Holomorphic Floer Theory, quantum wave functions and resurgence

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I start with a brief reminder on the general Riemann-Hilbert correspondence and its relation to HFT.

If $(M, \omega^{2,0})$ is a complex symplectic manifold then under some assumptions one can associate to it two categories depending on parameter $\hbar \in \mathbb{C}^*$: the Fukaya category $\mathcal{F}_\hbar(M) = \mathcal{F}(M, Re(\omega^{2,0}/\hbar) + iIm(\omega^{2,0}/\hbar))$ and the category of holonomic DQ-modules $Hol_\hbar(M)$ over the quantized algebra $\mathcal{O}_\hbar(M)$. The Riemann-Hilbert correspondence (conjectural in general) of Kontsevich and myself says that these categories are equivalent.

One can think of this statement as of categorical version of the comparison isomorphism of Betti and de Rham cohomology, where on the Betti side we use Morse complexes. In HFT we study complex symplectic manifolds and Floer complexes of holomorphic Lagrangian subvarieties. Therefore dependence on the parameter $\hbar$ is crucial: Floer differential is non-trivial only for special values of $\hbar$. Then Floer homology groups form local systems over $\mathbb{C}^*_\hbar$ and the corresponding Stokes isomorphisms can be described in terms of the wall-crossing formulas.

More precise statement of the RH-correspondence requires more technical details. For example one has to choose a partial compactification $\overline{M} \supset M$, so that the Fukaya category will be partially wrapped.

For more details on HFT and RH-correspondence I refer to several videos and slides of our talks and lecture series on this topic which are available online. E.g. my 2016 lecture at IAS in Princeton:

https://www.youtube.com/watch?v=HU3ZtGJ6ITY

I will not use those technical details today.

Today I am going to discuss the story which is related to a very special case of HFT, namely, $M = T^*X$, where X is a complex manifold, say, smooth complex affine algebraic variety, $dim_{\mathbb{C}} X = n$. We will be interested in $Hom(L_0, L_1)$ taken in the Fukaya category of M. Here $L_0 = X, L_1 = graph(df)$, where $f \in \mathcal{O}(X)$. More generally one can take instead of $graph(df)$ the graph of a closed 1-form, as in Maxim's talk. Furthermore we will rescale the symplectic form and the function f by $\hbar$, as I mentioned on the previous slide.

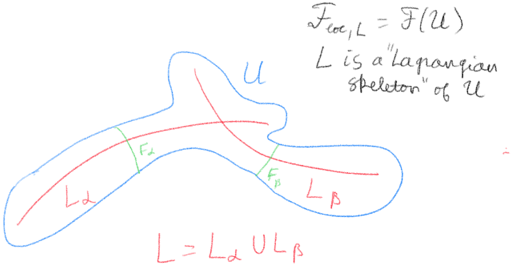
This special case of HFT controls the geometric and analytic properties of the exponential integral $I(\hbar) := \int_C e^{f/\hbar} vol$. Here the real n-dimensional integration cycle C belongs to an appropriate class of chains (say, positive integer combination of real n-dimensional oriented submanifolds), and vol is a chosen top degree "volume form" on X.

The (twisted) global de Rham cohomology is defined by $H^{\bullet}_{DR,glob}(X,f) = \mathbb{H}^{\bullet}(X_{Zar}, (\Omega^{\bullet}_X, d_f := d + df \wedge (\bullet)))$, while the global Betti cohomology is defined as the stabilization of the relative cohomology group $H_{\bullet}^{Betti,glob}((X,f,\mathbb{Z}) := H_{\bullet}((X, f^{-1}(c), \mathbb{Z}), c < 0$, and $|c|$ is sufficiently large. One can make de Rham and Betti cohomology into sheaves of graded abelian groups over $\mathbb{C}^*_{\hbar}$ by replacing f by $f/\hbar$. In fact one should work with an appropriate compactification $\overline{X}$, since X is non-compact. In the end one has a comparison isomorphism of Betti and de Rham cohomology. The integral $I(\hbar)$ can be thought of as the corresponding exponential period, i.e. the pairing of a closed de Rham class with the dual to a closed Betti cohomology class.

