The nilpotent orbit theorem, old and new

Christian Schnell | Stony Brook University

The nilpotent orbit theorem, old and new

Christian Schnell | Stony Brook University

Outline of the talk

The nilpotent orbit theorem was proved by Schmid in 1973.

It describes the asymptotic behavior of the Hodge filtration in a polarized variation of Hodge structure on the punctured disk.

Plan for today:

- Sketch Schmid's original proof (when the eigenvalues of the monodromy operator are roots of unity).
- Explain a new proof that works in general.

Outline of the talk

The nilpotent orbit theorem was proved by Schmid in 1973.

It describes the asymptotic behavior of the Hodge filtration in a polarized variation of Hodge structure on the punctured disk.

Plan for today:

- Sketch Schmid's original proof (when the eigenvalues of the monodromy operator are roots of unity).
- Explain a new proof that works in general.

Complex Hodge structures

Some basic terminology...

Recall that a Hodge structure of weight n on a complex vector space V is simply a decomposition

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A hermitian pairing $Q \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ is called a polarization if

1. The decomposition is orthogonal.
2. $\langle v, w \rangle = \sum_{p,q} (-1)^q Q(v^{p,q}, w^{p,q})$ is positive definite.

The Hodge structure is determined by the Hodge filtration

$$F^p = V^{p,q} \oplus V^{p+1,q-1} \oplus V^{p+2,q-2} \oplus \cdots$$

because of the identity

$$V^{p,q} = F^p \cap (F^{p+1})^{\perp}.$$

This is only true if we have a polarization!

Complex Hodge structures

Some basic terminology...

Recall that a Hodge structure of weight n on a complex vector space V is simply a decomposition

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A hermitian pairing $Q\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ is called a polarization if

1. The decomposition is orthogonal.
2. $\langle v, w \rangle = \sum_{p,q} (-1)^q Q(v^{p,q}, w^{p,q})$ is positive definite.

The Hodge structure is determined by the Hodge filtration

$$F^p = V^{p,q} \oplus V^{p+1,q-1} \oplus V^{p+2,q-2} \oplus \cdots$$

because of the identity

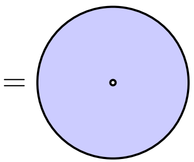
$$V^{p,q} = F^p \cap (F^{p+1})^{\perp}.$$

This is only true if we have a polarization!

The period mapping

Consider a polarized variation of Hodge structure on

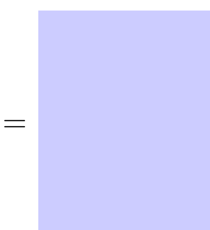
$$\Delta^* = \left\{ t \in \mathbb{C} \;\middle|\; 0 < |t| < 1 \right\} =$$



It can be described by its period mapping $\Phi\colon \mathbb{H} \to D$.

Pull back to the universal covering space

$$\mathbb{H} = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z < 0 \right\} =$$



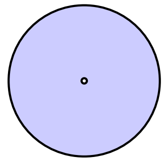
Let V be the vector space of multivalued flat sections.

- Hermitian pairing Q on V (from the polarization).
- Monodromy operator $T \in \mathrm{O}(V, Q)$ (from $z \mapsto z + 2\pi i$)
- At each point $z \in \mathbb{H}$, we get a Hodge structure of weight n on V, polarized by the hermitian pairing Q.
- We denote its Hodge filtration by the symbol $\Phi(z)$.

The period mapping

Consider a polarized variation of Hodge structure on

$$\Delta^* = \left\{ t \in \mathbb{C} \;\middle|\; 0 < |t| < 1 \right\} =$$



It can be described by its period mapping $\Phi \colon \mathbb{H} \to D$.

Pull back to the universal covering space

$$\mathbb{H} = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z < 0 \right\} =$$

Let V be the vector space of multivalued flat sections.

- Hermitian pairing Q on V (from the polarization).
- Monodromy operator $T \in \mathrm{O}(V, Q)$ (from $z \mapsto z + 2\pi i$)
- At each point $z \in \mathbb{H}$, we get a Hodge structure of weight n on V, polarized by the hermitian pairing Q.
- We denote its Hodge filtration by the symbol $\Phi(z)$.

The period mapping

The period domain D and its so-called compact dual $\check{D}$ are both complex manifolds and homogeneous spaces:

- $\check{D}$ parametrizes filtrations on V (with $\dim F^p$ fixed)
- $\check{D}$ is homogeneous under the complex Lie group $\mathrm{GL}(V)$
- $D \subseteq \check{D}$ parametrizes Hodge filtrations (of Hodge structures of weight n on V, polarized by Q)
- D is homogeneous under the real Lie group $\mathrm{O}(V, Q)$

The period mapping is holomorphic and

$$\Phi(z + 2\pi i) = T \cdot \Phi(z).$$

The period mapping

The period domain D and its so-called compact dual $\check{D}$ are both complex manifolds and homogeneous spaces:

- $\check{D}$ parametrizes filtrations on V (with $\dim F^p$ fixed)
- $\check{D}$ is homogeneous under the complex Lie group $\mathrm{GL}(V)$
- $D \subseteq \check{D}$ parametrizes Hodge filtrations (of Hodge structures of weight n on V, polarized by Q)
- D is homogeneous under the real Lie group $\mathrm{O}(V, Q)$

The period mapping is holomorphic and

$$\Phi(z + 2\pi i) = T \cdot \Phi(z).$$

The untwisted period mapping

Now we make the period mapping single-valued...

The eigenvalues of T satisfy $|\lambda| = 1$. Therefore

$$T = T_s \cdot T_u = e^{2\pi i S} \cdot e^{2\pi i N},$$

where $N \in \mathrm{End}(V)$ is nilpotent and $S \in \mathrm{End}(V)$ is semisimple with real eigenvalues (in a fixed interval of length < 1).

The expression $e^{-z(S+N)}\Phi(z)$ is invariant under $z \mapsto z + 2\pi i$:

$$\begin{aligned} e^{-(z+2\pi i)(S+N)}\Phi(z + 2\pi i) &= e^{-z(S+N)} e^{-2\pi i(S+N)} T\Phi(z) \\ &= e^{-z(S+N)}\Phi(z). \end{aligned}$$

It therefore descends to a holomorphic mapping

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)}\Phi(z),$$

called the untwisted period mapping.

The untwisted period mapping

Now we make the period mapping single-valued...

