1. Operations on Hochschild and cyclic complexes (AMS Meeting, Purdue) p. 1-24

2. Operations, Getzler-Gauss-Manin connection, NC cryst. complexes (U Miami) 3/22 p. 25-41

What do (dg) (n-)categories / (shifted) symplectic mflds form?

...

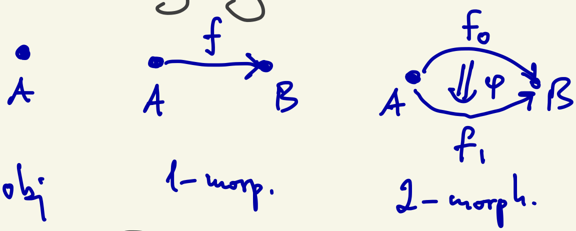
And what does this imply?

A) (dg) categories form:

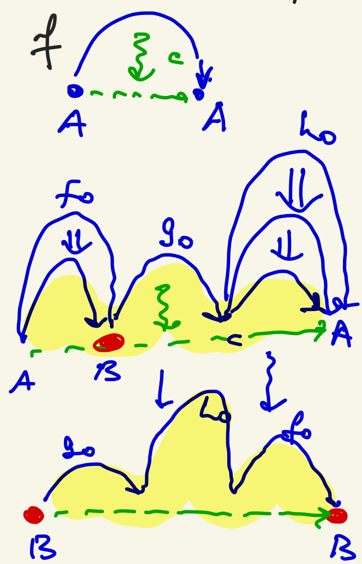
A 2-category with a trace functor

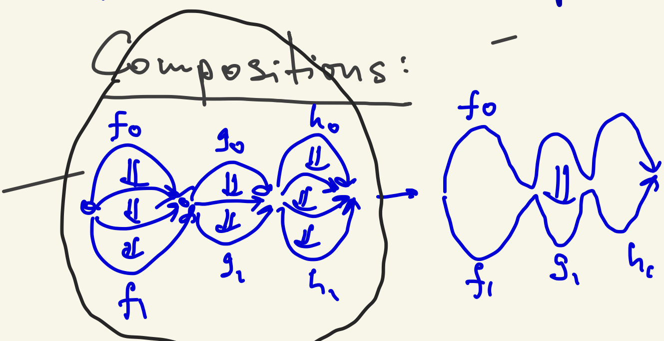
[In what sense?...]

2-category:



With a trace functor





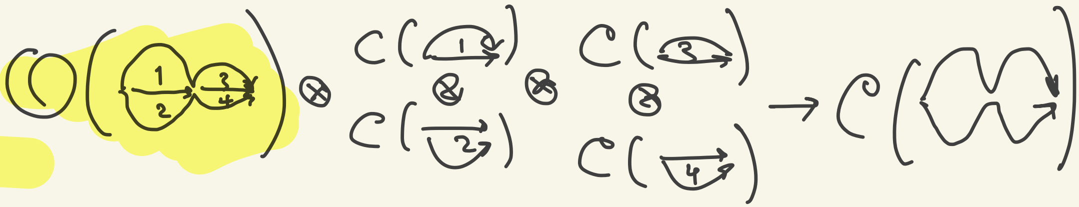
"Up to homotopy" – in what sense?

Batanin/Tamarkin:

Formalism of 2-operads

$\mathcal{O}\Big($$\Big) \in ob(\mathcal{C})$ $\qquad \mathcal{C} = \text{Compx}, \ldots$ $\quad (\mathcal{C}, \otimes)$

Operations: as if



and the way of composing them.

Tamarkin: (What do dg Cat form?) (cofibrant replacement of) the constant 2-opd

acts on: $\quad \text{graph}(B) = {}_fB \quad A \dashrightarrow {}_fB \rightsquigarrow B \quad R\mathrm{Hom}_{A-B}({}_gB, {}_fB) \simeq C^\bullet(A, {}_fB_g)$

$\bullet A$ algebra

$A \xrightarrow{f} B$ morphism

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Hochschild cochain cplx

Recently, Bottman: (similar/related version of 2-operads)

(M, ω) $\bullet$

$M_1 \xrightarrow{L_{12}} M_2$

$L \subset M_1^- \times M_2$

$$\begin{tikzcd} M_1 \arrow[r, bend left, "L_{12}"] \arrow[r, bend right, "L'_{12}"'] \arrow[r, phantom, "\Downarrow"] & M_2 \end{tikzcd}$$

$CF^*(L_{12}, L'_{12})$

(for me):

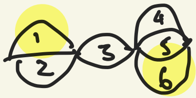
Morally, $\mathrm{Fuk}(M) \approx$ some sort of modules over the quantized fns on M...

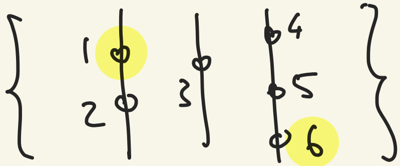
(microlocal methods in sympl. geom.)