Since $L_0 := X$ and $L_{1,\hbar} = graph(df/\hbar)$ are complex Lagrangians in T^*X, for generic $\hbar$ there are no pseudo-holomorphic discs with the boundary on $L_0 \cup L_{1,\hbar}$. They appear for finitely many Stokes directions in $\mathbb{C}^*_\hbar$. In terms of $I(\hbar)$ this is reflected in the jumps of $I(\hbar)$ as $\hbar$ crosses a Stokes rays (i.e. when several critical values of f belong to such a ray). These jumps are described by the wall-crossing formulas of Cecotti-Vafa type, which I will recall later in the talk.

Furthermore, there are local versions of de Rham and Betti cohomology, when one considers instead of the pair $(X, f^{-1}(-\infty))$ the pair $(f^{-1}(D(z_i)), f^{-1}(-\infty_i))$, where $D(z_i)$ is a small disc with the center at the critical value z_i of f and $-\infty_i = \partial D(x_i) \cap \mathbb{R}_{<0}$. With all these groups (or rather sheaves on $\mathbb{C}^*_\hbar$) defined, there are four comparison isomorphisms: Betti global-to-local, de Rham global-to-local, Betti global-to-de Rham global, Betti local-to-de Rham local. At the categorical level one can define local versions of the Fukaya category and the category of holonomic DQ-modules, $\mathcal{F}_{\hbar,loc} := \oplus_{z_i \in Critval(f)} \mathcal{F}_{z_i,loc,\hbar}(M)$ and $Hol_{\hbar,loc} := Hol_{\hbar,loc}(M)$. For that one considers a small tubular neighborhood of $L := L_0 \cup L_1$ instead of T^*X. Recall that we also have global categories $\mathcal{F}_\hbar = \mathcal{F}_\hbar(M)$ and $Hol_\hbar = Hol_\hbar(M)$ connected by the global RH-correspondence. All four comparison isomorphisms of cohomology have their categorical upgrades. The above-mentioned RH-correspondence is the categorical upgrade of Betti global-to-de Rham global isomorphism. Similarly we have the local RH-correspondence.

Fukaya category.pdf



The following equivalences hold:
1') (Fukaya local-to-global). Outside of Stokes rays in $\mathbb{C}^*_\hbar$ we have an isomorphism of analytic families of categories:

$$\mathcal{F}_\hbar \simeq \mathcal{F}_{\hbar,loc}.$$

Here Stokes rays are those rays $Arg(\hbar) = const$ for which there exist pseudo-holomorphic discs with boundaries on L_0 and L_1.
2') (Holonomic local-to-global).

$$Hol_\hbar \otimes \mathbb{C}((\hbar)) \simeq Hol_{loc},$$

where in the LHS the notation means that corresponding category over $\mathbb{C}[[\hbar]]$ with inverted $\hbar$.
3') (Global Riemann-Hilbert correspondence). We have an equivalence of analytic families of categories over $\mathbb{C}^*_\hbar$:

$$Hol_\hbar \simeq \mathcal{F}_\hbar.$$

4') (Local Riemann-Hilbert correspondence).

$$Hol_{\hbar,loc} \simeq \mathcal{F}_{\hbar,loc} \otimes \mathbb{C}((\hbar)).$$

Categorical equivalences imply the corresponding isomorphisms of cohomology groups. Consider e.g. the category $\mathcal{F}_\hbar(T^*X)$, the Fukaya category associated with the real symplectic form $Re(\omega^{2,0}_{T^*X}/\hbar)$ and the B-field $Im(\omega^{2,0}_{T^*X}/\hbar)$. Then $L_0 = X$ and $L_1 = graph(df)$ are objects as long as we endow them with the trivial rank 1 local systems.

Theorem

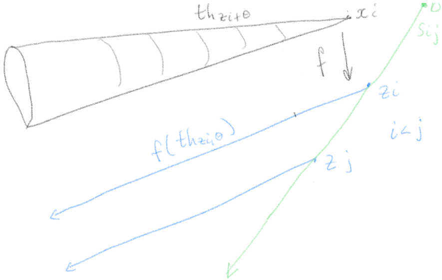
In the above notation for any $\hbar \in \mathbb{C}^*$ there is a natural isomorphism

$$Ext^\bullet_{\mathcal{F}_\hbar(T^*X)}(L_0, L_1) \simeq H^\bullet(X, f^{-1}(z), \mathbb{Z})$$

as long as z belongs to the ray $\mathbb{R}_{<0} \cdot \hbar$ and $|z|$ is sufficiently large.

