Canonical wall structures via punctured Gromov-Witten theory

Miami 04/22 (w/ M. Gross)

GS 03: Mirrors of degenerate CY's as log-spaces

$$\begin{array}{ccc} (B,\mathcal{P},\varphi) & \xleftrightarrow[1:1]{DLT} & (\check{B},\check{\mathcal{P}},\check{\varphi}) \\ \updownarrow 1:1 & & \updownarrow 1:1 \\ (X_0, \mathcal{M}_{X_0}) & & (Y_0, \mathcal{M}_{Y_0}) \end{array}$$

GS 07: $h^{p,q}(X_0^+) = h^{n-p,q}(Y_0^+)$

GS 07: Unique smoothing via wall structures (algorithmic construction – scattering)

$$\begin{array}{ccc} X_0 \hookrightarrow \mathfrak{X} & & Y_0 \hookrightarrow \mathcal{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C}[\![t]\!] = S & & \check{S} \end{array}$$

GPS 09: Enumerative interpretation of scattering (blown up toric surfaces)

GHK 11: canonical scattering diagram: $(Y,D) \rightsquigarrow \mathfrak{X}/\operatorname{Spec}\mathbb{C}[NE(Y)]$
dim = 2

(AGS 07)/ G 09/CPS 09/GHK 11/GHS 16: broken lines & theta functions

ACGS 19 punctured invts

$\left(\begin{array}{l}(B,P) + \text{consistent} \\ \text{wall structure}\end{array}\right) \rightsquigarrow \mathfrak{X} = \operatorname{Spec}/\operatorname{Proj} \bigoplus_{p\in B(\mathbb{Z})} A\,\vartheta_p$ fcts on $X_0 \mid$ codim = 2

GS 19: Intrinsic mirror symmetry (w/o walls – cf. SH^0)

$$\begin{array}{c}(Y,D)\\(\mathcal{Y},Y_0)\end{array} \rightsquigarrow \vartheta_p \cdot \vartheta_q = \sum_r N_{pq}^r \cdot \vartheta_r$$

punctured invts

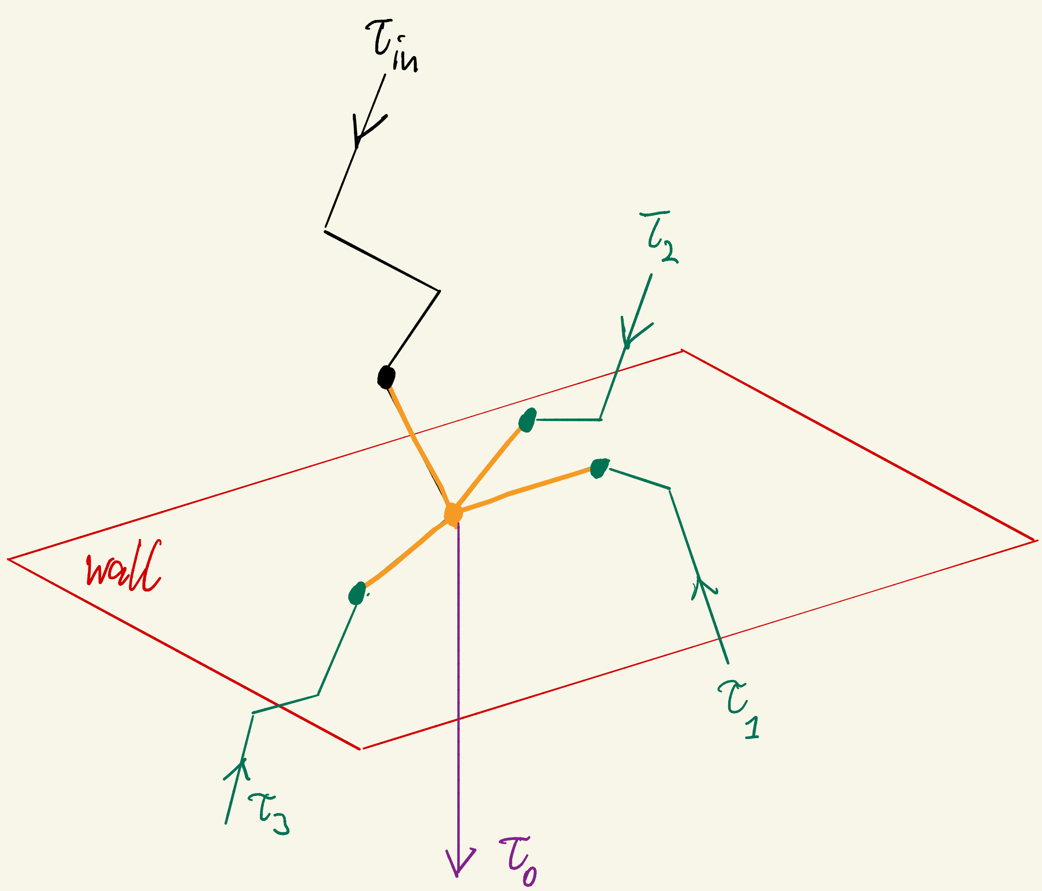
GS 21: walls and ϑ-fcts from punctured invts