Idea: $\mathcal{P}($$)$

$\downarrow ?$

k $\quad \boxed{\text{(const 2-op)}}$

$\mathcal{P}\Big($$\Big)$ = FM_2 compactification of



Bottmann's realisation of this idea:

2-associahedra

For Alg: from what actually acts on $C^\bullet(A,B)$ etc., need go to recognition principles, etc.

But Sympl mflds: 2-associah actually parametrize holomorphic curves, etc.!

① Algebras form a (strict) category in cocategories.

$$\underset{A}{\bullet} \qquad \underset{A}{\bullet} \xrightarrow{f} \underset{B}{\bullet} \qquad \begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet \end{tikzcd} \quad C^{\bullet}(A, {}_fB_g)$$

alg

form a dg category

$$C^{\bullet}(A,B)$$

$$\mathcal{B}(A,B)$$
$$\|$$

Now:

$$\mathrm{Bar}\, C^{\bullet}(A,B) \quad \text{dg cocategory}$$

$$m_{ABC} : \mathcal{B}(A,B) \otimes \mathcal{B}(B,C) \to \mathcal{B}(A,C)$$

dg functor; associative.

(Getzler–Jones, Gerst.–Voronov, ...)

How to pass to 2-categories (i.e. categories in categories)?

Apply Cobar.

$A \rightrightarrows B \quad \rightsquigarrow \quad \mathrm{Cobar}\,\mathrm{Bar}\, C^\bullet(A,B) \xrightarrow{\sim} C^\bullet(A,B)$

<u>weaker</u> 2-category structure (due to T. Leinster).

From Leinster's $\xrightarrow{\text{to}}$ Batanin/Tamarkin's

$\downarrow$ to

Joyal-Lurie ...

Definitely true, not sure where it is written.

Add Hochschild chains?

Several ways to do this. The results somehow diverge.

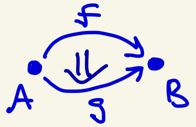
(hyperbolic situation)

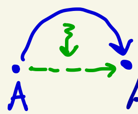
"Дороги расползались, как раки."

(Н.В.Гоголь)

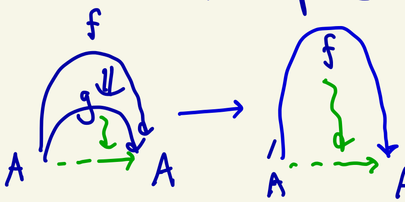
"The roads crawled away like crawfish" (N.V.Gogol).

Adding Hochschild chains, 1: The trace functor

As above:  $\rightsquigarrow C^\bullet(A, {}_f B_g)$

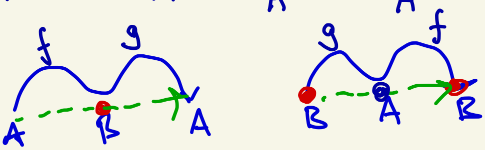
 $\rightsquigarrow C_{-\bullet}(A, {}_f A)$

Two examples of operations:



$$C^\bullet(A, {}_f A_g) \otimes C_{-\bullet}(A, {}_g A) \to C_{-\bullet}(A, {}_f A)$$

$$\iota_\varphi(a_0 \otimes \dots \otimes a_n) = \pm \varphi(a_{k+1}, \dots, a_n)\, a_0 \otimes a_1 \otimes \dots \otimes a_k$$



$$C_{-\bullet}(A, {}_{gf}A) \to C_{-\bullet}(B, {}_{fg}B)$$

$$a_0 \otimes \dots \otimes a_n \mapsto f(a_0) \otimes \dots \otimes f(a_n)$$

Thm (T.-Wei) ([NT], Cyclic homology; R. Wei's thesis)

Algebras form a category in cocategs with a trace functor in the following sense.

a) As above: $A, B \mapsto$ dg cocat $\mathcal{B}(A,B)$

b) $m_{ABC} : \mathcal{B}(A,B) \otimes \mathcal{B}(B,C) \to \mathcal{B}(A,C)$
strictly associative.

c) $A \mapsto TR(A)$ dg comod over $\mathcal{B}(A,A)$

d) $A, B \mapsto T_{AB} : m^*_{ABA} TR(A) \xrightarrow{\sim} m^*_{BAB} TR(B)$

$$\begin{tikzcd} & \mathcal{B}(A,B) \otimes \mathcal{B}(B,A) \arrow[dl, "m_{ABA}"'] \arrow[dr, "m_{BAB}"] & \\ \mathcal{B}(A,A) & & \mathcal{B}(B,B) \end{tikzcd}$$

e) For any A, B, C:

$$\begin{tikzcd} & \mathcal{B}(A,B) \otimes \mathcal{B}(B,C) \otimes \mathcal{B}(C,A) \arrow[dl, "m_{ABCA}"'] \arrow[d, "m_{BCAB}"] \arrow[dr, "m_{CABC}"] & \\ \mathcal{B}(A,A) & \mathcal{B}(B,B) & \mathcal{B}(C,C) \end{tikzcd}$$

homotopy btw id and "T^3" which are two endomorphisms of $m^*_{ABCA} TR(A)$.
The homotopy is central in all poss. ways. NO higher homots

Remarks.