The eigenvalues of T satisfy $|\lambda| = 1$. Therefore

$$T = T_s \cdot T_u = e^{2\pi i S} \cdot e^{2\pi i N},$$

where $N \in \mathrm{End}(V)$ is nilpotent and $S \in \mathrm{End}(V)$ is semisimple with real eigenvalues (in a fixed interval of length < 1).

The expression $e^{-z(S+N)}\Phi(z)$ is invariant under $z \mapsto z + 2\pi i$:

$$\begin{aligned} e^{-(z+2\pi i)(S+N)}\Phi(z+2\pi i) &= e^{-z(S+N)}e^{-2\pi i(S+N)}T\Phi(z) \\ &= e^{-z(S+N)}\Phi(z). \end{aligned}$$

It therefore descends to a holomorphic mapping

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)}\Phi(z),$$

called the untwisted period mapping.

The nilpotent orbit theorem

One of the main results in the theory.

- Convergence
- Approximation

Nilpotent Orbit Theorem

1. Ψ_S extends holomorphically across the origin.
2. The period mapping Φ is close to the nilpotent orbit

$$\Phi_{nil} \colon \mathbb{C} \to \check{D}, \quad \Phi_{nil}(z) = e^{zN}F_{lim}.$$

More precisely, one has $\Phi_{nil}(z) \in D$ for $\mathrm{Re}\, z \ll 0$, and

$$d_D\big(\Phi(z), \Phi_{nil}(z)\big) \leq C|\mathrm{Re}\, z|^m e^{-\delta|\mathrm{Re}\, z|},$$

for certain constants $C > 0$, $m \in \mathbb{N}$, and $\delta > 0$.

Here $F_{lim} \in \check{D}$ is the so-called limiting Hodge filtration,

$$F_{lim} = \lim_{\mathrm{Re}\, z \to -\infty} e^{-zN}\Phi(z).$$

It is obtained from the filtration $\Psi_S(0) \in \check{D}$ by making it compatible with the decomposition into eigenspaces

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s).$$

The nilpotent orbit theorem

One of the main results in the theory.

- Convergence
- Approximation

Nilpotent Orbit Theorem

1. Ψ_S extends holomorphically across the origin.
2. The period mapping Φ is close to the nilpotent orbit

$$\Phi_{nil}\colon \mathbb{C} \to \check{D}, \quad \Phi_{nil}(z) = e^{zN} F_{lim}.$$

More precisely, one has $\Phi_{nil}(z) \in D$ for $\operatorname{Re} z \ll 0$, and

$$d_D\Big(\Phi(z), \Phi_{nil}(z)\Big) \leq C|\operatorname{Re} z|^m e^{-\delta|\operatorname{Re} z|},$$

for certain constants $C > 0$, $m \in \mathbb{N}$, and $\delta > 0$.

Here $F_{lim} \in \check{D}$ is the so-called limiting Hodge filtration,

$$F_{lim} = \lim_{\operatorname{Re} z \to -\infty} e^{-zN}\Phi(z).$$

It is obtained from the filtration $\Psi_S(0) \in \check{D}$ by making it compatible with the decomposition into eigenspaces

$$V = \bigoplus_{|\lambda|=1} E_\lambda(T_s).$$

Sketch of Schmid's original proof

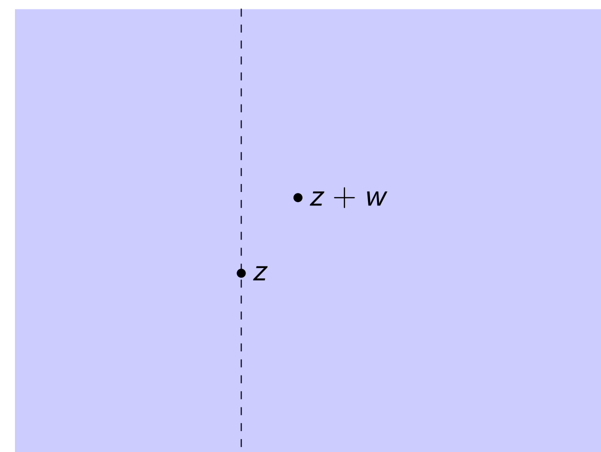
Schmid assumes that the eigenvalues of T are roots of unity.

- Automatic for Hodge structures defined over $\mathbb{Q}$.
- Reduce to $T = e^{2\pi i N}$ unipotent (by $t \mapsto t^m$).

With this assumption, the untwisted period mapping becomes

$$\Psi\colon \Delta^* \to \check{D}, \quad \Psi(e^z) = e^{-zN}\Phi(z).$$

Let me sketch the proof for why Ψ extends across the origin.



Sketch of Schmid's original proof

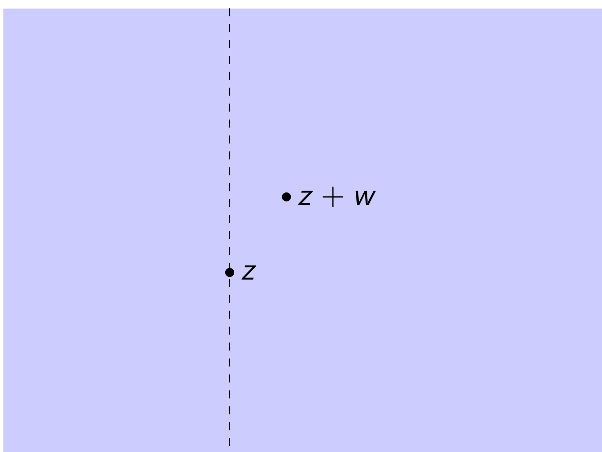
Schmid assumes that the eigenvalues of T are roots of unity.

- Automatic for Hodge structures defined over $\mathbb{Q}$.
- Reduce to $T = e^{2\pi i N}$ unipotent (by $t \mapsto t^m$).

With this assumption, the untwisted period mapping becomes

$$\Psi\colon \Delta^* \to \check{D}, \quad \Psi(e^z) = e^{-zN}\Phi(z).$$

Let me sketch the proof for why Ψ extends across the origin.



Sketch of Schmid's original proof

Idea: Show that the derivative of

$$\mathbb{H} \to \check{D}, \quad z \mapsto e^{-zN}\Phi(z)$$

is of order $|\mathrm{Re}\, z|^m e^{-\varepsilon|\mathrm{Re}\, z|}$, and then integrate.

For fixed $z \in \mathbb{H}$, consider the holomorphic mapping

$$\Phi_z\colon \mathbb{C} \to \check{D}, \quad w \mapsto e^{-wN}\Phi(z+w).$$

It is again invariant under $w \mapsto w + 2\pi i$.