For (X,D) log CY construct (rel. case: $(X,D)=(\mathcal{X},X_0)$)

I) wall structure on $(B,\mathcal{P})$ [dual intersection cplx]

II) broken lines via punctured invts

III) scattering of broken lines on walls via gluing of punctured invts



Intrinsic mirror symmetry setup

X projective $/\mathbb{C}$, $D \subset X$ snc,

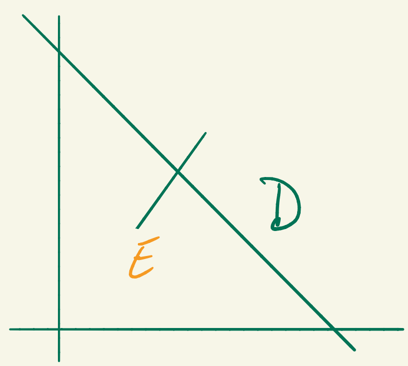
$$c_1(\Theta_X(\log D)) \equiv_{\mathbb{Q}} \sum a_i D_i \,, \quad a_i \geq 0$$

"good" divisors : D_i with $a_i = 0$.

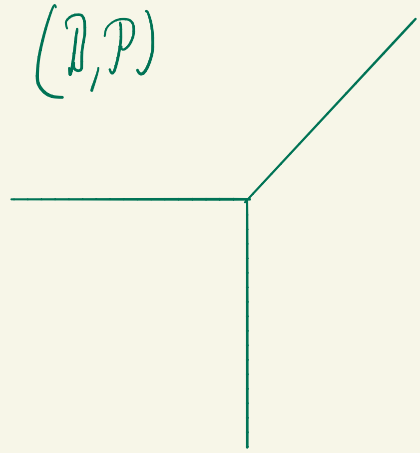
$\rightsquigarrow$ (B, P) = dual intersection complex for $\{D_i \mid a_i = 0\}$.

Simplicial complex P, $B = |P|$. (KS-skeleton)

E.g. $Bl_1 \mathbb{P}^2$:



(B, P)



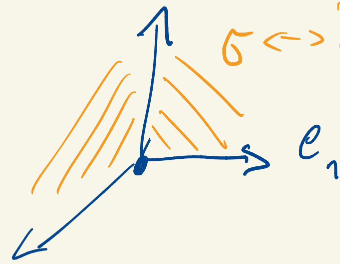
D with or without E

Affine structure

(B,P) has affine str. in codim $=1$ $\Longleftrightarrow$ $D_{i_1}\cap\dots\cap D_{i_{n-1}} = C$

then locally (B,P) : fan $\quad\langle e_2,\dots,e_{n-1}\rangle$

$\sigma' = D_1'\cap D_2\cap\dots\cap D_n$ $\qquad$ $\sigma \leftrightarrow D_1\cap D_2\cap\dots\cap D_n$