(1) The operations that arise on Hochschild chain complexes are obtained from the above as follows:

$$\iota_\varphi : C_\bullet(A, A) \to C_{\bullet - |\varphi|}(A, A)$$

from c);

$$L_\varphi : C_\bullet(A, A) \to C_{\bullet - |\varphi| + 1}(A, A)$$

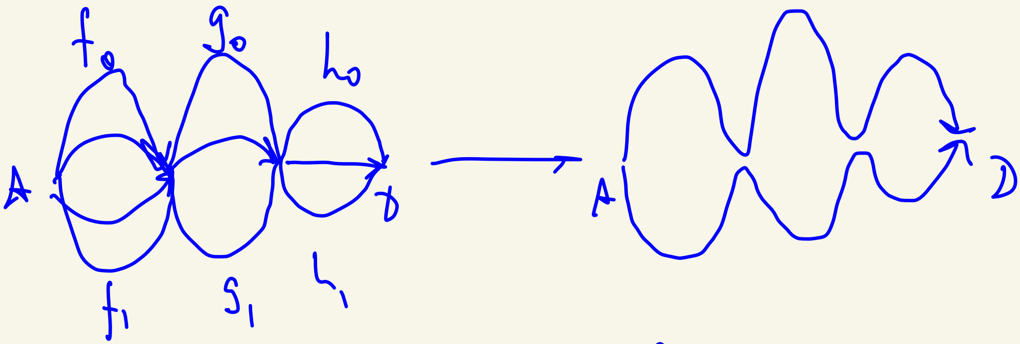
from d);

$$B : C_\bullet(A) \longrightarrow C_{\bullet + 1}(A)$$

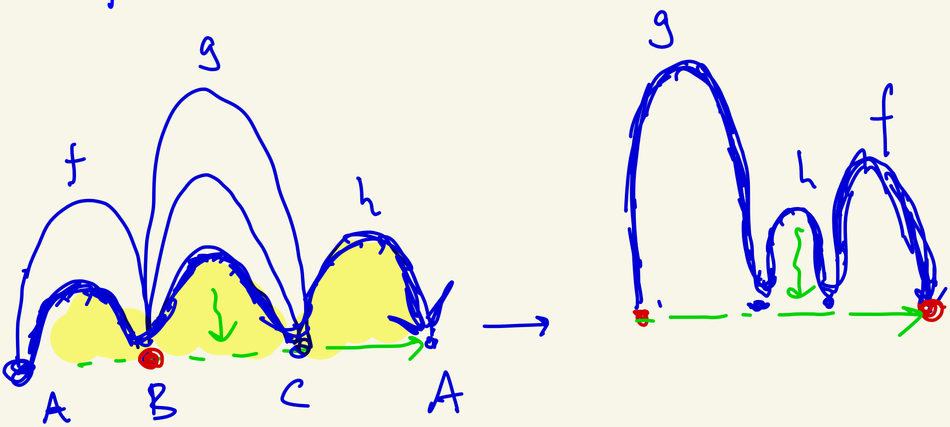
from e)

Seems significant that the existence of the cyclic (NC DR) differential is due to non-strictness.

(2.) Not sure how to relate this to a Batanin-Tamarkin (Bottman) approach with a version of a 2-operad that describes operations

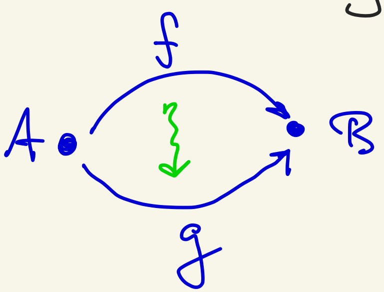


and



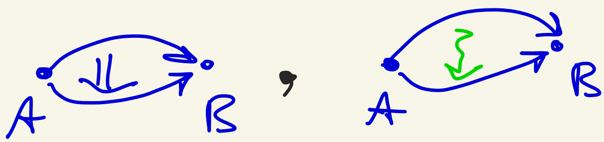
Adding Hochschild chains, 2: Twisted tetramodule.

It may be more natural to consider more general Hochschild chains



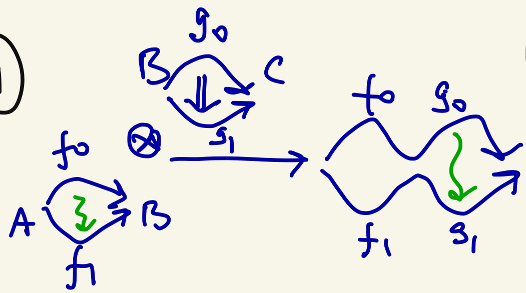
$$C_{-\bullet}(A,B)_{f\,g}$$

There is a nontrivial algebraic structure on

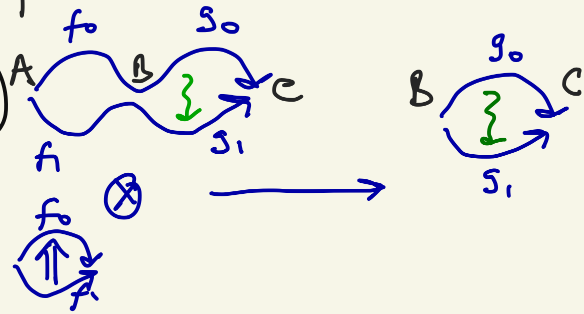


Two types of operations:

1



2



In [NT]:

$A \underset{g}{\overset{f}{\Downarrow}} B \mapsto C^\bullet(A, {}_fB_g)$; $A \underset{g}{\overset{f}{\xi}} B \mapsto C_\bullet(A, {}_fB_g)$

form "a twisted bimodule over a category in cocategories"

For example: $\mathrm{Bar}\, C^\bullet(A,A)$ is a bialg (in fact a Hopf alg)

Consider (a bit formally) a dual bialgebra U.