Moreover, $\Phi_z(w)$ stays close to $\Phi(z)$ for $|\mathrm{Re}\, w| \leq \varepsilon|\mathrm{Re}\, z|$:

- Period mappings are "distance decreasing":

$$d_D\Big(\Phi(z+w), \Phi(z)\Big) \leq \frac{C|w|}{|\mathrm{Re}\, z|}$$

- The distance between $\Phi(z)$ and a fixed base point in D is bounded by $C|\mathrm{Re}\, z|^m$.
- The nilpotent operator N is also of order $|\mathrm{Re}\, z|^{-1}$.

Sketch of Schmid's original proof

Idea: Show that the derivative of

$$\mathbb{H} \to \check{D}, \quad z \mapsto e^{-zN}\Phi(z)$$

is of order $|\operatorname{Re} z|^m e^{-\varepsilon|\operatorname{Re} z|}$, and then integrate.

For fixed $z \in \mathbb{H}$, consider the holomorphic mapping

$$\Phi_z \colon \mathbb{C} \to \check{D}, \quad w \mapsto e^{-wN}\Phi(z+w).$$

It is again invariant under $w \mapsto w + 2\pi i$.

Moreover, $\Phi_z(w)$ stays close to $\Phi(z)$ for $|\operatorname{Re} w| \leq \varepsilon|\operatorname{Re} z|$:

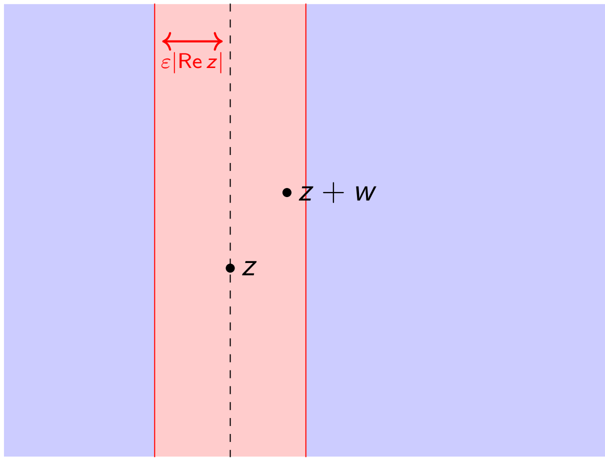
- Period mappings are "distance decreasing":

$$d_D\big(\Phi(z+w), \Phi(z)\big) \leq \frac{C|w|}{|\operatorname{Re} z|}$$

- The distance between $\Phi(z)$ and a fixed base point in D is bounded by $C|\operatorname{Re} z|^m$.
- The nilpotent operator N is also of order $|\operatorname{Re} z|^{-1}$.

Sketch of Schmid's original proof

Conclusion: $\Phi_z(w)$ stays in a coordinate neighborhood of $\Phi(z)$, for w in a vertical strip of width $\varepsilon|\operatorname{Re} z|$.



If $f \colon U_{\varepsilon|\operatorname{Re} z|} \to \mathbb{C}$ is holomorphic and $f(w+2\pi i) = f(w)$, then

$$|f'(0)| \leq e^{-\varepsilon|\operatorname{Re} z|} \sup_{w \in U_{\varepsilon|\operatorname{Re} z|}} |f(w)|.$$

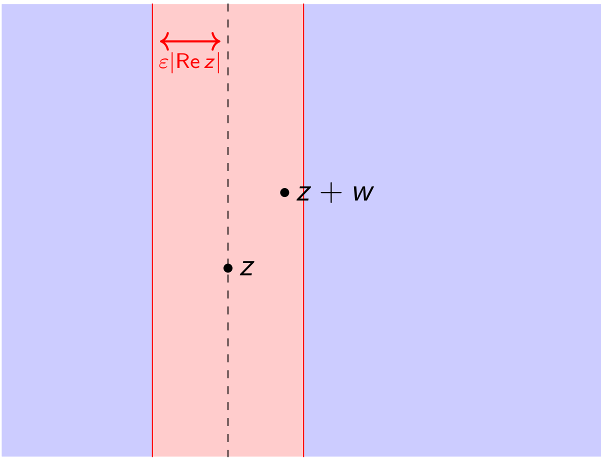
The derivative of Φ_z at $w = 0$ is therefore of order

$$C|\operatorname{Re} z|^m e^{-\varepsilon|\operatorname{Re} z|},$$

and the extra factor e^{-zN} can be handled by increasing m.

Sketch of Schmid's original proof

Conclusion: $\Phi_z(w)$ stays in a coordinate neighborhood of $\Phi(z)$, for w in a vertical strip of width $\varepsilon|\operatorname{Re} z|$.



If $f\colon U_{\varepsilon|\operatorname{Re} z|} \to \mathbb{C}$ is holomorphic and $f(w+2\pi i) = f(w)$, then

$$|f'(0)| \leq e^{-\varepsilon|\operatorname{Re} z|} \sup_{w \in U_{\varepsilon|\operatorname{Re} z|}} |f(w)|.$$

The derivative of Φ_z at $w = 0$ is therefore of order

$$C|\operatorname{Re} z|^m e^{-\varepsilon|\operatorname{Re} z|},$$

and the extra factor e^{-zN} can be handled by increasing m.

What happens when T is not unipotent?

Recall that, in the general case, we have

$$\Psi_S\colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)}\Phi(z),$$

where S is semisimple with real eigenvalues. The expression

$$w \mapsto e^{-w(S+N)}\Phi(z+w),$$

is still holomorphic and invariant under $w \mapsto w + 2\pi i$, but the eigenvalues of the extra factor e^{-wS} are of size $e^{-\alpha|\operatorname{Re} w|}$.

- This contributes terms that are exponential in $|\operatorname{Re} z|$.
- But the initial estimates only give $|\operatorname{Re} z|^{-1}$.

Therefore Schmid's argument breaks down completely.

What happens when T is not unipotent?

Recall that, in the general case, we have

$$\Psi_S \colon \Delta^* \to \check{D}, \quad \Psi_S(e^z) = e^{-z(S+N)}\Phi(z),$$

where S is semisimple with real eigenvalues. The expression

$$w \mapsto e^{-w(S+N)}\Phi(z+w),$$

is still holomorphic and invariant under $w \mapsto w + 2\pi i$, but the eigenvalues of the extra factor e^{-wS} are of size $e^{-\alpha|\mathrm{Re}\, w|}$.