$$e_1' = -e_1 - \sum_{\mu=2}^{n}(D_{i_\mu}\cdot C)\cdot e_\mu$$

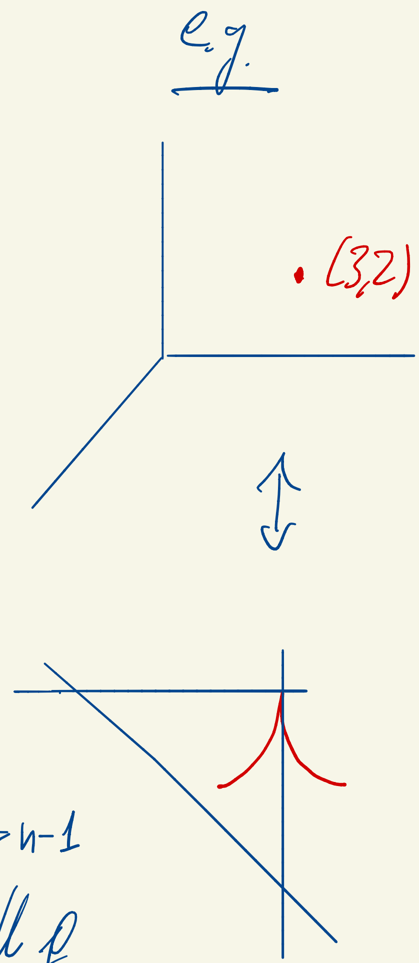
Prop: (Cf. Nicaise-Xu-Yu) $C \simeq \mathbb{P}^1$
logarithmic strata : $\{0,\infty\}$.

Cor: $(C, \mathcal{M}_X|_C)$ is idealized log smooth

Walls from 1-punctured invariants I

$\mathcal{B}(\mathbb{Z}) = \{$ contact orders ≥ 0 with $D \}$

e.g.



$A \in H_2(X, \mathbb{Z})$, $u \in \mathcal{B}(\mathbb{Z})$

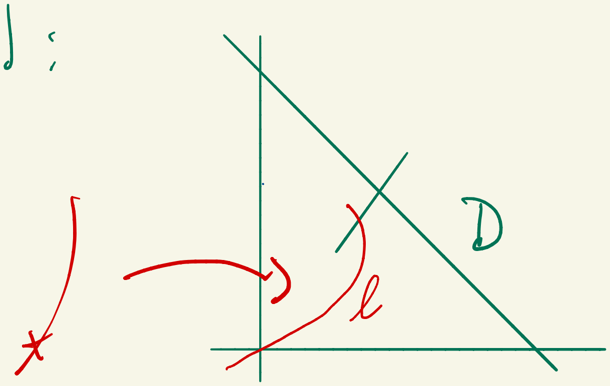
Consider 1-pointed GW-invt $N_{A,u,\tau}$

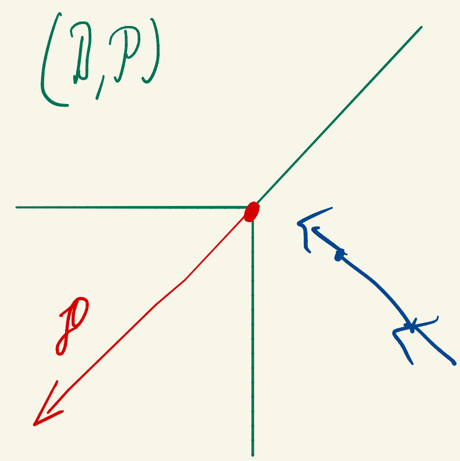
- class A
- contact order $u \in \Lambda_\sigma$, $\dim \sigma = n-1, n$
- tropical type τ of $\dim = n-2$

$\Rightarrow$ end pt of tropicalization covers wall ρ ($\dim = n-1$)

Expl (GHK):

$Bl_1 \mathbb{P}^2$





Walls from 1-punctured invariants II

Consider 1-pointed GW-invt $N_{A,u,\tau}$

- class A
- contact order $u \in \Lambda_\sigma$, $\dim = n-1, n$
- tropical type τ of $\dim = n-2$