A tetramodule (B. Shoikhet):

A bialgebra structure on $U \oplus \varepsilon M$, $\varepsilon^2 = 0$.

I.e.: M is a U bicomodule;

$U \otimes M \otimes U \to M$

morphism of bicomodules.

What we get is a <u>twisted</u> tetramodule:

$\mathcal{U}$-bimodule $\mathcal{M}$;

$$\mathcal{M} \xrightarrow{\Delta} \mathcal{U} \otimes \mathcal{M} \otimes \mathcal{U}^{+}$$

$\mathcal{U}$ and $\mathcal{U}^{+}$ are both $\mathcal{U}$ but with different bicomodule structures: on the left, via Δ on $\mathcal{U}$; on the right, via Δ + the antipode S on $\mathcal{U}$.

Q) Where else does such a structure appear? What do we get if we pass from Hopf algebra to a Poisson-Lie grp?

Adding Hochschild chains, 3:
Applying the functor $CC^{\bullet}_{-}(-)$

Recall from 1: dg functors of dg coalgebras

$$\mathcal{B}(A,B)\otimes^{\cdot}\mathcal{B}(B,C)\to\mathcal{B}(A,C)$$

and also:

$$\mathcal{B}(A,B)\otimes \mathrm{Bar}(A)\longrightarrow \mathrm{Bar}(B)$$

all associative

(where $\mathcal{B}(A,B)=\mathrm{Bar}\,C^{\bullet}(A,B)$)

Apply to that the functor

$$CC^{\bullet}_{\mathrm{II}}(-):\mathrm{Coalgs}\longrightarrow \mathrm{Cplxs}$$

($CC^{\bullet}_{\mathrm{II}}$ - cyclic <u>CO</u>chain complex of a <u>CO</u>algebra/category)

Thm (J. Jones; Quillen; [Feigin–T])

$$CC^\bullet_{II}(\mathrm{Bar}(C)) \simeq C_\bullet(C)$$

$$C^\bullet_{II}(\mathrm{Bar}(C)) \simeq CC^-_\bullet(C)$$

[Koszul duality preserves Hochschild; intertwines cyclic with negative cyclic].

Corollary: we get an A_∞ category over $k[[u]]$:

Objects: algebras A

Morphisms: $CC^-_\bullet(C^\bullet(A,B))$

its red. modu: $C_\bullet(C^\bullet(A,B))$ ☆

Or, strictification:

$CC^\bullet_{II}(\mathrm{Bar}(C^\bullet(A,B)))$

And: A_∞ (resp. dg) functor from it to Cplxs:

$$A \longmapsto CC_\bullet^-(A)$$

This is ($\approx$ '99) Khalkhali's idea for studying operations on $CC_\bullet^-$

Remark: Applying $C_\bullet$ to the 2-category of symplectic manifolds figures prominently in Bottman, relates to symplectic cohomology SH^*.

Application of 3: rigidity and Getzler's Gauss-Manin connection.

Construction:

$$\mathfrak{g}_A^\bullet = C^{\bullet+1}(A,A), \delta, [\,,\,]_{Gerst}$$

$$U(\mathfrak{g}^\bullet_A) \longrightarrow \mathrm{Bar}\, C^\bullet(A, A)(\mathrm{id}_A, \mathrm{id}_A)$$

morphism of dg bialgebras

From "at:

- $CC^\bullet_{\mathrm{II}}(U(\mathfrak{g}^\bullet_A))$ is an A_∞ algebra
- $CC^\bullet_{\mathrm{II}}(\mathrm{Bar}(A)) \simeq CC^-_\bullet(A)$

  is an A_∞ module.

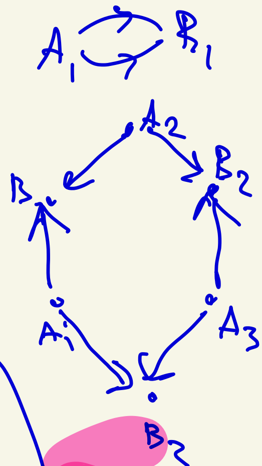
cyclic cohomology of $U(\mathfrak{g}^\bullet_A)$ viewed as a <u>co</u>algebra.

$CC^\bullet_{\mathrm{II}}(U(\mathfrak{g}^\bullet))$ (as an A_∞-alg) is computed in [NT] for any dg Lie algebra $\mathfrak{g}^\bullet$.