- This contributes terms that are exponential in $|\mathrm{Re}\, z|$.
- But the initial estimates only give $|\mathrm{Re}\, z|^{-1}$.

Therefore Schmid's argument breaks down completely.

New proof

In the remainder of the talk, I will try to describe a new proof for the nilpotent orbit theorem that works in general.

Nilpotent Orbit Theorem

1. Ψ_S extends holomorphically across the origin.
2. The period mapping Φ is close to the nilpotent orbit

$$\Phi_{nil} \colon \mathbb{H} \to \check{D}, \quad \Phi_{nil}(z) = e^{zN} F_{lim}.$$

More precisely, one has $\Phi_{nil}(z) \in D$ for $\mathrm{Re}\, z \ll 0$, and

$$d_D\Big(\Phi(z), \Phi_{nil}(z)\Big) \leq C|\mathrm{Re}\, z|^m e^{-\delta|\mathrm{Re}\, z|},$$

for certain constants $C > 0$, $m \in \mathbb{N}$, and $\delta > 0$.

The main ingredients are

- Curvature properties of the Hodge metric.
- Hörmander's L^2-estimates.

New proof

In the remainder of the talk, I will try to describe a new proof for the nilpotent orbit theorem that works in general.

Nilpotent Orbit Theorem

1. Ψ_S extends holomorphically across the origin.
2. The period mapping Φ is close to the nilpotent orbit

$$\Phi_{nil} \colon \mathbb{H} \to \check{D}, \quad \Phi_{nil}(z) = e^{zN} F_{lim}.$$

More precisely, one has $\Phi_{nil}(z) \in D$ for $\operatorname{Re} z \ll 0$, and

$$d_D\Big(\Phi(z), \Phi_{nil}(z)\Big) \leq C |\operatorname{Re} z|^m e^{-\delta |\operatorname{Re} z|},$$

for certain constants $C > 0$, $m \in \mathbb{N}$, and $\delta > 0$.

The main ingredients are

- Curvature properties of the Hodge metric.
- Hörmander's L^2-estimates.

Complex variations of Hodge structure

First some background...

Let E be smooth vector bundle on a complex manifold, together with a flat connection $d \colon A^0(E) \to A^1(E)$.

A variation of Hodge structure (VHS) of weight n on E is a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

such that the flat connection d takes $A^0(E^{p,q})$ into

$$A^{1,0}(E^{p,q}) \oplus A^{1,0}(E^{p-1,q+1}) \oplus A^{0,1}(E^{p,q}) \oplus A^{0,1}(E^{p+1,q-1}).$$

This gives a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$. The operator

$$\theta \colon A^0(E^{p,q}) \to A^{1,0}(E^{p-1,q+1})$$

is called the Higgs field.

Note. The Higgs field is important because it is the derivative of the period mapping. (More on this later.)

Complex variations of Hodge structure

First some background...

Let E be smooth vector bundle on a complex manifold, together with a flat connection $d\colon A^0(E) \to A^1(E)$.

A variation of Hodge structure (VHS) of weight n on E is a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

such that the flat connection d takes $A^0(E^{p,q})$ into

$$A^{1,0}(E^{p,q}) \oplus A^{1,0}(E^{p-1,q+1}) \oplus A^{0,1}(E^{p,q}) \oplus A^{0,1}(E^{p+1,q-1}).$$

This gives a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$. The operator

$$\theta\colon A^0(E^{p,q}) \to A^{1,0}(E^{p-1,q+1})$$

is called the Higgs field.

Note. The Higgs field is important because it is the derivative of the period mapping. (More on this later.)

Complex variations of Hodge structure

Alternative holomorphic description:

- The operator $d'' = \bar{\partial} + \theta^*$ turns E into a holomorphic vector bundle $\mathscr{E}$.
- The Hodge bundles $F^pE = E^{p,q} \oplus E^{p+1,q-1} \oplus \cdots$ are holomorphic subbundles $F^p\mathscr{E}$.
- The operator $d' = \partial + \theta$ defines a holomorphic connection ∇ on $\mathscr{E}$.
- The Hodge bundles satisfy $\nabla(F^p\mathscr{E}) \subseteq \Omega^1 \otimes F^{p-1}\mathscr{E}$.
- The operator $\bar{\partial}$ turns $E^{p,q}$ into a holomorphic vector bundle $\mathscr{E}^{p,q}$, and $\mathscr{E}^{p,q} \cong F^p\mathscr{E}/F^{p+1}\mathscr{E}$.
- The Higgs field θ is just the holomorphic operator

$$F^p\mathscr{E}/F^{p+1}\mathscr{E} \to \Omega^1 \otimes F^{p-1}\mathscr{E}/F^p\mathscr{E}$$

induced by ∇.

Complex variations of Hodge structure

First some background. . .

Let E be smooth vector bundle on a complex manifold, together with a flat connection $d\colon A^0(E) \to A^1(E)$.

A variation of Hodge structure (VHS) of weight n on E is a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

such that the flat connection d takes $A^0(E^{p,q})$ into

$$A^{1,0}(E^{p,q}) \oplus A^{1,0}(E^{p-1,q+1}) \oplus A^{0,1}(E^{p,q}) \oplus A^{0,1}(E^{p+1,q-1}).$$

This gives a decomposition $d = \partial + \theta + \bar{\partial} + \theta^*$. The operator

$$\theta\colon A^0(E^{p,q}) \to A^{1,0}(E^{p-1,q+1})$$

is called the Higgs field.

Note. The Higgs field is important because it is the derivative of the period mapping. (More on this later.)

Complex variations of Hodge structure

A polarization of a VHS E is a hermitian pairing

$$Q\colon A^0(E) \otimes_{A^0} \overline{A^0(E)} \to A^0$$

that is flat with respect to d, such that

$$h_E(v, w) = \sum_{p+q=n} (-1)^q Q(v^{p,q}, w^{p,q})$$

is a positive definite hermitian metric on E, and the Hodge decomposition becomes orthogonal.

This hermitian metric on E is called the Hodge metric.

Complex variations of Hodge structure

A polarization of a VHS E is a hermitian pairing

$$Q\colon A^0(E) \otimes_{A^0} \overline{A^0(E)} \to A^0$$

that is flat with respect to d, such that

$$h_E(v, w) = \sum_{p+q=n} (-1)^q Q(v^{p,q}, w^{p,q})$$

is a positive definite hermitian metric on E, and the Hodge decomposition becomes orthogonal.

This hermitian metric on E is called the Hodge metric.