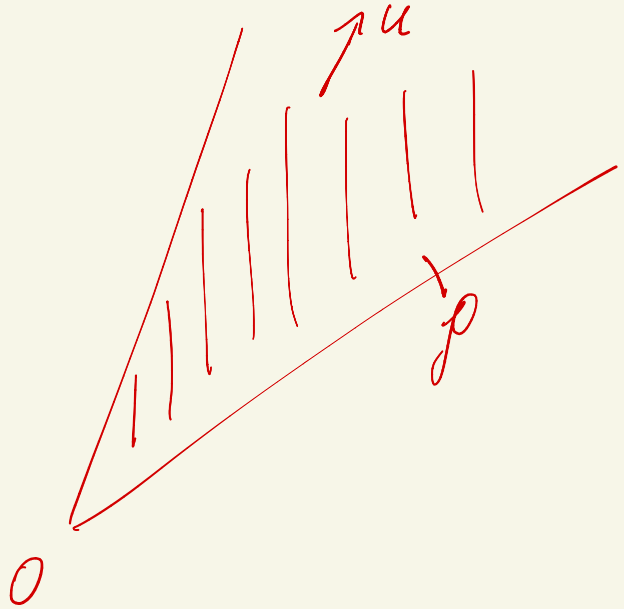
$$ev : \tilde{\tau} \to \sigma$$

<u>Fct. carried by wall:</u>

$$\boxed{f_\rho = \exp\left(k_\tau N_{A,u,\tau} t^A z^{-u}\right)}$$

$$k_{\tilde{\tau}} = |\mathrm{coker}(ev : \Lambda_{\tilde{\tau}} \to \Lambda_\sigma)_{tor}|$$

$(n=3)$



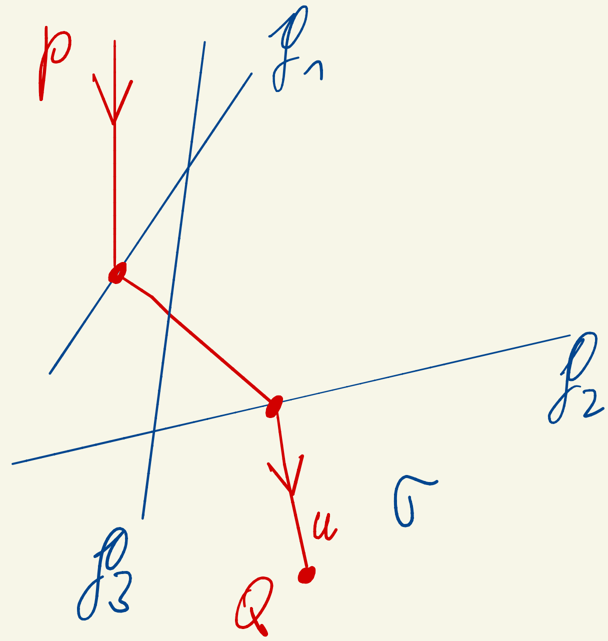
Broken lines from 2-punctured invariants

Depends on 2 contact orders:

- p (incoming) ≥ 0
- u (outgoing) $\in \Lambda_\sigma$, $\sigma \in \mathcal{P}$

and

- $A \in H_2(X, \mathbb{Z})$
- tropical type τ



Punctured GW-inv: $N_{A,p,u,\tau}$

Theta fcts:

$$\boxed{\vartheta^{\log}_{p,Q} = \sum_{A,u,\tau} k_\tau N_{A,p,u,\tau} \, t^A z^{-u}}$$

Wall structure consistent $\overset{[GHS]}{\Longrightarrow}$ defines homogeneous coord. ring of mirror.