More about this: in the end of these notes.

Pictures

$$\begin{tikzcd} B_1 & A_2 \arrow[l, "f_{21}"'] \arrow[d, "f_{22}"] \\ A_1 \arrow[u, "f_{11}"] \arrow[r, "f_{12}"'] & B_2 \end{tikzcd}$$

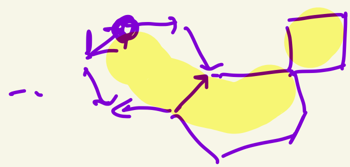


$$C^{\bullet}\left(\bigotimes_1^m A_i, \bigotimes^m B_i\right)$$

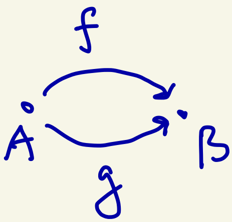
Those should be acted upon by $\mathbb{k}$

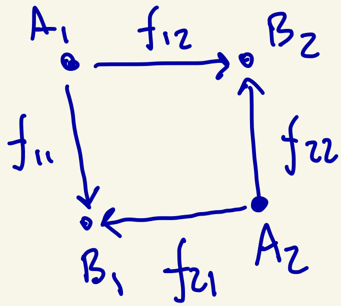
Conjecture

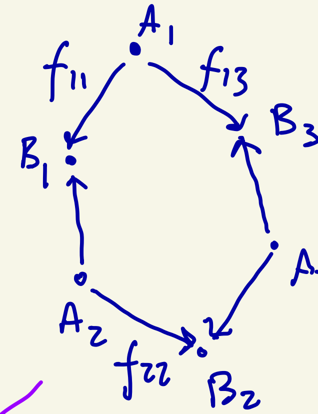
a) notion similar to Batanin (or Bottman) 2-operad:



A more general structure:





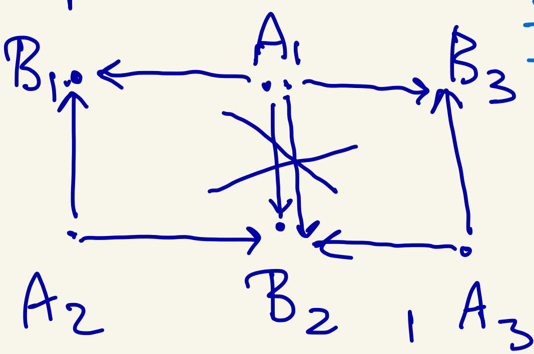


2n-gons.

$$C^\bullet(\otimes A_j \curvearrowright, \otimes B_j \curvearrowleft)$$

two $\otimes A_j$-bimod strres

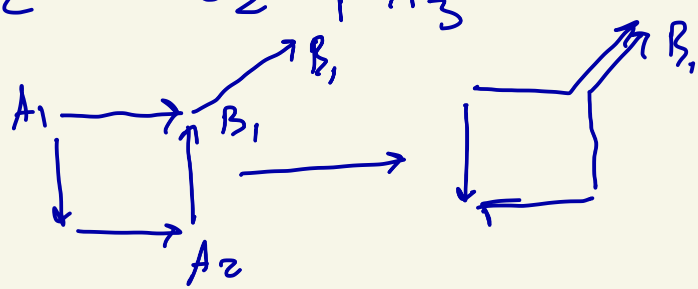
Operations:



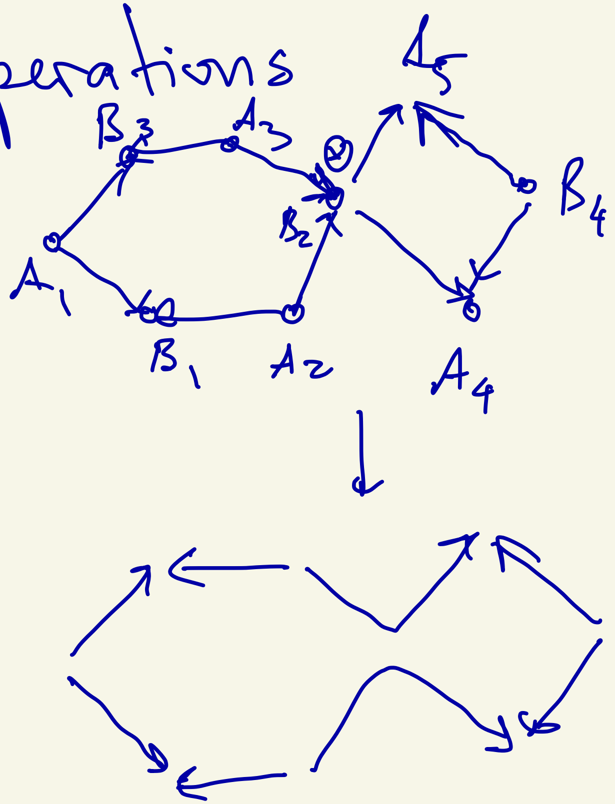
4-gon × 4-gon

↓

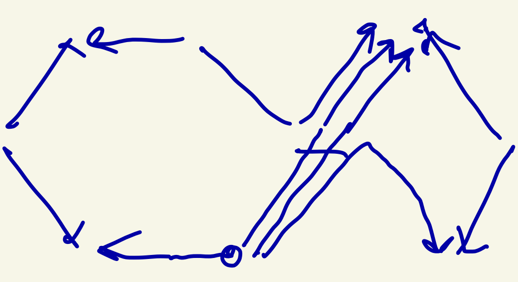
6-gon



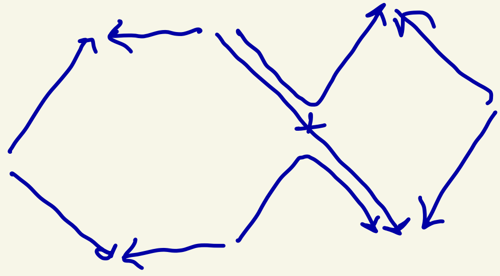
There are also brace-like operations



They are homotopies between



and



Applying them at all possible pairs of vertices, we get the Kontsevich-Vlassopoulos bracket.

When all $A_i, B_i = A$ and all $f_{ij} = id_A$, we get operations on $C^\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n})$ where $\alpha: A^{\otimes n} \circlearrowleft$ is the cyclic shift (which is an algebra isomorphism).

MC elements for the KV bracket are pre-CY structures.

<u>Also</u>: including Hochschild <u>chains</u> $C_\bullet\left(\bigotimes_j A_j, \bigotimes_n B_{jn}\right)$

What is the structure on chains <u>and</u> cochains?

<u>Note</u>: $C_\bullet(A^{\otimes n}, A^{\otimes n})$

$\cong$

$C_\bullet(A, A)$

(not so for cochains).

Plus: there is "the Tate diagonal", or Frobenius.

The structure should incorporate both Frobenius and KV bracket.

Operations on Hochschild complexes and rigidity / Getzler-GM / NC crystalline complexes

(U Miami, March '22)

Recall: operations on Hochschild and cyclic complexes.

Assoc (dg) algebra $A \rightsquigarrow$

1) Hochschild complex $(C_\bullet(A), b)$

2) Negative cyclic complex

$$CC_\bullet^-(A), b+uB$$

$$CC_\bullet^-(A) = C_\bullet(A)[[u]]$$

and also

$$CC_\bullet(A), b+uB$$

$$CC_\bullet(A) = C_\bullet(A)((u)) / u\, C_\bullet(A)[[u]]$$

$$CC_\bullet^{per}(A) = C_\bullet(A)((u))$$

Dually, for a dg coalgebra B:

$C^\bullet_{II}(B)$, $CC^\bullet_{II}(B)$

$\boxed{\text{II refers to: using } \oplus \text{ instead of } \prod \text{ in defs}}$

Motivation/relation to Hodge Thy:

When A is commutative/ k,

$\mathbb{Q} \subset k$:

$$\text{HKR}: \quad \begin{array}{ccc} C_\bullet(A) & \longrightarrow & \Omega^\bullet_{A/k} \\ \shortparallel & & \cup \\ A^{\otimes \bullet+1} & & \\ \ni & & \\ a_0 \otimes \cdots \otimes a_n & \longmapsto & \frac{1}{n!} a_0\, da_1 \cdots da_n \end{array}$$

Intertwines: b with 0
B with d_{DR}

Is a quasi-isom for $C_\bullet$, $CC_\bullet$, etc. when A is regular.

$$\underset{b+uB}{CC_\bullet^-(A)} \xrightarrow{\sim} \underset{0+ud_{DR}}{\Omega^\bullet_{A/k}[[u]]}$$

Dgla of Hochschild cochains:

(a.k.a. deformation complex of A)

$C^\bullet(A,A) = \mathrm{Hom}_k(A^{\otimes \bullet}, A)$ $\quad \delta : C^\bullet \to C^{\bullet+1}$

$[\,,\,] = [\,,\,]_{Gerst}$

makes $C^{\bullet+1}(A,A)$ a dgla $\mathfrak{g}^\bullet_A$

Thm [NT]

$CC^\bullet_{II}(\bar{U}(\mathfrak{g}^\bullet_A))$ is an A_∞ algebra

$CC^-_\bullet(A)$ is an A_∞ module

($\bar{U}$ is an augmentation ideal; the A_∞ alg structure ind. by $\bar{U} \otimes \bar{U} \xrightarrow[\text{mult}]{} \bar{U}$).

$\ker \delta \cap \mathfrak{g}^0_A = \mathrm{Der}(A)$

$H^0(\mathfrak{g}^\bullet_A, \delta) = \mathrm{Der}(A) / \mathrm{Der}^{in}(A)$

Ex. $\mu \in C^2(A,A)$ $\quad m : A^{\otimes 2} \to A$

$$\mu(a,b) = (-1)^{|a|}\, a * b$$

in the graded case

$$[\mu,\mu] = 0 \iff * \text{ associative}$$

(if no 2-torsion)

In particular: $m(a,b) = (-1)^{|a|}\, ab$

$$\delta = [m, \cdot]$$

Also: $\mathfrak{g}_A^\bullet$ acts on $C_\bullet(A)$

$CC_\bullet^-(A) \ldots$

$\varphi \mapsto L_\varphi$

$\cap$

$\mathfrak{g}_A^\bullet$

$$b = L_m$$

B independent on mult.