Curvature of the Hodge metric

The curvature operator of the Hodge metric on $E^{p,q}$ is

$$\Theta = -(\theta\theta^* + \theta^*\theta).$$

Unfortunately, the expression

$$h_E\Big(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}} u, u\Big) = h_E\Big(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u\Big) - h_E\Big(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u\Big)$$

is, in general, neither positive nor negative definite.

Basic Estimate (Simpson)

Viewing $\theta_{\partial/\partial t}$ as a section of $\mathrm{End}(E)$, one has

$$h_{\mathrm{End}(E)}\Big(\theta_{\partial/\partial t}, \theta_{\partial/\partial t}\Big) \leq \frac{C_0}{|t|^2(-\log|t|)^2}.$$

It follows that metrics of the form $h_E \cdot |t|^a(-\log|t|)^b$ have

- positive curvature for $b \gg 0$,
- negative curvature for $b \ll 0$.

This is the main technical point! (Cornalba-Griffiths, Simpson)

Curvature of the Hodge metric

The curvature operator of the Hodge metric on $E^{p,q}$ is

$$\Theta = -(\theta\theta^* + \theta^*\theta).$$

Unfortunately, the expression

$$h_E\Big(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}} u, u\Big) = h_E\Big(\theta_{\partial/\partial t} u, \theta_{\partial/\partial t} u\Big) - h_E\Big(\theta^*_{\partial/\partial \bar{t}} u, \theta^*_{\partial/\partial \bar{t}} u\Big)$$

is, in general, neither positive nor negative definite.

Basic Estimate (Simpson)

Viewing $\theta_{\partial/\partial t}$ as a section of $\mathrm{End}(E)$, one has

$$h_{\mathrm{End}(E)}\Big(\theta_{\partial/\partial t}, \theta_{\partial/\partial t}\Big) \leq \frac{C_0}{|t|^2(-\log|t|)^2}.$$

It follows that metrics of the form $h_E \cdot |t|^a(-\log|t|)^b$ have

- positive curvature for $b \gg 0$,
- negative curvature for $b \ll 0$.

This is the main technical point! (Cornalba-Griffiths, Simpson)

Broad outline of the new proof

We have an induced variation of Hodge structure

$$\mathrm{End}(E) = \bigoplus_k \mathrm{End}(E)^{k,-k}.$$

View $\theta_{\partial/\partial t}$ as a section of $\mathrm{End}(E)^{-1,1}$; holomorphic because

$$[\bar{\partial}, \theta] = 0.$$

But as a section of $\mathrm{End}(E)$, it is **not** holomorphic:

$$[d'', \theta] = [\bar{\partial} + \theta^*, \theta] = [\theta^*, \theta] \neq 0.$$

Step 1. We lift $t\theta_{\partial/\partial t}$ to a holomorphic section ϑ of the Hodge bundle $F^{-1}\,\mathrm{End}(E)$, such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0\,\mathrm{End}(E).$$

We also obtain an estimate of the form

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(\vartheta, \vartheta)|t|^a(-\log|t|)^b d\mu \leq C.$$

For this, we use Hörmander's L^2-estimates; the key point is that $h_{\mathrm{End}(E)} \cdot |t|^a(-\log|t|)^b$ has positive curvature for $b \gg 0$.

Broad outline of the new proof

We have an induced variation of Hodge structure

$$\mathrm{End}(E) = \bigoplus_k \mathrm{End}(E)^{k,-k}.$$

View $\theta_{\partial/\partial t}$ as a section of $\mathrm{End}(E)^{-1,1}$; holomorphic because

$$[\bar{\partial}, \theta] = 0.$$

But as a section of $\mathrm{End}(E)$, it is **not** holomorphic:

$$[d'', \theta] = [\bar{\partial} + \theta^*, \theta] = [\theta^*, \theta] \neq 0.$$

Step 1. We lift $t\theta_{\partial/\partial t}$ to a holomorphic section ϑ of the Hodge bundle $F^{-1}\,\mathrm{End}(E)$, such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0\,\mathrm{End}(E).$$

We also obtain an estimate of the form

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(\vartheta, \vartheta)|t|^a(-\log|t|)^b d\mu \leq C.$$

For this, we use Hörmander's L^2-estimates; the key point is that $h_{\mathrm{End}(E)} \cdot |t|^a(-\log|t|)^b$ has positive curvature for $b \gg 0$.

Broad outline of the new proof

Step 2. Pulling back to $\mathbb{H}$, we get a holomorphic mapping

$$\vartheta\colon \mathbb{H} \to \mathrm{End}(V)$$

such that $\vartheta(z + 2\pi i) = T\vartheta(z)T^{-1}$. Untwisting gives

$$B\colon \Delta^* \to \mathrm{End}(V), \quad B(e^z) = e^{-z(S+N)}\vartheta(z)e^{z(S+N)}.$$

For suitable $a > -2$ and $b \gg 0$, the L^2-estimate implies that B extends holomorphically across the origin.

Step 3. The tangent space to $\check{D}$ at the point $\Phi(z) \in \check{D}$ is

$$T_{\Phi(z)}\check{D} \cong \mathrm{End}(V)/F^0\,\mathrm{End}(V)_{\Phi(z)}.$$

The derivative of the period mapping $\Phi\colon \mathbb{H} \to \check{D}$ is

$$\theta_{\partial/\partial z} \mod F^0\,\mathrm{End}(V)_{\Phi(z)}.$$

Therefore the derivative of $z \mapsto \Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$ is

$$\begin{aligned} &e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \\ &\qquad \equiv B(e^z) - (S+N) \mod F^0\,\mathrm{End}(V)_{\Psi_S(e^z)}. \end{aligned}$$

The operator on the right-hand side is holomorphic!

Broad outline of the new proof

Step 2. Pulling back to $\mathbb{H}$, we get a holomorphic mapping

$$\vartheta\colon \mathbb{H} \to \operatorname{End}(V)$$

such that $\vartheta(z+2\pi i) = T\vartheta(z)T^{-1}$. Untwisting gives

$$B\colon \Delta^* \to \operatorname{End}(V), \quad B(e^z) = e^{-z(S+N)}\vartheta(z)e^{z(S+N)}.$$

For suitable $a > -2$ and $b \gg 0$, the L^2-estimate implies that B extends holomorphically across the origin.

