Equivariant birational geometry of $\mathbb{P}^3$

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Hans Frederick Blichfeldt

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Blichfeldt's short biography

Blichfeldt was born on January 9, 1873 in Iller (Denmark).

In 1888, his family moved to the US.

Blichfeldt worked for several years as a lumberman and a railway worker.

In 1894, Blichfeldt became a student at Stanford University, which did not charge tuition at the time.

He got BSc degree in 1896, and MSc degree in 1897.

Then Blichfeldt moved to Leipzig University and completed a Ph.D. there in 1898. His advisor was Sophus Lie.

Returning to Stanford, Blichfeldt became a full professor by 1913, and HoS from 1927 until 1938.

Blichfeldt represented the US at the ICM in 1932 and 1936.

Blichfeldt served as vice-president of the AMS in 1912.

Blichfeldt remained unmarried throughout his life.

He died on November 16, 1945 in Palo Alto, California.

Finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$

In 1917, Blichfeldt wrote his magnum opus

Finite Collineation Groups

which classifies finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$.

This book is a standard reference for finite collineation groups.

Blichfeldt has split finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$ into 4 classes.

In geometric language, these classes can be described as follows:

(I) intransitive groups fix a point or leave a line invariant,

(II) transitive groups are groups that are not intransitive,

(III) imprimitive groups are transitive groups that
- either leave a union of two skew lines invariant,
- or have an orbit of length 4 (monomial subgroups),

(IV) primitive groups are transitive groups that are not imprimitive.

Theorem

$\mathrm{PGL}_4(\mathbb{C})$ *contains finitely many primitive finite subgroups.*

Equivariant birational rigidity

Fix a finite subgroup $G \subset \mathrm{PGL}_4(\mathbb{C})$.

Problem

Describe G-birational maps from $\mathbb{P}^3$ to G-Mori fibre spaces.

Problem

Describe G-Sarkisov links that start at $\mathbb{P}^3$.

If there are no G–Sarkisov links that start at $\mathbb{P}^3$, we say that

$\mathbb{P}^3$ is *G-birationally super-rigid*.

If every G–Sarkisov link that starts at $\mathbb{P}^3$ ends at $\mathbb{P}^3$, we say that

$\mathbb{P}^3$ is *G-birationally rigid*.

$\mathbb{P}^3$ is G-birationally rigid if and only if $\mathbb{P}^3$ cannot be G-birationally transformed into other G-Mori fiber space.

When is $\mathbb{P}^3$ G-birationally rigid?

Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$.

Then $\mathbb{P}^3$ is G-birationally rigid if and only if the following conditions are satisfied:

1. $\nexists$ G-map $\mathbb{P}^3 \dashrightarrow S$ whose general fiber is a rational curve;
2. $\nexists$ G-map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ whose general fiber is a rational surface;
3. $\nexists$ G-map $\mathbb{P}^3 \dashrightarrow X$ such that
   - X is a Fano threefold with terminal singularities,
   - the G-invariant class group of X is of rank 1,
   - X is not G-isomorphic to $\mathbb{P}^3$.

Theorem (Cheltsov,Shramov)

$\mathbb{P}^3$ is G-birationally rigid if and only if G is a primitive group that is not isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$.

Intransitive groups

Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$.

If G fixes a point, then there exists the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\pi}{\swarrow} & & \overset{\phi}{\searrow} & \\ \mathbb{P}^3 & & \overset{\rho}{\dashrightarrow} & & \mathbb{P}^2 \end{array}$$

where π is the blow up of the G-fixed point, ρ is a projection from the G-fixed point, and ϕ is a $\mathbb{P}^1$-bundle.

If G leaves line invariant, then there exists the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\pi}{\swarrow} & & \overset{\phi}{\searrow} & \\ \mathbb{P}^3 & & \overset{\rho}{\dashrightarrow} & & \mathbb{P}^1 \end{array}$$

where π is the blow up of the G-invariant line, ρ is a projection from the G-invariant line, and ϕ is a $\mathbb{P}^2$-bundle.

Imprimitive groups

Let G be a finite imprimitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$.

If G leaves invariant a pair of skew lines in $\mathbb{P}^3$, there is diagram

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^3 & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where π is the blow up of these lines, and ϕ is a $\mathbb{P}^1$-bundle.