We can make these considerations very explicit. Assume now that X is Kähler and f is Morse. Let $\theta = Arg(\hbar)$. We define a thimble $th_{z_i,\theta+\pi}$ as the union of gradient lines (for the Kähler metric) of the function $Re(e^{-i\theta}f)$ outcoming from the critical point $x_i \in X$ such that $f(x_i) = z_i$. These gradient lines are also integral curves for the Hamiltonian function $Im(e^{-i\theta}f)$ with respect to the *symplectic* structure. Hence $f(th_{z_i,\theta+\pi})$ is a ray $Arg(z) = \theta + \pi$ outcoming from the critical value $z_i \in S$.

Thimbles



Let us assume that X carries a holomorphic volume form vol and consider the following exponential integral as a function of $\hbar \in \mathbb{C}^*$ such that $\hbar$ does not belong to Stokes rays $Arg(\hbar) = Arg(z_i - z_j), i \neq j$:

$$I_i(\hbar) = \int_{th_{z_i,\theta+\pi}} e^{f/\hbar} vol.$$

Assume that the set of critical values $S = \{z_1, ..., z_k\}$ is in generic position in the sense that no straight line contains three points from S. Then a Stokes ray contains two different critical values which can be ordered by their proximity to the vertex.

Wall-crossing formulas

It is easy to see that if in the $\hbar$-plane we cross the Stokes ray $s_{ij} := s_{\theta_{ij}}$ containing critical values $z_i, z_j, i < j$, then the integral $I_i(\hbar)$ changes such as follows:

$$I_i(\hbar) \mapsto I_i(\hbar) + n_{ij} I_j(\hbar),$$

where $n_{ij} \in \mathbb{Z}$ is the number of gradient trajectories of the function $Re(e^{i(Arg(z_i - z_j)/\hbar)} f)$ joining critical points x_i and x_j (same as the number of pseudo-holomorphic discs with boundary on $L_0 \cup L_1$ mentioned before).

Let us modify the exponential integrals such as follows:

$$I_i^{mod}(\hbar) := \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-z_i/\hbar} I_i(\hbar).$$

Then as $\hbar \to 0$ the stationary phase expansion ensures that as a formal series

$$I_i^{mod}(\hbar) = c_{i,0} + c_{i,1}\hbar + ... \in \mathbb{C}[[\hbar]],$$

where $c_{i,0} \neq 0$. The jump of the modified exponential integral across the Stokes ray s_{ij} is given by $\Delta(I_i^{mod}(\hbar)) = n_{ij} I_j^{mod}(\hbar) e^{-(z_i - z_j)/\hbar}$.

RH problem

Therefore the vector $\bar{I}^{mod}(\hbar) = (I_1^{mod}(\hbar), ..., I_k^{mod}(\hbar)), k = |S|$ satisfies the Riemann-Hilbert problem on $\mathbb{C}$ with known jumps across the Stokes rays and known asymptotic expansion as $\hbar \to 0$ (notice that because of our ordering of the points in S, the function $e^{-(z_i - z_j)/\hbar}$ has trivial Taylor expansion as $\hbar \to 0$ along the Stokes ray s_{ij}).

In abstract terms, we consider the Riemann-Hilbert problem for a sequence of $\mathbb{C}^k$-valued functions (here k is the rank of the Betti cohomology, which is under our assumptions is equal to the cardinality $|S| = k$) $\Psi_1(\hbar), ..., \Psi_k(\hbar)$ on $\mathbb{C}^* - \cup(Stokes\,rays)$ each of which has a formal power asymptotic expansion in $\mathbb{C}[[\hbar]]$ as $\hbar \to 0$, and which satisfy the following jumping conditions along the Stokes rays s_{ij}:

$$\Psi_j \mapsto \Psi_j,$$

$$\Psi_i \mapsto \Psi_i + n_{ij} e^{-\frac{z_i - z_j}{\hbar}} \Psi_j.$$

This collection $(\Psi_i)_{1 \le i \le k}$ gives rise to a holomorphic vector bundle on $\mathbb{C}^*_\hbar$. Formal expansions at $\hbar = 0$ of its holomorphic sections are resurgent series (i.e. they are Borel resummable).