Step 3. The tangent space to $\check{D}$ at the point $\Phi(z) \in \check{D}$ is

$$T_{\Phi(z)}\check{D} \cong \operatorname{End}(V)/F^0\operatorname{End}(V)_{\Phi(z)}.$$

The derivative of the period mapping $\Phi\colon \mathbb{H} \to \check{D}$ is

$$\theta_{\partial/\partial z} \mod F^0\operatorname{End}(V)_{\Phi(z)}.$$

Therefore the derivative of $z \mapsto \Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$ is

$$\begin{aligned} &e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \\ &\qquad \equiv B(e^z) - (S+N) \mod F^0\operatorname{End}(V)_{\Psi_S(e^z)}. \end{aligned}$$

The operator on the right-hand side is holomorphic!

Broad outline of the new proof

Step 4. Let $g\colon \mathbb{H} \to GL(V)$ be the unique (holomorphic) solution of the initial value problem

$$g'(z) = \Big(B(e^z) - (S+N)\Big)\cdot g(z), \quad g(-1) = \mathrm{id}.$$

Then $g(z)^{-1}\Psi_S(e^z)$ is constant, and therefore

$$\Psi_S(e^z) = g(z)\cdot\Psi_S(e^{-1}).$$

The ODE has a regular singular point at $t = 0$, and so

$$g(z) = M(e^z)\cdot e^{Az}$$

with $M\colon \Delta^* \to GL(V)$ meromorphic and $A \in \operatorname{End}(V)$. Since $\Psi_S(e^z)$ is single-valued, we get

$$\Psi_S(t) = M(t)\cdot\Psi_S(e^{-1}),$$

and because $\check{D}$ is projective, it follows that Ψ_S extends.

Disadvantage of this proof

Unlike Schmid's proof, the argument so far does not give a good estimate for how quickly $\Psi_S(t)$ converges to $\Psi_S(0)$.

Broad outline of the new proof

Step 4. Let $g\colon \mathbb{H} \to \mathrm{GL}(V)$ be the unique (holomorphic) solution of the initial value problem

$$g'(z) = \Big(B(e^z) - (S+N)\Big) \cdot g(z), \quad g(-1) = \mathrm{id}\,.$$

Then $g(z)^{-1}\Psi_S(e^z)$ is constant, and therefore

$$\Psi_S(e^z) = g(z) \cdot \Psi_S(e^{-1}).$$

The ODE has a regular singular point at $t = 0$, and so

$$g(z) = M(e^z) \cdot e^{Az}$$

with $M\colon \Delta^* \to \mathrm{GL}(V)$ meromorphic and $A \in \mathrm{End}(V)$. Since $\Psi_S(e^z)$ is single-valued, we get

$$\Psi_S(t) = M(t) \cdot \Psi_S(e^{-1}),$$

and because $\check{D}$ is projective, it follows that Ψ_S extends.

Disadvantage of this proof

Unlike Schmid's proof, the argument so far does not give a good estimate for how quickly $\Psi_S(t)$ converges to $\Psi_S(0)$.

Broad outline of the proof

Now we derive such estimates...

Step 5. Since $\Psi_S\colon \Delta \to \check{D}$ is holomorphic, the derivative of

$$z \mapsto \Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$$

is of order $|e^z| = e^{-|\mathrm{Re}\, z|}$.

Recall that the derivative is also equal to

$$e^{-z(S+N)}\theta_{\partial/\partial z} e^{z(S+N)} - (S+N) \mod F^0 \,\mathrm{End}(V)_{\Psi_S(e^z)}.$$

By analyzing this expression, one can show that

$$\|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z)} + \sum_{k\leq -2} \|N^{k,-k}\|^2_{\Phi(z)} \leq C|\mathrm{Re}\, z|^{2m} e^{-2\delta|\mathrm{Re}\, z|}$$
$$\sum_{k\leq -1} \|P_\lambda^{k,-k}\|^2_{\Phi(z)} \leq C|\mathrm{Re}\, z|^{2m} e^{-2\delta|\mathrm{Re}\, z|}$$

for $|\mathrm{Re}\, z| \gg 0$. Here $P_\lambda \in \mathrm{End}(V)$ is the projection to the λ-eigenspace of T_s, and $\|-\|_{\Phi(z)}$ is the Hodge norm.

- The constants $\delta > 0$ and $m \in \mathbb{N}$ only depend on (V, T).
- But no information about the value of $C > 0$!
- Also no information about how big $|\mathrm{Re}\, z|$ needs to be!

Broad outline of the proof

Now we derive such estimates...

Step 5. Since $\Psi_S \colon \Delta \to \check{D}$ is holomorphic, the derivative of

$$z \mapsto \Psi_S(e^z) = e^{-z(S+N)}\Phi(z)$$

is of order $|e^z| = e^{-|\operatorname{Re} z|}$.

Recall that the derivative is also equal to

$$e^{-z(S+N)}\theta_{\partial/\partial z}e^{z(S+N)} - (S+N) \mod F^0 \operatorname{End}(V)_{\Psi_S(e^z)}.$$

By analyzing this expression, one can show that

$$\|\theta_{\partial/\partial z} - N^{-1,1}\|^2_{\Phi(z)} + \sum_{k\leq -2} \|N^{k,-k}\|^2_{\Phi(z)} \leq C|\operatorname{Re} z|^{2m}e^{-2\delta|\operatorname{Re} z|}$$

$$\sum_{k\leq -1} \|P_\lambda^{k,-k}\|^2_{\Phi(z)} \leq C|\operatorname{Re} z|^{2m}e^{-2\delta|\operatorname{Re} z|}$$

for $|\operatorname{Re} z| \gg 0$. Here $P_\lambda \in \operatorname{End}(V)$ is the projection to the λ-eigenspace of T_s, and $\|-\|_{\Phi(z)}$ is the Hodge norm.

- The constants $\delta > 0$ and $m \in \mathbb{N}$ only depend on (V, T).
- But no information about the value of $C > 0$!
- Also no information about how big $|\operatorname{Re} z|$ needs to be!

Broad outline of the proof

Step 6. We can use the maximum principle to show that the above inequalities actually hold for

$$\operatorname{Re} z \leq x < 0,$$

with a constant $C > 0$ that only depends on x, $\operatorname{rk} E$, and on the Hodge norms $\|N\|_{\Phi(-1)}$ and $\|P_\lambda\|_{\Phi(-1)}$.

The two operators

$$\theta_{\partial/\partial z} - \sum_{k\leq -1} N^{k,-k} \quad \text{and} \quad \sum_{k\leq -1} P_\lambda^{k,-k}$$

are holomorphic sections of the bundle $\operatorname{End}(E)/F^0\operatorname{End}(E)$.

The key point is that the metric $h_{\operatorname{End}(E)} \cdot (-\log|t|)^b$ on this bundle has negative curvature for $b \ll 0$.