Suppose that $\mathbb{P}^3$ contains a G-orbit Σ_4 of length 4.
Let $\mathcal{M}$ be the linear system that consists of sextic surfaces in $\mathbb{P}^3$ singular along each line passing through two points in Σ_4.
Then $\mathcal{M}$ defines a map $\psi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$. Let $X_{24} = \overline{\mathrm{im}(\psi)}$. Then

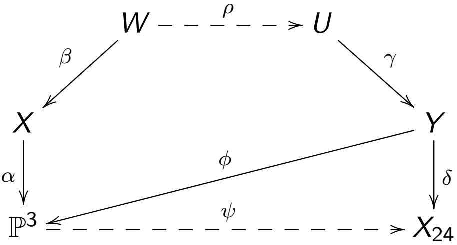
(i) the induced map $\mathbb{P}^3 \dashrightarrow X_{24}$ is G-birational,

(ii) $X_{24} \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 / \langle \tau \rangle$ for an involution τ that fixes 8 points,

(iii) the Fano threefold X_{24} is a G-Mori fiber space over a point.

Toric Fano–Enriques threefold of degree 24

Let $\Sigma_4 = \{P_1, P_2, P_3, P_3\}$, let L_{ij} be the line in $\mathbb{P}^3$ that passes though P_i and P_j, let Π_i be the plane in $\mathbb{P}^3$ that passes through all points in Σ_4 except for P_i. Let $\alpha\colon X \to \mathbb{P}^3$ be the blow up of the points P_i, let $\beta\colon W \to X$ be the blow up of the proper transform of the lines L_{ij}. There is a commutative diagram



where ρ is a composition of 12 Atiyah flops, the map γ is the contraction of the proper transforms of the α-exceptional surfaces to singular points of type $\frac{1}{2}(1,1,1)$, ϕ is the G-birational extraction of the curve $\sum L_{ij}$, and δ is the contraction of the proper transform of the planes Π_i to singular points of type $\frac{1}{2}(1,1,1)$.

Primitive subgroups isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$

There are two primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to $\mathfrak{A}_5$.

One of them leaves a quadric surface invariant. Its action on $\mathbb{P}^3$ comes from the irreducible four-dimensional representation of the icosahedral group.

Another one preserves a twisted cubic curve, so that its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of the binary icosahedral group.

There are two primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to $\mathfrak{S}_5$.

One of them preserves a quadric surface, and its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of the group $\mathfrak{S}_5$.

Another one leaves invariant a pair of disjoint twisted cubic curves. Its action on $\mathbb{P}^3$ comes from an irreducible four-dimensional representation of a central extension of the group $\mathfrak{S}_5$.

First primitive subgroups isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$

Let G be a primitive finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that $G \cong \mathfrak{S}_5$ or $G \cong \mathfrak{A}_5$, and there exists a G-invariant quadric surface in $\mathbb{P}^3$.

Then $\mathbb{P}^3$ contains a G-orbit Σ_5 of length 5.

Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of this orbit.

Then there exists a G-commutative diagram

$$\begin{array}{ccccccc} & & X & \overset{\chi}{\dashrightarrow} & Y & & \\ & \swarrow^{\pi} & & \searrow^{\alpha} \quad \swarrow^{\beta} & & \searrow^{\phi} & \\ \mathbb{P}^3 & & & V_3 & & & S \end{array}$$

Here α is the contraction of the proper transforms of the 10 lines in $\mathbb{P}^3$ passing through pairs of points in Σ_5, χ is a composition of Atiyah flops in these 10 curves, β is a flopping contraction, ϕ is a $\mathbb{P}^1$-bundle, V_3 is the Segre cubic hypersurface in $\mathbb{P}^4$, and S is the smooth del Pezzo surface of degree 5.

Second primitive subgroup isomorphic to $\mathfrak{A}_5$

Let G be a primitive finite subgroup $\mathfrak{A}_5$ in $\mathrm{PGL}_4(\mathbb{C})$ such that $\mathbb{P}^3$ contains a G-invariant twisted cubic curve $\mathcal{C}$.

Let $\pi\colon X \to \mathbb{P}^3$ be the blow-up of the curve $\mathcal{C}$.

Then there exists a G-commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\pi} & & \searrow^{\phi} & \\ \mathbb{P}^3 & & \overset{\rho}{\dashrightarrow} & & \mathbb{P}^2 \end{array}$$

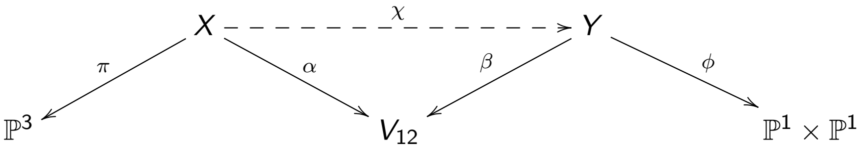
Here ϕ is a $\mathbb{P}^1$-bundle whose fibers are proper transforms of secants of the curve $\mathcal{C}$, and the map ρ is given by the linear system of quadric surfaces that contain $\mathcal{C}$.