Let me say few words about the conceptual reason for the resurgence. The above considerations are a special case of the general theory of analytic wall-crossing structures introduced jointly with Maxim in arXiv: 2005.10651. In the case relevant to our considerations the wall-crossing structure is given by a local system over $\mathbb{C}^*_\hbar$ of $\Gamma = \mathbb{Z}^{\# Critval(f)}$-graded Lie algebras $Vect_\Gamma$ on the character torus $\mathbf{T}_\Gamma = Hom(\Gamma, \mathbb{C}^*)$ together with some additional data. We showed that this structure gives rise to a formal scheme which contains a wheel of projective lines inside. Analyticity of WCS means that this formal scheme comes from a $\mathbb{C}$-analytic one. Among other things we formulated in that paper the resurgence conjecture for sections of a certain holomorphic fiber bundle with the fiber $\mathbf{T}_\Gamma$. The fiber bundle is associated with the given analytic wall-crossing structure. This fiber bundle can be replaced by a vector bundle in the case of exponential integrals, as we explained above. The wall-crossing structure corresponding to the exponential integral is analytic. Then the resurgence conjecture explains the well-known fact about the resurgence of the Taylor series of $I^{mod}(\hbar)$. In what follows we would like to use the same approach in the infinite-dimensional case.

Next, I am going to discuss our proposal in the case of the Feynman integral over the space of smooth paths in the complex symplectic manifold $(M, \omega^{2,0})$ with prescribed Lagrangian boundary conditions. For simplicity I will assume that $M = \mathbb{C}^{2n}$, the standard symplectic vector space. Symbolically, the Feynman path integral is written as

$$\int e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi.$$

The infinite-dimensional cycle of integration is not specified here. Sometimes the integration cycle can be chosen as an integer linear combination of the homology classes of Lefschetz thimbles which are well-defined.

Boundary conditions for the path integral with the action $S(\varphi)$ are specified by a choice of two holomorphic Lagrangian submanifolds $L_0, L_1 \subset \mathbb{C}^{2n}$. This means that we consider the space of smooth maps $\varphi : [0,1] \to \mathbb{C}^{2n}$ such that $\varphi(0) \in L_0, \quad \varphi(1) \in L_1$. This is an infinite-dimensional complex symplectic manifold, and we may assume that $S(\varphi)$ is holomorphic.

Let

$$S(\varphi) = \int_0^1 \sum_i p_i(t) \frac{dq_i(t)}{dt} + \int_0^1 H(\mathbf{q}(t), \mathbf{p}(t), t) dt$$

where

$$H : \mathbb{C}^{2n} \times [0,1] \to \mathbb{C}$$

is holomorphic in complex coordinates $(\mathbf{q}, \mathbf{p})$ and C^∞ in real coordinate $t \in [0,1]$.

Critical points of $S(\varphi)$ are solutions of the Euler-Lagrange equation

$$\frac{\delta S(\varphi)}{\delta \varphi} = 0 \iff \dot{p}_i = \frac{\partial H}{\partial q_i}, \dot{q}_i = -\frac{\partial H}{\partial p_i}$$

Then we have a family of partially defined holomorphic symplectomorphisms

g_t : (an open dense subset in $\mathbb{C}^{2n}$) $\to \mathbb{C}^{2n}$, $\quad t \in [0,1]$, $g_0 =$ identity map.

Critical points are identified with intersection points

$$\mathrm{Crit}(S) \simeq g_1(L_0) \cap L_1$$

Typically all intersections are transversal, and there are countably many of them. Let φ_α denote the critical point of S, where the index α labels intersection points of Lagrangians (same as critical points of S).

