$\hat{Z}-$invariants and universal abelian covers

joint work in progress with L. Katzarkov, K.S. Lee and S. Gukov

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Introduction

Setup

- We consider a plumbed 3-manifold M which is $\mathbb{Q}-$homology sphere.

$$H^*(M,\mathbb{Q}) = H^*(S^3,\mathbb{Q})$$

- $H = H_1$ is finite Abelian group.

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- All examples today: Seifert fibered spaces:

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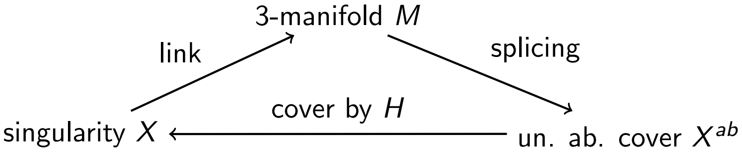
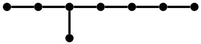
- A natural source of such M: link of an isolated normal surface singularity X.
- X quasihomogeneous $\Leftrightarrow$ plumbing graph is a star.
- X quasihomogeneous $\Rightarrow$ covering X^{ab} by H of Brieskorn ICIS type.

Relations

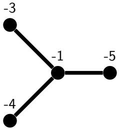

3-manifold M
link
splicing
cover by H
singularity X
un. ab. cover X^{ab}

Links of singularities

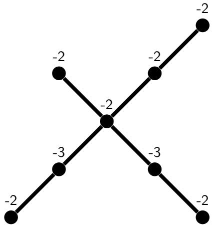
$$E_8 : x^2 + y^3 + z^5 = 0 \longrightarrow M(-2, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{4}{5}) \longleftrightarrow$$



$$S_{12} : x^3y + y^2z + xz^2 = 0 \longrightarrow M(-1, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}) \longleftrightarrow$$



$$x^4 + y^5 + z^6 = 0 \longrightarrow M(-2, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{2}{5}) \longleftrightarrow$$



False theta functions

Denote $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. Let $m \in \mathbb{Z}^+$ and r a class modulo $2m$. *False theta function (or Eichler integral)* of modulus m and residue r is

$$\Psi_{m,r}(\tau) = \sum_{\substack{n\in\mathbb{Z} \\ n=r \bmod 2m}} \operatorname{sgn}(n)\, q^{n^2/4m}. \tag{1}$$

We can also write:

$$\Psi_{m,r}(\tau) = \sum_{n=0}^{\infty} \chi_{m,r}(n) q^{n^2/4m}$$

where

$$\chi_{m,r}(n) = \begin{cases} 1 & \text{if } n = r \mod 2m \\ -1 & \text{if } n = -r \mod 2m \\ 0 & \text{otherwise} \end{cases}$$

False theta functions occur naturally as components of invariants of 3-manifolds and have interesting number theoretical properties.

Theta functions and generating functions

We can package the character in terms of a generating function:

$$f_{m,r}(x) = \sum_{n=0}^{\infty} \chi_{m,r}(n) x^n = \frac{x^{m-r} - x^{-(m-r)}}{x^m - x^{-m}}.$$

$\widehat{Z}-$ invariants

Given a plumbed manifold, we can use plumbing graph data to construct a q-series invariants (GPPV).
Let M be the plumbing matrix of 3-manifold M.
Formula from GPPV:

$$\widehat{Z}_a(q) = (-1)^{\pi} q^{\frac{3\sigma - \sum_v m_v}{4}} \cdot \text{v.p.} \oint_{|z_v|=1} \prod_{v \in \text{Vert}} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v} \right)^{2-\deg(v)} \cdot \Theta_a^{-M}(\vec{z})$$

where

$$\Theta_a^{-M}(\vec{z}) = \sum_{\vec{\ell} \in 2M\mathbb{Z}^s + \vec{a}} q^{-\frac{(\vec{\ell}, M^{-1}\vec{\ell})}{4}} \prod_{v \in \text{Vert}} z_v^{\ell_v}.$$

$\widehat{Z}_b$ invariants of Seifert manifolds

For Seifert manifold with 3 singular fibres:

$$\widehat{Z}_b(q) = \Psi_m^{r_1} + \Psi_m^{r_2} + \Psi_m^{r_3} + \cdots + \Psi_m^{r_n}$$

What are m and $r_1, r_2, \ldots, r_n$?
We write

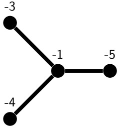
$$F_b(x) = f_{m,r_1} + f_{m,r_2} + f_{m,r_3} + \cdots + f_{m,r_n}$$

and

$$F_{SU(2)}(x) = \sum_b F_b(x).$$

Example S_{12}

Let M be the link of the singularity S_{12} given by $x^3y + y^2z + xz^2 = 0$.

- $M = (-1, 1/3, 1/4, 1/5) = (-2/3, 1/4, 1/5)$
- 

- $e = -\frac{13}{60}$
- $H_1(M) = H = \mathbb{Z}/13\mathbb{Z}$
- $X^{ab} : x^3 + y^4 + z^5 = 0$ (with diagonal action of H).