Thm (GS20):

a) $\vartheta^{\log}_{P,Q} = \vartheta^{GHS}_{P,Q}$

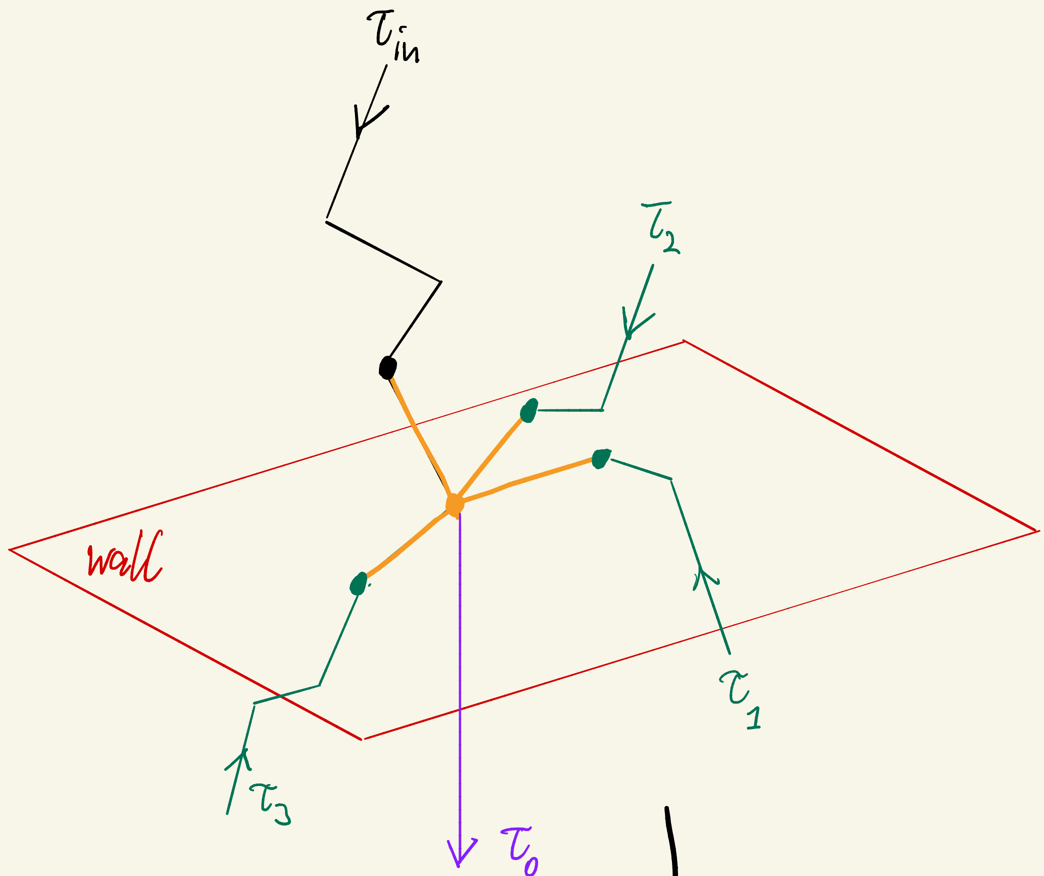
b) log-theoretic wall structure is consistent

c) structure coefficients are as in "Intrinsic Mirror Symmetry"

Rest of talk:

Log-theoretic broken lines = broken lines from scattering, as in [GHS]