Let us think of the initial Feynman integral as of an isomorphism of global de Rham and global Betti cohomology (both ill-defined). Then the sum over critical points (e.g. flat connections for the complexified Chern-Simons functional) can be interpreted as a composition of this isomorphism with the global-to-local Betti isomorphism. Its finite-dimensional analog is the sum of exponential integrals over local thimbles coming out of the critical points of $S(\varphi)$. The previously mentioned holomorphic fiber bundle is obtained from this local story by adding to it Stokes automorphisms. Hence the problem splits into two: define local expressions corresponding to critical points and then glue the global holomorphic fiber bundle over $\mathbb{C}^*_\hbar$ using Stokes automorphisms (i.e. solving the RH problems as we saw before).
Hence would like to make sense (for each critical point φ_α, and, say, assuming transversality) of the following expansion:

$$\int_{\substack{\text{local Lefschetz thimble} \\ \text{outcoming from } \varphi_\alpha}} e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi \underset{\hbar \to 0}{\sim} e^{\frac{S(\varphi_\alpha)}{\hbar}} \hbar^{-\frac{n}{2}} \cdot (c_{0,\alpha} + c_{1,\alpha}\hbar + c_{2,\alpha}\hbar^2 + \dots),$$

Let us explain the idea.

Quantum mechanics identifies the Feynman integral with the expression $\langle\psi_0|e^{\widehat{H}}|\psi_1\rangle$, where ψ_0 and ψ_1 are quantum wave functions corresponding to L_0 and L_1 respectively, and $\widehat{H}$ is the quantization of the Hamiltonian H. For simplicity I am going to assume that $H = 0$, i.e. the theory is topological. We will define in terms of the deformation quantization the quantum wave function associated to a complex Lagrangian submanifold, and then make sense of the expression $\langle\psi_0|\psi_1\rangle$. In the case when $H \neq 0$ there is an upgrade of that story, where $\widehat{H}$ is used for the transport of quantum wave functions along the Hamiltonian flow of H. In the end we get answers in terms of finite-dimensional data thanks to our generalized RH-correspondence.

Let us discuss some details.

With a complex symplectic manifold $(M, \omega^{2,0}), dim_{\mathbb{C}} M = 2n$ one can associate (after Kashiwara and Kontsevich) a sheaf of categories of DQ-modules. Let us assume that its global sections is the category of modules of finite type over an associative algebra $\mathcal{O}_{\hbar}(M)$ over $\mathbb{C}[[\hbar]]$ endowed with the $*$-product, which is the deformation quantization of the algebra $\mathcal{O}(M)$. Physicists would call $\mathcal{O}_{\hbar}(M)$ the canonical coisotropic brane. The previously mentioned category $Hol_{\hbar}(M)$ of holonomic DQ-modules is a subcategory of this one. Under these assumptions, it is known after the works of Agnolo, Kashiwara, Schapira and others that to every holomorphic Lagrangian submanifold $L \subset M$ endowed with a choice of $K_L^{1/2}$ one can assign canonically an object $E_L^{DR,\hbar} := E_{L,K_L^{1/2}}$ in the category of holonomic $\mathcal{O}_{\hbar}(M)$-modules. For that one should invert $\hbar$ and make some other choices. In some situations $\hbar$ can be considered as a fixed complex number, which we will assume for simplicity.

Then having two, say, transversal holomorphic Lagrangian submanifolds L_0, L_1 we can write the following sequence of maps

$$Ext^n(E_{L_0}^{DR,\hbar}, \mathcal{O}_\hbar(M)) \otimes Ext^0(\mathcal{O}_\hbar(M), E_{L_1}^{DR,\hbar}) \to Ext^n(E_{L_0}^{DR,\hbar}, E_{L_1}^{DR,\hbar}) \to$$

$$Ext^n_{\mathcal{F}_{loc,\hbar}}(E_{L_0}^{Betti,\hbar,loc}, E_{L_1}^{Betti,\hbar,loc}) \to \mathbb{C}((\hbar^{1/2})).$$

Elements of the first tensor factor are called left quantum wave functions associated to L_0. Second tensor factor consists of right quantum wave functions associated to L_1. Ext-groups can be taken in the category of DQ-modules. Passing from first line to the second one involves the global-to-local equivalence for holonomic DQ-modules as well as the local RH-correspondence. We denote by $E_L^{Betti,loc,\hbar} := RH_\hbar(E_L^{DR,\hbar})$ the object corresponding to $E_L^{DR,\hbar}$ under this correspondence. These are objects of the local Fukaya category $\mathcal{F}_{\hbar,loc}$ are supported on L. Hence the final Ext-group is the sum over all intersection points $L_0 \cap L_1$ of some local expressions, i.e. a series in rational powers of $\hbar$ with complex coefficients. Roughly this is our proposal for the sum over critical points in the formal expansion of the Feynman integral.