$\widehat{Z}_b$ of S_{12}

$$\widehat{Z}_0 = q^{\frac{1}{2}}(1 - q^{52} - q^{91} - q^{130} + q^{273} + q^{338} + q^{429} - q^{767} + q^{793} + \dots$$

$$\widehat{Z}_1 = -\frac{1}{2}q^{\frac{233}{26}}(1 + q^{8} - q^{12} + q^{19} + q^{42} + q^{124} - q^{141} + q^{183} -$$

$$-q^{190} + q^{238} - q^{261} - q^{425} - q^{506} + q^{539} - q^{561} - q^{617} + q^{943} + \dots$$

$$\widehat{Z}_2, \widehat{Z}_3, \dots, \widehat{Z}_{12}.$$

$m = 780.$

Generating function for S_{12}

There are therefore 13 $\widehat{Z}_b$ which have modulus $60 \cdot 13 = 780$. When they sum-up to $Z_{SU(2)}$ we obtain the series

$$Z^{S12}_{SU(2)} = q^{\frac{1}{2}}(1 - q^{\frac{5}{13}} - q^{\frac{7}{13}} - q^{\frac{11}{13}} + q^{\frac{18}{13}} + q^{\frac{24}{13}} + q^{\frac{28}{13}} - q^{\frac{47}{13}} + q^{\frac{73}{13}} + \dots$$

This gives:

$$F_{SU(2)}(x) = \frac{(y^3 - y^{-3})(y^4 - y^{-4})(y^5 - y^{-5})}{y^{60} - y^{-60}}$$

where $y = x^{\frac{1}{13}}$.

It is natural to try to express invariants of M in terms of invariants of X^{ab} and action of H on it.
Poincaré series of the ring of functions of hypersurface

$$x^{a_1} + y^{a_2} + z^{a_3} = 0$$

is

$$P(x) = \frac{1 - x^a}{(1 - x^{\frac{a}{a_1}})(1 - x^{\frac{a}{a_1}})(1 - x^{\frac{a}{a_1}})},$$

where $a = LCM(a_1, a_2, a_3)$. There is also H-equivariant version.
Related to Steenbrink spectrum by theorem of Ebeling.

Two examples

Let $M_1 = M(-1, 1/7, 1/7, 4/7)$ and $M_2 = M(-1, 1/7, 2/7, 3/7)$.
$H_1(M_1) = H_1(M_2) = \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.
M_1 and M_2 are not homeomorphic (Orlik).

$$\begin{aligned} Z^{M_1}_{SU(2)} &= q^{5/7}(-1 + 3q^{5/7} - 3q^{12/7} + q^3 - q^{11} + 3q^{96/7} - 3q^{117/7} + \dots \\ &= q^{\frac{1}{7}}(3\Psi^6_7 - \Psi^4_7) \end{aligned}$$

$$\begin{aligned} Z^{M_2}_{SU(2)} &= q^{-4/7}(-1 + 3q^{5/7} - 3q^{12/7} + q^3 - q^{11} + 3q^{96/7} - 3q^{117/7} \dots \\ &= q^{-\frac{8}{7}}(3\Psi^6_7 - \Psi^4_7) \end{aligned}$$

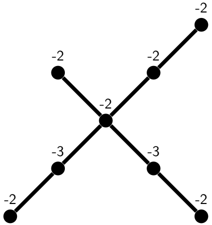
This gives:

$$f^{M_1}_{SU(2)}(x) = f^{M_1}_{SU(2)}(x) = \frac{3(x - x^{-1}) - (x^3 - x^{-3})}{x^7 - x^{-7}} = \frac{-(x - x^{-1})^3}{x^7 - x^{-7}}$$

Borel planes coincide!

Four singular fibres

We consider link of $x^4 + y^5 + z^6$. Studied in $CCFGH18$.

- 

- $M(-2, 1/2, 2/3, 2/5, 2/5)$
- $e = -1/30$, $H = \mathbb{Z}/5\mathbb{Z}$
- $\widehat{Z}_b$ are combinations of weight 1/2 and 3/2 false theta fuctions.

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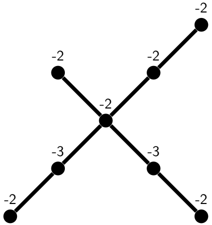
- 

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$$F_{SU(2)} = \frac{-(x^{15} - x^{-15})(x^{10} - x^{-10})(x^6 - x^{-6})^2(x^{30} + x^{-30})}{2(x^{30} - x^{-30})^2}$$

Generating function of $Z_{SU(2)}$

$$F_{SU(2)} = -\frac{1}{2}\frac{(x^{15}-x^{-15})(x^{10}-x^{-10})(x^{6}-x^{-6})^2(x^{30}+x^{-30})}{(x^{30}-x^{-30})^2}$$
$$= -\frac{1}{60}(x^{15}-x^{-15})(x^{10}-x^{-10})(x^{6}-x^{-6})^2\frac{d}{dx}\frac{1}{(x^{30}-x^{-30})^2}$$

Poincaré series of the universal Abelian cover:

$$P(x) = \frac{(1-x^{30})^2}{(1-x^{15})(1-x^{10})(1-x^{6})^2}.$$

Conclusion

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- There is an easy explicit formula for the generating function of $Z_{SU(2)}$ for star-shaped graphs. What does it mean conceptually?
- Manifolds with the same Borel plane. What does that mean for resurgence and Chern-Simons theory?
- How is the generating function related to invariants from singularity theory?
- What about $\widehat{Z}_b$? What about more complicated manifolds/graphs?

Conclusion

References

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Thank you for your attention!