Geom. motivation:

A/k commutative, $k \supset \mathbb{Q}$:

$$\Lambda^\bullet_A \operatorname{Der}(A) \xrightarrow{HKR} C^\bullet(A,A)$$

$$\theta \longleftrightarrow \delta$$

$$A \ni a \longmapsto a\,;$$

$$\Lambda^n \operatorname{Der}(A) \ni X_1 \wedge \dots \wedge X_n \longmapsto$$

$$\longmapsto \sum \pm \frac{1}{n!} D_{\sigma 1} \cup \dots \cup D_{\sigma n}$$

$$(D_{\sigma 1} \cup \dots \cup D_{\sigma n})(a_1, \dots, a_n) =$$

$$= D_{\sigma 1}(a_1) \dots D_{\sigma n}(a_n)$$

Thm Quasi-isom when A regular.

Note: $\Lambda^{\bullet+1}_A \operatorname{Der}(A)$, $\delta = 0$ itself a dgla.

Thm HKR intertwines Lie alg structures on $HH^\bullet(A)$
$$\begin{array}{c} HH^\bullet(A) \\ \shortparallel \\ H^\bullet(\delta) \end{array}$$
and $\Lambda^\bullet \mathrm{Der}(A)$ (easy).

HKR can be extended to an L_∞ quis of dgla

$$\left(\Lambda_A^{\bullet+1} \mathrm{Der}(A), 0, [,]\right) \to \mathfrak{g}_A^\bullet$$

(Kontsevich Formality, '97).

So: morally, $C_\bullet(A)$ is nc forms; $CC_\bullet^-(A)$, etc. nc DR; $C^\bullet(A,A)$ nc multivectors.

From operations to rigidity

⓪ The calculation of $CC^\bullet_{\Pi}(U(\mathfrak{g}))$

Start with the Hochschild cplx $C^\bullet_{\Pi}(U(\mathfrak{g}))$. There are two dg subalgebras: $U(\mathfrak{g})$ and $\mathrm{Cobar}\,\overline{U}(\mathfrak{g})$. Their cross product is $\simeq C^\bullet_{\Pi}(U(\mathfrak{g}))$.

Explicitly: the algebra is generated by: $X \in \mathfrak{g}$; $(a) \in \overline{U}(\mathfrak{g})$

Relations: $[X, (a)] = ([X, a])$;

$$XY - YX = [X, Y]_{\mathfrak{g}}$$

Differential: $\partial^{\mathrm{cobar}}$: $X \mapsto 0$;

$$(a) \longmapsto \sum (-1)^{|a^{(1)}|} (a^{(1)}) \cdot (a^{(2)})$$

When $(\mathfrak{g}, \delta)$ dgla, total diff: $\delta + \partial^{\mathrm{cobar}}$

$CC_\bullet^{II}(U(\mathfrak{g}))$ is a deformation of that:

<u>Generators</u>: $X, (a)$ as above; u central

<u>Differential</u>: $\delta + \partial^{cobar} + uB$

$B: (a) \mapsto a;\ X \mapsto 0$

We use the notation (for a dgla $(\mathfrak{g}, \delta)$):

$$C^\bullet_{II}(U(\mathfrak{g})) \simeq U(\mathfrak{g}) \ltimes \mathrm{cobar}(\bar{U}(\mathfrak{g}))$$

$$CC^\bullet_{II}(U(\mathfrak{g})) \simeq U(\mathfrak{g}) \ltimes_1 \mathrm{cobar}(\bar{U}(\mathfrak{g}))[u]$$

(1) $\mathfrak{a} = k \cdot R \qquad |R| = 2$

abelian dgLa

Find an element $x = x(R)$ of degree $\underline{1}$ in

$$U(\mathfrak{a}) \ltimes_1 \mathrm{Cobar}\, \overline{U}(\mathfrak{a})((u)) \quad \left(\simeq CC_\bullet^{II}(U(\mathfrak{a}))[u^{-1}] \right)$$

$$\partial^{\mathrm{cobar}} x + x^2 = -R \qquad (*)$$

<u>Answer</u>:

i) <u>Notation</u>: for a series in 2 variables

$$F(R, y) = \sum_{n=1}^{\infty} x_n(R) \cdot y^n$$

put

$$x_F(R) = \sum_{n=1}^{\infty} x_n(R) \cdot \underbrace{(R^n)}_{\text{free gen. in Cobar}}$$

ii) Fact:

$x_F(R)$ satisfies $(*)$ when

$$F(y,R) = -\sum_{n=1}^{\infty} \frac{u^{-n}}{n!} y(y-R)\dots(y-(n-1)R)$$

(2) This allows to straighten "connections with small curvature":

If $D \in C^{\bullet}_{\mathrm{II}}(g)^1$ and $D^2 = R$

then

$$(D + x(R))^2 = 0$$

Applications

1) Getzler's Gauss-Manin

2) NC crystalline complex.