Axiomatically the above data can be described such as follows:
1) an element $\mu \in Ext^n_{Hol_\hbar(M)}(E_0^{DR,\hbar}, E_1^{DR,\hbar})$ encoding the volume form vol_X, where $n = \frac{dim_{\mathbb{C}} M}{2}$;
2) a class $\gamma \in Ext^0_{\mathcal{F}_{\hbar,loc}}(E_0^{Betti,\hbar,loc}, E_1^{Betti,\hbar,loc})$ encoding the integration cycle.
Then the corresponding exponential integral considered as a formal power series in $\hbar$ (i.e. the perturbative expansion) is given by
$I_{form}(\hbar) = \langle RH_\hbar(\mu), \gamma \rangle$, where $\langle \bullet, \bullet \rangle$ is the "Calabi-Yau pairing" between Ext^n and Ext^0 in the n-dimensional Calabi-Yau category $\mathcal{F}_{\hbar,loc}$ and $RH_\hbar$ is the Riemann-Hilbert functor.

As I explained previously in the finite-dimensional case, in order to obtain from the formal series $I_{form}(\hbar)$ the analytic function $I(\hbar)$ one should apply the Stokes automorphisms and glue the holomorphic vector bundle over $\mathbb{C}^*_\hbar$ (it extends to $\mathbb{C}_\hbar$). The previous discussion ensures that we can do the same thing in the infinite-dimensional case. Also we need to show that the arising wall-crossing structure is analytic. One way to do that is Floer-theoretical: one can prove exponential bounds on the virtual numbers of pseudo-holomorphic discs with the boundary on $L_0 \cup L_1$. Then (assuming the resurgence conjecture) one claims the resurgence property of the perturbative series in $\hbar$.

All the above was just a sketch. More accurate definition of the quantum wave function structure on $(M, \omega^{2,0})$ and quantum wave function associated to $L, K_L^{1/2}$ can be made using the language of Harish-Chandra pairs and formal differential geometry in the sense of Gelfand-Kazhdan. Few explicit formulas illustrating the above discussion are given on the next slides.

In coordinates in $\mathbb{C}^{2n}$ the quantum wave function is the formal expression of the type

$$\psi_L = \hbar^{-n/4} \exp\left(\frac{1}{\hbar}\mathcal{F}_0\right) \cdot (G_0 + G_1\hbar + G_2\hbar^2 + \dots) \cdot (dq_1 \wedge dq_2 \wedge \dots \wedge dq_n)^{1/2}$$

where $G_0, G_1, \dots$ are analytic functions in $\mathbf{q} = (q_1, \dots, q_n)$.
If both L_0 and L_1 project one-to-one to $\mathbf{q}$-coordinates near the intersection point $(\mathbf{q}_\alpha, \mathbf{p}_\alpha) \in \mathbb{C}^{2n}$ we have

$$L_0 = \text{ graph of } d\mathcal{F}_0, \qquad L_1 = \text{ graph of } -d\mathcal{F}'_0$$

Then the corresponding quantum wave functions $|\psi\rangle$ and $\langle\psi'|$ can be understood as half-densities:

$$|\psi\rangle = \hbar^{-n/4} \exp\Big(\sum_{g \geq 0} \hbar^{g-1}\mathcal{F}_g\Big) \cdot (dq_1 \wedge dq_2 \wedge \dots \wedge dq_n)^{1/2},$$

$$\langle\psi'| = \hbar^{-n/4} \exp\Big(\sum_{g \geq 0} \hbar^{g-1}\mathcal{F}'_g\Big) \cdot (dq_1 \wedge dq_2 \wedge \dots \wedge dq_n)^{1/2},$$

and the function $\mathcal{F}_0 + \mathcal{F}'_0$ has Morse critical point at $\mathbf{q}_\alpha \in \mathbb{C}^n$.

