (The divisorial log structure)

Let $D \subset X$ be a divisor, and $j : X \setminus D \to X$. Define $\mathcal{M}(X, D) := j^*(\mathcal{O}_{X \setminus D}^\times) \cap \mathcal{O}_X$, and $\alpha_X : \mathcal{M}(X, D) \to \mathcal{O}_X$ to be the inclusion.

- $\mathcal{M}_{\mathbb{A}^1_t, 0} = \{ h \cdot t^n \mid h \in \mathcal{O}_{\mathbb{A}^1_t}^\times \}$.
- $\overline{\mathcal{M}}_{\mathbb{A}^1_t, 0, 0} \cong \mathbb{N}$, via the isomorphism $t^n \mapsto n$.

Example (The standard log point)

Let $X := \text{Spec} \mathbb{C}$, $\mathcal{M}_X := \mathbb{C}^\times \oplus \mathbb{N}$, and define $\alpha_X : \mathcal{M}_X \to \mathbb{C}$ as follows:

$$\alpha_X(x, n) := \begin{cases} x & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$
Stable log maps

Let \((S, \mathcal{M}_S)\) be a log point and let \((X, \mathcal{M}_X)\) be a log scheme over \((S, \mathcal{M}_S)\) (in applications, \((S, \mathcal{M}_S)\) will be either the trivial log point or the standard log point).

**Definition**

A **stable log map** with target \(X/S\) is a commutative diagram

\[
\begin{array}{ccc}
(C, \mathcal{M}_C) & \xrightarrow{f} & (X, \mathcal{M}_X) \\
\downarrow{\pi} & & \downarrow{} \\
(W, \mathcal{M}_W) & \rightarrow & (S, \mathcal{M}_S)
\end{array}
\]

where \((W, \mathcal{M}_W)\) is a log point, and \(\pi: (C, \mathcal{M}_C) \rightarrow (W, \mathcal{M}_W)\) is an integral log smooth curve, such that the underlying map of scheme \(f: C \rightarrow X\) is a stable map.

The local structure of \(\mathcal{M}_C\) is defined by Fumiharu Kato.
The **combinatorial type** $\tau$ of a stable log map $f : C/W \to X/S$ consists of:

1. The dual intersection graph $G = G_C$ of $C$, with set of vertices $V(G)$, set of edges $E(G)$, and set of legs $L(G)$.
2. The map $\sigma : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$ mapping $x \in C$ to $(\overline{M}_{X,f(x)})^\vee_{\mathbb{R}}$.
3. The contact data $u_p \in \overline{M}_{X,f(p)}^\vee = \text{Hom}(\overline{M}_{X,f(p)}, \mathbb{N})$ and $u_q \in \text{Hom}(\overline{M}_{X,f(q)}, \mathbb{Z})$ at marked points $p$ and nodes $q$ of $C$. 
Basic monoid

Definition

Given a combinatorial type $\tau$ of a stable log map $f : C/W \to X/S$, we define the associated **basic monoid** $Q$ by first defining its dual

$$Q^\vee_\tau = \left\{ ((V_\eta)_{\eta}, (e_q)_q) \in \bigoplus_{\eta} \overline{M}_{X,f(\eta)}^\vee \bigoplus_q \mathbb{N} \left| \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\} \right.$$  

where the sum is over generic points $\eta$ of $C$ and nodes $q$ of $C$. We then set

$$Q_\tau := \text{Hom}(Q^\vee_\tau, \mathbb{N}).$$

- $Q_\tau$ indeed only depends on the combinatorial type of $f : C/W \to X/S$.
- $Q^\vee_{\tau,\mathbb{R}} := \text{Hom}(Q_\tau, \mathbb{R}_{\geq 0})$ is the moduli cone of tropical curves of fixed combinatorial type.
Basic stable log maps

Given a stable log map $f: C/W \rightarrow X/S$, one can show that there is a canonical map $Q \rightarrow \overline{M}_W$, where $Q$ is the basic monoid defined by the combinatorial type of $f$.

**Definition**

A stable log map $f: C/W \rightarrow X/S$ is said to be **basic** if the natural map of monoids $Q \rightarrow \overline{M}_W$ is an isomorphism.

**Theorem (Abramovich–Chen, Gross–Siebert, 2011)**

*The moduli space $\mathcal{M}(X/S)$ of basic stable log maps with target $X/S$ is a Deligne-Mumford stack.*
Basic stable log maps

For every $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$ and $u = (u_1, \ldots, u_k)$ with $u_i \in |\Sigma(X)|$, we denote by $\mathcal{M}_{g,u}(X/S, \beta)$ the moduli space of genus $g$ basic stable log maps to $X/S$ of class $\beta$ and with $k$ marked points of contact data

$$u = u_1, \ldots, u_k.$$

**Theorem (Abramovich–Chen, Gross–Siebert, 2011)**

- If $X/S$ is proper, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ is a proper Deligne-Mumford stack.

- If $X/S$ is log smooth, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ admits a natural virtual fundamental class $[\mathcal{M}_{g,u}(X/S, \beta)]^{virt}$. 
Counts of rational log maps in \( n \)-dimensional toric varieties

Counts of tropical trees in \( \mathbb{R}^n \)

- We will work with “families” of tropical trees corresponding to log curves over basic monoids of rank equal to the dimension of the family!