1) S a scheme

$\mathcal{A}$ $\mathcal{O}_S$-algebra

Assume the $\mathcal{O}_S$-mod $\mathcal{A}$ has a connection $\nabla_{\mathcal{A}}$

(not flat; not as an algebra).

$$\nabla_{\mathcal{A}}^2 = R \in \Omega^2(S, \mathcal{E}nd\,\mathcal{A})$$

$$\nabla_{\mathcal{A}} m \in \Omega^1(S, \{\mathcal{A}^{\otimes 2} \to \mathcal{A}\})$$

$$\mathcal{R} = \nabla_{\mathcal{A}} m + R \in \Omega^\bullet(S, \underbrace{\mathfrak{g}^\bullet_{\mathcal{A}}}_{\text{bundle of dgla}})$$

$$\nabla_{GGM} = \nabla_{\mathcal{A}} + b + uB + \underbrace{\iota(\mathcal{R})}_{}$$

(the action of the element of $\Omega^\bullet(S, CC^{-}_\bullet(U(\mathfrak{g}^\bullet_{\mathcal{A}})))$ on $CC^{per}_\bullet$

is a flat superconnection, i.e.

$$\nabla_{GGM} = \nabla_A + \omega$$

$$\omega \in \Omega^\bullet(S, \mathcal{E}nd_{\mathcal{O}_S} CC^{per}_\bullet(A))$$

$$\nabla^2_{GGM} = 0$$

2) A_0 an algebra over $\mathbb{F}_p$.

Assume: A a $\mathbb{Z}$-module;

$$A/pA \xrightarrow{\sim} A_0.$$

Lift the product on A_0 to a bilinear (not associative) product on A.

$$m : A^{\otimes 2} \longrightarrow A$$

$$m_0(a, b) = ab \qquad m_0 : A_0^{\otimes 2} \longrightarrow A_0$$

$\frac{1}{2}[m, m] =: R$ $\quad R(a,b,c) =$

$= m(m(a,b),c) - m(a, m(b,c))$

$$R \in C^2(A, A)$$

(A viewed as an alg with 0 product)

$b_m + uB + \chi(R)$ well-defined

on $CC_\bullet^{per}(A)^\wedge$ ← p-adic completion

$$(b_m + uB + \chi(R))^2 = 0$$

<u>Note:</u> when m is associative, get $CC_\bullet^{per}$(lifting of A_0)

Independence on lifting (or on ∇_A in GGM case): along the same lines.

More precisely:

A_∞ functor

$$\mathbb{Z}_p \cdot \mathcal{C} \longrightarrow \left\{ \text{Complexes over } \mathbb{Z}_p \right\}$$

(resp. over k).

$\mathcal{C}$: objects = liftings m of m_0 to A

(resp. connections on the $\mathcal{O}_S$-module A_S).

Morphisms: unique morphism for any two objects.

$$m \longmapsto \left(CC_\bullet^{per}(A)^\wedge, b_m + uB \right)$$

$$\nabla_A \longmapsto \Omega^\bullet(S, CC_\bullet^{per}(A_S)), \nabla_{GGM}$$

The D-module analogy

Given $X \xrightarrow{f} \mathbb{A}^1$, a parameter ρ, and a $\mathcal{D}_X$-module $\mathcal{M}$, consider two $\mathcal{D}_X[s]$-modules:

a) $e^{-f/h} \cdot \mathcal{M}[h^{-1}] \cdot \delta_0$

s acts by $-\rho h \frac{\partial}{\partial h}$.

b) $f^{s+\mathbb{Z}} \cdot \mathcal{M}[s] \cdot \delta_1$

two formal symbols

When f is invertible, there is a morphism

$$e^{-f/h} \cdot \mathcal{M}[h^{-1}]\,\delta_0 \longrightarrow f^{s+\mathbb{Z}}\,\mathcal{M}[s] \cdot \delta_1$$

$$e^{-f/h} \cdot (1/h)^n \cdot m \cdot \delta_0 \mapsto s(s-\rho)\dots(s-(n-1)\rho) \cdot f^{s-n} \cdot m\,\delta_1$$

or, formally: putting $s = y/h$, $\rho = R/h$:

$$\delta_0 \mapsto \sum_{n=0}^{\infty} \frac{h^{-n}}{n!}\, y(y-R)\dots(y-(n-1)R)\cdot f^s\cdot \delta_1$$

$$= \left(1+\frac{R}{h}\right)^{y/R}\cdot f^s \delta_1$$

In the language of direct images:

$$\underset{x}{i : X} \longrightarrow \underset{x}{X} \times \underset{t}{\mathbb{A}^1} \qquad T^*(X\times\mathbb{A}^1): \begin{matrix} x & t \\ \xi & \tau \end{matrix}$$

$$i_+ \mathcal{M} \cong \underset{\tau = \frac{1}{h}}{\mathcal{M}[\tau]} \simeq e^{-f/h}\cdot \mathcal{M}\left[\frac{1}{h}\right]$$