The local pairing at $(\mathbf{q}_\alpha, \mathbf{p}_\alpha) \in L_0 \cap L_1 \subset \mathbb{C}^{2n}$ is defined as a formal integral over the local Lefschetz thimble:

$$\langle \psi' | \psi \rangle_{(\mathbf{q}_\alpha, \mathbf{p}_\alpha)} :=$$

$$\hbar^{-n/2} \int_{\substack{\text{Lefschetz thimble} \\ \text{near } \mathbf{q}_\alpha}} \exp\Big(\sum_{g \geq 0} \hbar^{g-1} (\mathcal{F}_g + \mathcal{F}'_g)\Big) \cdot (dq_1 \wedge dq_2 \wedge \cdots \wedge dq_n).$$

Next three slides: pairing of quantum wave functions represents the evaluation of the WKB series at a point.

Consider the following differential equation depending analytically on $\hbar$:

$$\left[-\left(\hbar\frac{d}{dx}\right)^2 + (x^4+1)\right]\psi(x) = 0, \qquad x = q = q_1$$

We normalize the WKB solution by the condition at $+\infty$:

$$\lim_{x\to+\infty} \hbar^{1/4}\psi(x)\cdot \exp\left(\frac{x^3}{3\hbar}\right) = 1$$

as series in $\hbar$.
In other words,

$$\mathcal{F}_0(x) = -x^3/3, x \to +\infty, \lim_{x\to+\infty} \mathcal{F}_g(x) = 0, g \geq 1.$$

This gives a multi-valued analytic wave function on the elliptic curve

$$L_0 := \{(q, p) \in \mathbb{C}^2 | p^2 = q^4 + 1\}$$

with branches which differ from each other by the value of quantum periods:

$$H_1(L_0, \mathbb{Z}) \to \mathbb{C}[[\hbar]].$$

Quantum periods are equal modulo $\hbar$ to the classical periods of the 1-form $p\, dq_{|L_0}$.

If we *fix* $x = x_0 \in \mathbb{C}$ then the formal expression in $\hbar$ (which is the "value" of WKB solution at $x = x_0$):

$$\hbar^{-1/2} \exp\Big(\sum_{g \geq 0} \mathcal{F}_g(x_0)\, \hbar^{g-1}\Big)$$

can be identified with $\langle \psi' | \psi \rangle$ where ψ' informally corresponds to $\hbar^{-1/2}$ "$\delta(x - x_0)$", i.e. it is an analytic right wave function associated with the Lagrangian submanifold

$$L_1 := \{(q, p) \in \mathbb{C}^2 | q = x_0\}$$

More precisely, it is the wave function obtained by applying to the most basic function (constant)

$$\psi(x) := \hbar^{-1/4} = \hbar^{-1/4} \exp(0), \quad \mathcal{F}_0 = \mathcal{F}_1 = \cdots = 0$$

an appropriate element of $SL(2, \mathbb{C}) \ltimes \mathbb{C}^2$ which moves the line $q_2 = 0$ to the line $q_1 = x_0$.

Relation to the Chern-Simons theory

For the Chern-Simons theory on a compact 3-manifold M_3 with the simple simply-connected compact gauge group G_c and level $k \in \mathbb{Z}$, one can try to apply our proposal in the following way. First, we complexify the theory by taking the complexification G of G_c. Then consider a knot $K \subset M_3$. Its small tubular neighborhood in M_3 gives rise to a complex symplectic manifold (more precisely, symplectic stack) of flat G-connections on the boundary of the tubular neighborhood. It contains two Lagrangian subvarieties L_{in} (resp. L_{out}) consisting of those connections which can be extended inside (resp. outside) of the tubular neighborhood of K. The intersection $L_{in} \cap L_{out}$ can be interpreted as the set of flat G-connections on M_3.

Since the Chern-Simons functional is multivalued as a function, the above considerations should be lifted to the universal cover of the space of all G-connections. Then one should assign to each intersection point of L_{in} and L_{out} a series in $1/k$ (let us ignore for a moment that k is integer), and combine the series for different intersection points in a single one, which is a perturbative invariant of M_3. This is a story about path integral with the boundary conditions discussed above. Our proposal should give the local answers in terms of the pairing of the corresponding analytic wave functions. In the finite-dimensional case this would be the exponential integral over the linear combination of local thimbles.

Then the formal expansions at intersection points should be combined with the wall-crossing formulas which describe the interaction between the intersection points.