Second primitive subgroup isomorphic to $\mathfrak{S}_5$

Let G be a primitive subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that $G \cong \mathfrak{S}_5$, and G leaves invariant a pair of disjoint twisted cubic curves C_1 and C_2.

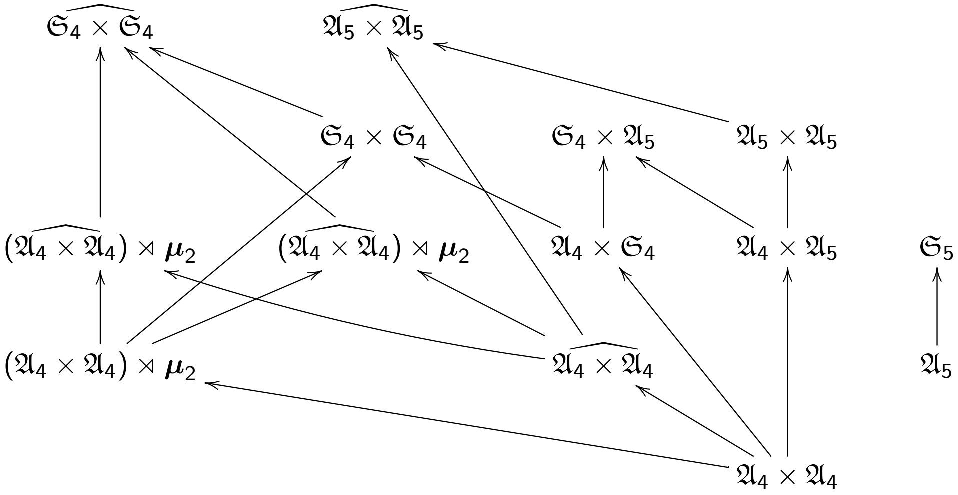
Let $\pi\colon X \to \mathbb{P}^3$ be the blow up of the curves C_1 and C_2.

Then there is a G-commutative diagram

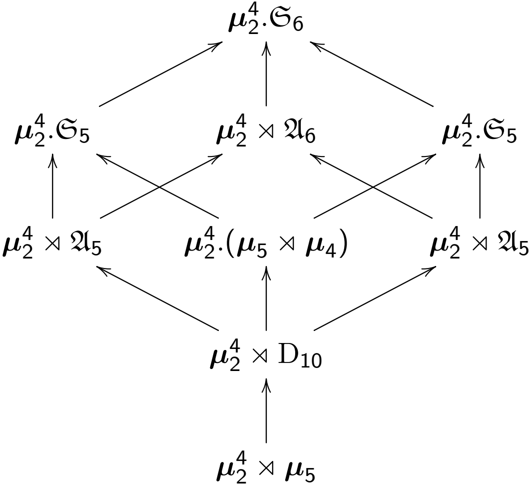


Here V_{12} is a divisor of bi-degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with ten isolated terminal Gorenstein singularities, α is a small birational morphism that contracts proper transforms of ten common secants of the curves C_1 and C_2, χ is a composition of ten flops, β is a flopping contraction, and ϕ is a $\mathbb{P}^1$-bundle.

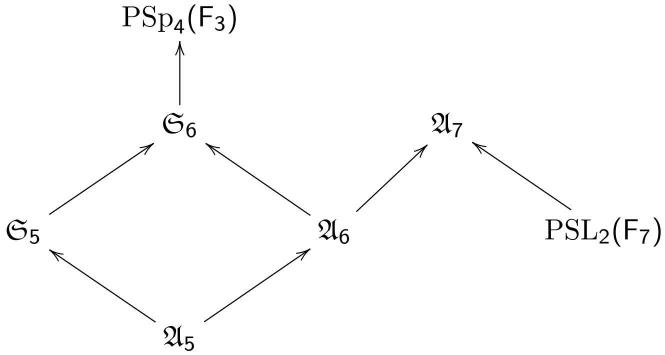
Primitive subgroups that leave invariant a quadric



Primitive subgroups that contain a subgroup $\boldsymbol{\mu}_2^4$



Remaining primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$



Kollár's question

Let X be a Fano variety with terminal singularities.