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Attractor trees to families of tropical curves

- Construction of a \((d - 2)\)-dimensional family \(\rho_T\) of tropical curves in \(M_{\mathbb{R}}\) from an attractor tree \(T\):
  - Extend the root of \(T\) to infinity to obtain a tropical curve with leaves constrained to lie in the hyperplanes \(\gamma_i^\perp\).
  - Deform this tropical curve while preserving the combinatorial type and the constraints on the leaves.
The description of $N_{\rho_T, H}^{\text{toric}}(X_\Sigma)$.

**Lemma (A-Bousseau)**

For general constraints $H = (H_1, \ldots, H_r)$, the moduli space of genus 0 log curves in $X_\Sigma$ matching $H$, and with tropicalization the $(d - 2)$-dimensional family of tropical curves $\rho_T$, is finite. Denote by $N_{\rho_T, H}^{\text{toric}}(X_\Sigma)$ the number of such log curves.

**Theorem (A-Bousseau)**

The coefficients $F_{r, T}^\theta(\gamma_1, \ldots, \gamma_r)$ expressing the contribution to $F_r^\theta(\gamma_1, \ldots, \gamma_r)$ of an attractor tree $T$ satisfy

$$F_{r, T}^\theta(\gamma_1, \ldots, \gamma_r) = N_{\rho_T, H}^{\text{toric}}(X_\Sigma).$$
Summary of the proof

- Construct a toric degeneration $\mathcal{X} \to \mathbb{A}^1$ of $X_\Sigma$ and of the constraints $H$ (similar as in Nishinou-Siebert).

- Degeneration formula: express the invariants $N^\text{toric}_{\rho_T,H}(X_\Sigma)$ of the general fibers $X_\Sigma$ as a sums of invariants $N^\text{toric}_{\rho_S}(X_0)$ of the special fiber $X_0$, where $S$ are binary trees in $M_\mathbb{R}$ deforming $T$.

- Show that

$$N^\text{toric}_{\rho_S}(X_0) = \prod_v |\langle \gamma_v', \gamma_v'' \rangle|$$

Key technical point: theory of punctured log maps [Abramovich-Chen-Gross-Siebert] to produce log curves by gluing.

- By the flow tree formula,

$$F^\theta_{r,T}(\gamma_1, \ldots, \gamma_r) = \sum_S \prod_v |\langle \gamma_v', \gamma_v'' \rangle|$$
A correspondence between quiver DT and log curves in cluster varieties?
Cluster varieties and $\mathbb{A}^1$ curves

- $Q, N = \mathbb{Z}^{Q_0} = \bigoplus_{i \in Q_0} \mathbb{Z}s_i, M_\mathbb{R} = \text{Hom}(N, \mathbb{R}), \nu_i := \langle s_i, - \rangle \in M$.
  - Fan in $M_\mathbb{R}$ containing the rays $\mathbb{R}_{\geq 0}\nu_i$.
  - Toric variety $\overline{X}$, toric boundary $\overline{D}$, components $(\overline{D}_i)_{i \in Q_0}$.
- $X$: blow-up of $\overline{X}$ along the codimension two loci $(1 + z^{s_i} = 0)|_{\overline{D}_i}$.
- $D$: strict transform of $\overline{D}$, $(X, D)$: log Calabi-Yau pair.

Complement $U = X \setminus D$, Poisson cluster variety $U = \bigcup (\mathbb{C}^*)^{\mid Q_0 \mid}$.

---

A1-curves: rational curves in $X$ meeting $D$ in a single point.

A1-curves come in $(d - 2)$-dimensional families, where $d = |Q_0| = \dim X$.

$M_\beta$: compactification of the moduli space of A1-curves of class $\beta \in H_2(X, \mathbb{Z})$.


Such counts are punctured log Gromov-Witten invariants of Abramovich–Chen–Gross–Siebert, counting A1-curves in $(X, D)$ of class $\beta$, with degeneration pattern $\tau$. ²

²Argüz–Gross, The Higher Dimensional Tropical Vertex, Geometry & Topology 26 (5), 2135-2235
Quiver-cluster

- Algebra: $\Omega_{\gamma}^{\theta}$ of the quiver $Q$ are Euler characteristics of moduli spaces of $\theta$-stable representations of $Q$ of dimension $\gamma$.
- Geometry: $GW_{\beta}^{T}$ of the cluster variety $(X, D)$ attached to $Q$ are counts of “maximally degenerate $\mathbb{A}^1$-curves in $(X, D)$ of class $\beta$.

**Theorem (A-Bousseau)**

*Assume that the DT attractor invariants of $Q$ are trivial. Then, there exists an explicit correspondence $\beta \rightarrow \gamma$, such that*

$$\sum_{\tau} GW_{\beta}^{T} = \Omega_{\gamma}^{\theta}.$$  

*where the sum is over all curves whose tropicalization have type $\tau$, containing one marked leg, tracing out a subspace of $M_{\mathbb{R}}$ containing $\theta$.*

- This is compatible with the previous quiver DT-toric log GW correspondence, the cluster variety $(X, D)$ degenerates to the toric variety $(\overline{X}, \overline{D})$, and the log GW invariants are related.
Heuristic picture of the proof.

- $U = X \setminus D$ admits a Lagrangian torus fibration with base $M_{\mathbb{R}}$.
- Counts $GW^\tau_\beta$ of $\mathbb{A}^1$-curves in $(X, D)$ are computed by tropical curves in $M_{\mathbb{R}}$.
- $M_{\mathbb{R}}$ is also the space of stability parameters for DT invariants and the same tropical curves describe the wall-crossing behavior of DT invariant $DT^\theta_\gamma$!
Thank you for your attention!