Negative curvature (in the sense of Griffiths)

A metric h on a bundle E has negative curvature iff

$$\log h(s,s)$$

is (pluri-)subharmonic for every holomorphic section s.

Broad outline of the proof

Step 6. We can use the maximum principle to show that the above inequalities actually hold for

$$\operatorname{Re} z \leq x < 0,$$

with a constant $C > 0$ that only depends on x, $\operatorname{rk} E$, and on the Hodge norms $\|N\|_{\Phi(-1)}$ and $\|P_\lambda\|_{\Phi(-1)}$.

The two operators

$$\theta_{\partial/\partial z} - \sum_{k \leq -1} N^{k,-k} \quad \text{and} \quad \sum_{k \leq -1} P_\lambda^{k,-k}$$

are holomorphic sections of the bundle $\operatorname{End}(E)/F^0 \operatorname{End}(E)$.

The key point is that the metric $h_{\operatorname{End}(E)} \cdot (-\log|t|)^b$ on this bundle has negative curvature for $b \ll 0$.

Negative curvature (in the sense of Griffiths)

A metric h on a bundle E has negative curvature iff

$$\log h(s,s)$$

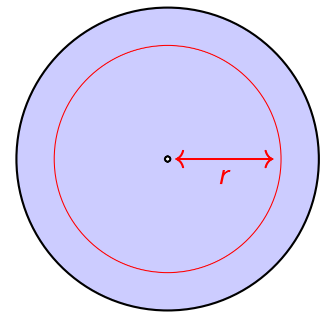
is (pluri-)subharmonic for every holomorphic section s.

The reasoning in a toy example

Let φ be a subharmonic function on Δ^*, and suppose that

$$\varphi \leq C - \delta \log|t|^2$$

on a small neighborhood of the origin, for some $C > 0$.



Then $\varphi + \delta \log|t|^2$ is subharmonic and bounded near $t = 0$. By the maximum principle, it achieves its maximum over any disk $|t| \leq r$ somewhere along the boundary.

From this, we get the more precise estimate

$$\varphi + \delta \log|t|^2 \leq \max_{|t|=r} \varphi(t) + \delta \log r^2,$$

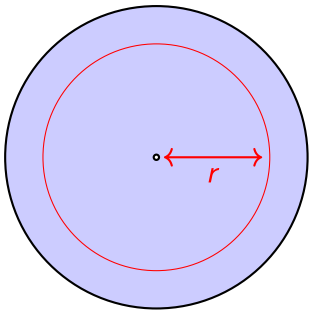
and so we can replace C by an explicit constant!

The reasoning in a toy example

Let φ be a subharmonic function on Δ^*, and suppose that

$$\varphi \leq C - \delta \log|t|^2$$

on a small neighborhood of the origin, for some $C > 0$.



Then $\varphi + \delta \log|t|^2$ is subharmonic and bounded near $t = 0$. By the maximum principle, it achieves its maximum over any disk $|t| \leq r$ somewhere along the boundary.

From this, we get the more precise estimate

$$\varphi + \delta \log|t|^2 \leq \max_{|t|=r} \varphi(t) + \delta \log r^2,$$

and so we can replace C by an explicit constant!

Broad outline of the proof

Step 7. Recall the limiting Hodge filtration

$$F_{lim} = \lim_{\operatorname{Re} z \to -\infty} e^{-zN}\Phi(z) \in \check{D}.$$

The estimates in Step 5 control the derivative of the curve

$$[0, \infty) \to \check{D}, \quad x \mapsto e^{-xN}\Phi(z + x),$$

which connects $\Phi(z)$ and $\Phi_{nil}(z) = e^{zN}F_{lim}$.

After integration, we obtain a distance estimate of the form

$$d_D\Big(\Phi(z), \Phi_{nil}(z)\Big) \leq C|\operatorname{Re} z|^m e^{-\delta|\operatorname{Re} z|}$$

(with different constants).

Broad outline of the proof

Step 7. Recall the limiting Hodge filtration

$$F_{lim} = \lim_{\operatorname{Re} z \to -\infty} e^{-zN}\Phi(z) \in \check{D}.$$

The estimates in Step 5 control the derivative of the curve

$$[0, \infty) \to \check{D}, \quad x \mapsto e^{-xN}\Phi(z + x),$$

which connects $\Phi(z)$ and $\Phi_{nil}(z) = e^{zN}F_{lim}$.

After integration, we obtain a distance estimate of the form

$$d_D\Big(\Phi(z), \Phi_{nil}(z)\Big) \leq C|\operatorname{Re} z|^m e^{-\delta|\operatorname{Re} z|}$$

(with different constants).

More details about Step 1

We have an induced variation of Hodge structure

$$\operatorname{End}(E) = \bigoplus_k \operatorname{End}(E)^{k,-k}.$$

View $\theta_{\partial/\partial t}$ as a section of $\operatorname{End}(E)^{-1,1}$, holomorphic because

$$[\bar\partial, \theta] = 0.$$

But as a section of $\operatorname{End}(E)$, it is **not** holomorphic:

$$[d'', \theta] = [\bar\partial + \theta^*, \theta] = [\theta^*, \theta] \neq 0.$$

Step 1. We lift $t\theta_{\partial/\partial t}$ to a holomorphic section ϑ of the Hodge bundle $F^{-1}\operatorname{End}(E)$, such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0 \operatorname{End}(E).$$

We also obtain an estimate of the form

$$\int_{\Delta^*} h_{\operatorname{End}(E)}(\vartheta, \vartheta)|t|^a(-\log|t|)^b d\mu \leq C.$$

For this, we use Hörmander's L^2-estimates; the key point is that $h_{\operatorname{End}(E)} \cdot |t|^a(-\log|t|)^b$ has positive curvature for $b \gg 0$.

More details about Step 1

We have an induced variation of Hodge structure

$$\mathrm{End}(E) = \bigoplus_k \mathrm{End}(E)^{k,-k}.$$

View $\theta_{\partial/\partial t}$ as a section of $\mathrm{End}(E)^{-1,1}$, holomorphic because

$$[\bar\partial, \theta] = 0.$$

But as a section of $\mathrm{End}(E)$, it is **not** holomorphic:

$$[d'', \theta] = [\bar\partial + \theta^*, \theta] = [\theta^*, \theta] \neq 0.$$

Step 1. We lift $t\theta_{\partial/\partial t}$ to a holomorphic section ϑ of the Hodge bundle $F^{-1}\,\mathrm{End}(E)$, such that

$$\vartheta \equiv t\theta_{\partial/\partial t} \mod F^0\,\mathrm{End}(E).$$

We also obtain an estimate of the form

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(\vartheta, \vartheta)|t|^a(-\log|t|)^b d\mu \leq C.$$

For this, we use Hörmander's L^2-estimates; the key point is that $h_{\mathrm{End}(E)} \cdot |t|^a(-\log|t|)^b$ has positive curvature for $b \gg 0$.