Let G be a finite subgroup in $\mathrm{Aut}(X)$.

Let H be a subgroup in G.

Suppose that

$$\mathrm{rk}\,\mathrm{Cl}^G(X) = \mathrm{rk}\,\mathrm{Cl}^H(X) = 1.$$

Then X is a G-Mori fibre space, and X is a H-Mori fibre space.

Lemma

If X is H-birationally super-rigid, then X is G-birationally super-rigid.

Question

If X is H-birationally rigid, is X always G-birationally rigid?

Kollár's question for $\mathbb{P}^3$

Let $H \subset G$ be primitive subgroups in $\mathrm{PGL}_4(\mathbb{C})$.

Lemma

If $\mathbb{P}^3$ is H-birationally rigid, then $\mathbb{P}^3$ is G-birationally rigid.

So, to prove Theorem it is enough to prove the following lemmas:

Lemma (Cheltsov,Shramov)

If $G \cong \mathfrak{A}_4 \times \mathfrak{A}_4$, then $\mathbb{P}^3$ is G-birationally rigid.

Lemma (Cheltsov,Shramov)

If $G \cong \boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, then $\mathbb{P}^3$ is G-birationally rigid.

Lemma (Cheltsov,Shramov)

If $G \cong \mathrm{PSL}_2(\mathsf{F}_7)$, then $\mathbb{P}^3$ is G-birationally rigid.

Lemma (Cheltsov,Shramov)

If $G \cong \mathfrak{A}_6$, then $\mathbb{P}^3$ is G-birationally rigid.

Equivariant birational solidity

Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$.

If $\mathbb{P}^3$ is not G–birational to a G-Mori fibre space with positive dimensional base, we say that

$\mathbb{P}^3$ is *G-solid*.

Observe that $\mathbb{P}^3$ is G-solid if and only if the following conditions are satisfied:

1. $\nexists$ G-map $\mathbb{P}^3 \dashrightarrow S$ whose general fiber is a rational curve;
2. $\nexists$ G-map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ whose general fiber is a rational surface.

If $\mathbb{P}^3$ is G-birationally rigid, then $\mathbb{P}^3$ is G-solid.

When is $\mathbb{P}^3$ G-solid?

Let G be a finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$.

If $\mathbb{P}^3$ is G-solid, then we proved that

- G is transitive,
- $\mathbb{P}^3$ does not contain G-invariant unions of two skew lines,
- neither $G \cong \mathfrak{A}_5$ nor $G \cong \mathfrak{S}_5$.

Theorem (Cheltsov, Dubouloz, Kishimoto)

Suppose that the following conditions are satisfied:

- *G is transitive,*
- *$\mathbb{P}^3$ does not contain G-invariant unions of two skew lines,*
- $|G| \geqslant 10616832$.

Then $\mathbb{P}^3$ is G-solid, G has unique G-orbit of length 4, and the only G-Mori fibre spaces G-birational to $\mathbb{P}^3$ are $\mathbb{P}^3$ and the toric Fano–Enriques threefold of degree 24 described earlier.

Small monomial subgroups in $\mathrm{PGL}_4(\mathbb{C})$

Let $G_{48,3} \cong \boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $G_{96,72} \cong \boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_6$ be subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let G be subgroup $G'_{324,160} \cong \boldsymbol{\mu}_3^3 \rtimes \mathfrak{A}_4$ in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If G is one of these three subgroups, then $\mathbb{P}^3$ is G-birational to a del Pezzo fibration.

Main Theorem

Theorem (Cheltsov,Sarikyan)

Let G be an imprimitive finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$ such that $\mathbb{P}^3$ does not have G-invariant unions of two skew lines.
Suppose that G is not conjugated to $G_{48,3}$, $G_{96,72}$, $G'_{324,160}$.
Then $\mathbb{P}^3$ is G-solid, and the only G-Mori fibre spaces G-birational to $\mathbb{P}^3$ are $\mathbb{P}^3$ and the toric Fano–Enriques threefold of degree 24 described earlier.

Corollary

Let G be an arbitrary finite subgroup in $\mathrm{PGL}_4(\mathbb{C})$. Then $\mathbb{P}^3$ is G-solid if and only if the following conditions are satisfied:

(a) *G does not fix a point,*

(b) *G does not leave a pair of two skew lines invariant,*

(c) *G is not isomorphic to $\mathfrak{A}_5$ or $\mathfrak{S}_5$,*

(d) *G is not conjugate to $G_{48,3}$, $G_{96,72}$ or $G'_{324,160}$.*