More details about Step 1

The operator $[d'', -]$ makes $\mathrm{End}(E)$ into a holomorphic vector bundle, and the derivative of $t\theta_{\partial/\partial t}$ is

$$f = [d''_{\partial/\partial \bar t}, t\theta_{\partial/\partial t}] = [\theta^*_{\partial/\partial \bar t}, t\theta_{\partial/\partial t}] \in A^0\Big(\Delta^*, \mathrm{End}(E)^{0,0}\Big),$$

It is therefore enough to solve the $\bar\partial$-equation

$$d''_{\partial/\partial \bar t} u = f$$

for $u \in A^0\Big(\Delta^*, F^0\,\mathrm{End}(E)\Big)$, subject to the condition that

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(u, u)|t|^a(-\log|t|)^b d\mu \leq C.$$

Then $\vartheta = t\theta_{\partial/\partial t} - u$ is the desired holomorphic section.

The input is again Simpson's basic estimate

$$h_{\mathrm{End}(E)}\Big(\theta_{\partial/\partial t}, \theta_{\partial/\partial t}\Big) \leq \frac{C_0}{|t|^2(-\log|t|)^2},$$

because it implies, for $a > -2$ and $b \geq 2$, that

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(f, f)|t|^{a+2}(-\log|t|)^{b+2} d\mu \leq \int_{\Delta^*} \frac{2C_0^2|t|^2}{|t|^4(-\log|t|)^4}|t|^{a+2}(-\log|t|)^{b+2} d\mu < +\infty.$$

More details about Step 1

The operator $[d'', -]$ makes $\mathrm{End}(E)$ into a holomorphic vector bundle, and the derivative of $t\theta_{\partial/\partial t}$ is

$$f = [d''_{\partial/\partial\bar{t}}, t\theta_{\partial/\partial t}] = [\theta^*_{\partial/\partial\bar{t}}, t\theta_{\partial/\partial t}] \in A^0\Big(\Delta^*, \mathrm{End}(E)^{0,0}\Big),$$

It is therefore enough to solve the $\bar{\partial}$-equation

$$d''_{\partial/\partial\bar{t}} u = f$$

for $u \in A^0\Big(\Delta^*, F^0 \,\mathrm{End}(E)\Big)$, subject to the condition that

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(u,u)|t|^a(-\log|t|)^b d\mu \leq C.$$

Then $\vartheta = t\theta_{\partial/\partial t} - u$ is the desired holomorphic section.

The input is again Simpson's basic estimate

$$h_{\mathrm{End}(E)}\Big(\theta_{\partial/\partial t}, \theta_{\partial/\partial t}\Big) \leq \frac{C_0}{|t|^2(-\log|t|)^2},$$

because it implies, for $a > -2$ and $b \geq 2$, that

$$\int_{\Delta^*} h_{\mathrm{End}(E)}(f,f)|t|^{a+2}(-\log|t|)^{b+2} d\mu \leq \int_{\Delta^*} \frac{2C_0^2|t|^2}{|t|^4(-\log|t|)^4}|t|^{a+2}(-\log|t|)^{b+2} d\mu < +\infty.$$

Hörmander's L^2-estimates in one dimension

Let E be a smooth vector bundle (on a domain $\Omega \subseteq \mathbb{C}$), with holomorphic structure given by $d'' \colon A^0(E) \to A^{0,1}(E)$.

Given $f \in A^0(\Omega, E)$, we want to solve the $\bar{\partial}$-equation

$$d''_{\partial/\partial\bar{t}} u = f.$$

Suppose E has a hermitian metric h with positive curvature: there is a positive function ρ such that

$$h\Big(\Theta_{\partial/\partial t \wedge \partial/\partial\bar{t}}\, \alpha, \alpha\Big) \geq \rho^2 h(\alpha, \alpha)$$

for every compactly supported $\alpha \in A^0_c(\Omega, E)$.

L^2-Estimates (Hörmander)

Under these assumptions, the $\bar{\partial}$-equation $d''_{\partial/\partial\bar{t}} u = f$ has a solution $u \in A^0(\Omega, E)$ that satisfies the L^2-estimate

$$\int_\Omega h(u,u) d\mu \leq \int_\Omega \frac{1}{\rho^2} h(f,f) d\mu,$$

provided that the right-hand side is finite.

In our setting, the metric $h_{\mathrm{End}(E)} \cdot |t|^a(-\log|t|)^b$ with $b \gg 0$ meets these conditions with $1/\rho^2 = |t|^2(-\log|t|)^2$.

(Explains the $a+2$ and $b+2$ on the previous slide...)

Hörmander's L^2-estimates in one dimension

Let E be a smooth vector bundle (on a domain $\Omega \subseteq \mathbb{C}$), with holomorphic structure given by $d''\colon A^0(E) \to A^{0,1}(E)$.
Given $f \in A^0(\Omega, E)$, we want to solve the $\bar{\partial}$-equation

$$d''_{\partial/\partial\bar{t}} u = f.$$

Suppose E has a hermitian metric h with positive curvature: there is a positive function ρ such that

$$h\Big(\Theta_{\partial/\partial t \wedge \partial/\partial \bar{t}}\, \alpha, \alpha\Big) \geq \rho^2 h(\alpha, \alpha)$$

for every compactly supported $\alpha \in A^0_c(\Omega, E)$.

L^2-Estimates (Hörmander)

Under these assumptions, the $\bar{\partial}$-equation $d''_{\partial/\partial\bar{t}} u = f$ has a solution $u \in A^0(\Omega, E)$ that satisfies the L^2-estimate

$$\int_\Omega h(u,u)\, d\mu \leq \int_\Omega \frac{1}{\rho^2} h(f,f)\, d\mu,$$

provided that the right-hand side is finite.

In our setting, the metric $h_{\mathrm{End}(E)} \cdot |t|^a (-\log|t|)^b$ with $b \gg 0$ meets these conditions with $1/\rho^2 = |t|^2(-\log|t|)^2$.

(Explains the $a+2$ and $b+2$ on the previous slide...)