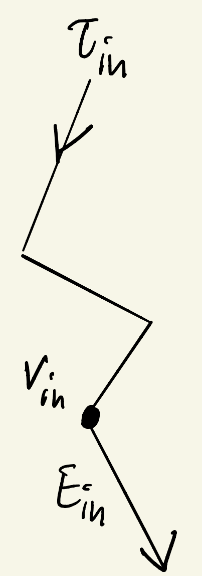
Scattering analysis via gluing

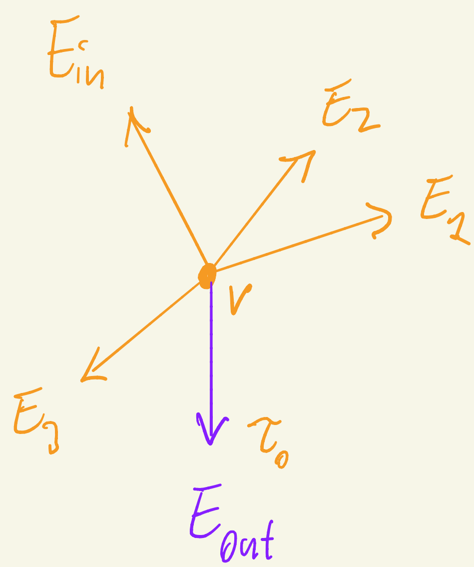


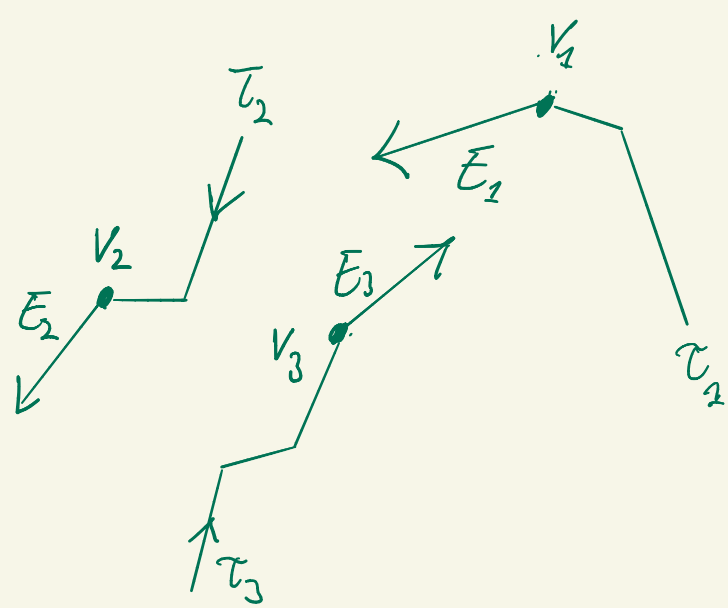
split

incoming broken line

wall types







Yixian Wu's formula

minimal cell σ containing g:

$\dim \sigma = n \Rightarrow$ gluing stratum $=$ pt

$\dim \sigma = n-1 \Rightarrow$ " $= \mathbb{P}^1$

<u>Thm</u> (Yixian Wu) Split tropical type τ along edges $E_1, \ldots, E_r$ and let $\delta : \mathcal{M}(X,\tau) \longrightarrow \prod_{i=1}^{s} \mathcal{M}(X,\tau_i)$ be the splitting morphism. Assume toric gluing strata, w a generic displacement vector.

$$\Rightarrow \quad \delta_*[\mathcal{M}(X,\tau)]^{virt} = \sum_{(\rho_i)\in\Delta(w)} \frac{m(\rho)}{\mathrm{Aut}(\rho)} \cdot \prod_i [\mathcal{M}(X,\rho_i)]^{virt}.$$

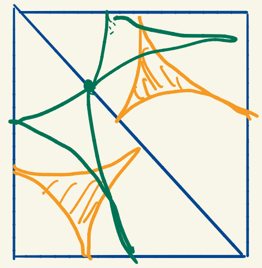
$\Delta(w) = \{(\rho_i)\}$, (ρ_i) perturbation of (τ_i), displaced by w

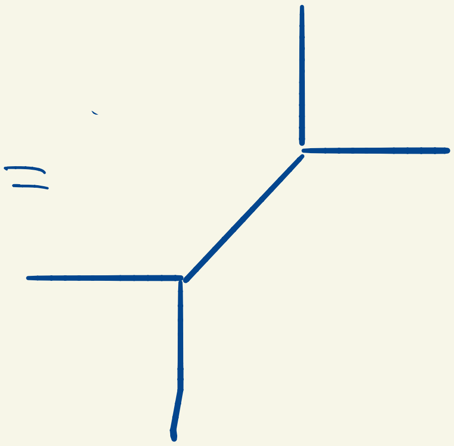
minimal-dim'l: $\sum \dim \tilde{\rho}_i = \dim N_E + \dim \tilde{\tau}$

$\tilde{\rho}_i = \rho_i +$ evaluation pts at gluing legs

$m(\rho) =$ lattice index

Expl:

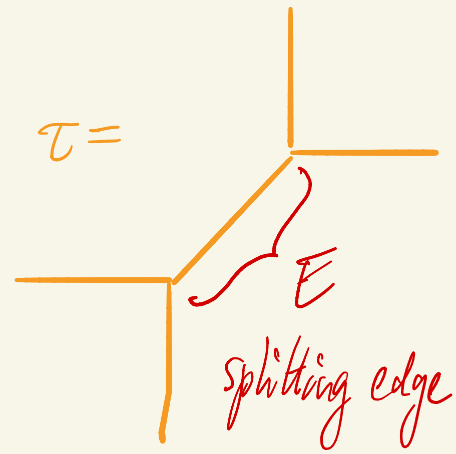
$X =$ 

$\Sigma(X) =$ 

Glue two lines in $\mathbb{P}^2 \cup \mathbb{P}^2$

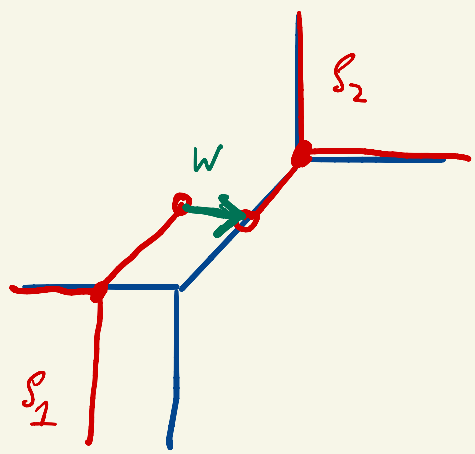
Gluing stratum: $\mathbb{P}^1$

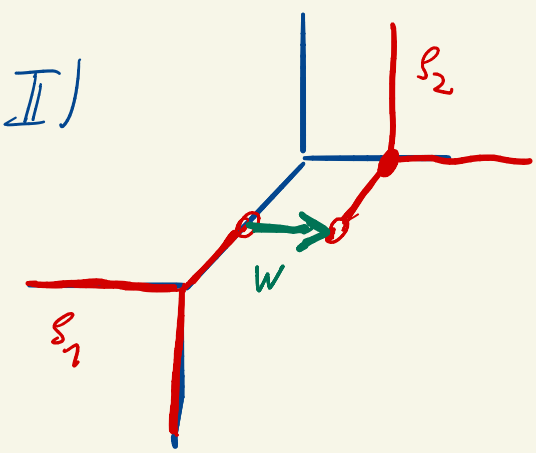
τ: type of rigid tropical curve defined by 1-skeleton of $\Sigma(X)$.

$\tau =$ 

w: say $(1,0)$ $\longrightarrow$

$\Delta(w)$: has two elements:

I) 

II) 

dimension formula: $\sum \dim \tilde{\rho}_i = \dim N_E + \dim \tilde{\tau}$

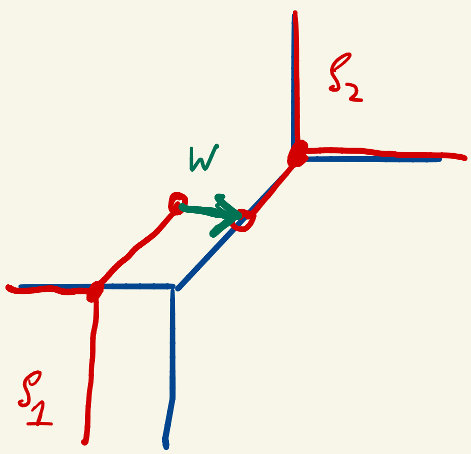
e.g. in (I): $\underbrace{(2+1)}_{\dim \rho_1} + \underbrace{(1+1)}_{\dim \rho_2} = 3 + (1+1)$ ✓

Geometrically:

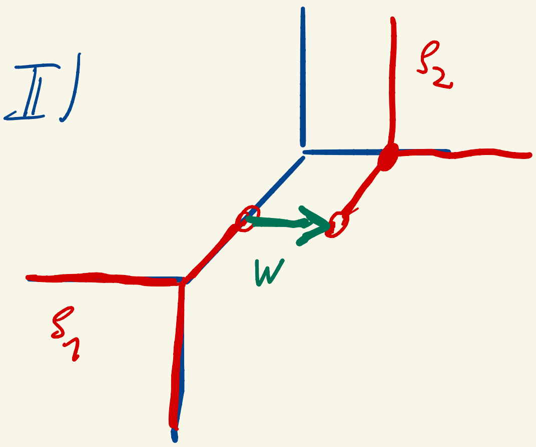
w-displacement vector $\longleftrightarrow$ $\mathbb{G}_m$-action on $\prod \mathcal{M}(X_i, \tau_i)$

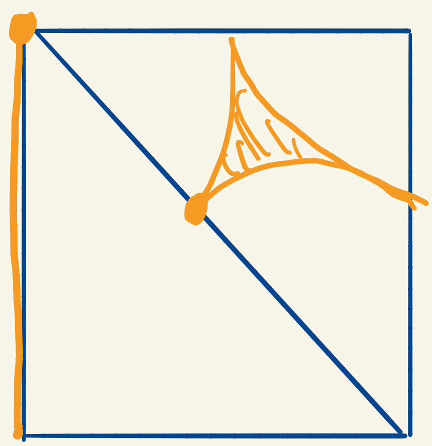
[or rather on $\prod \mathfrak{M}^{ev}(\mathcal{X}_i, \tau_i)$]

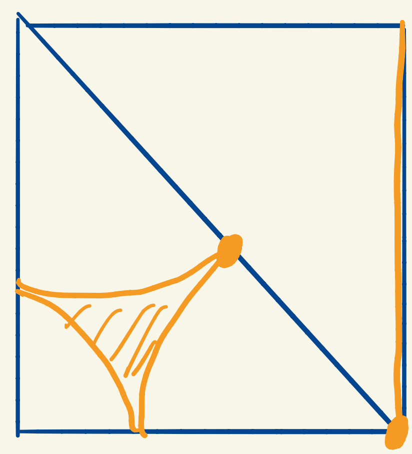
I)



II)

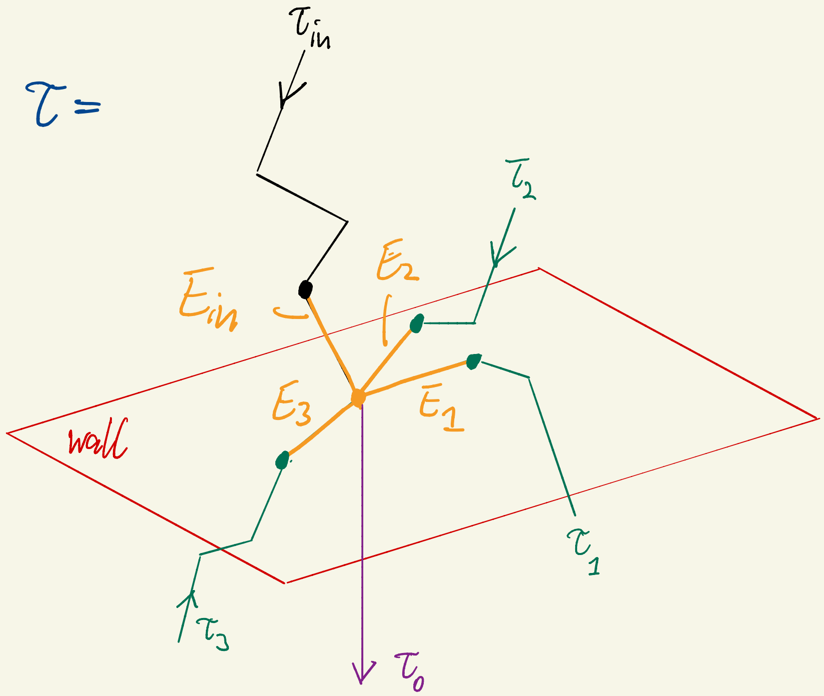




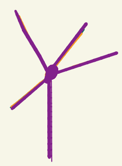


For broken line rule:

$\tau =$



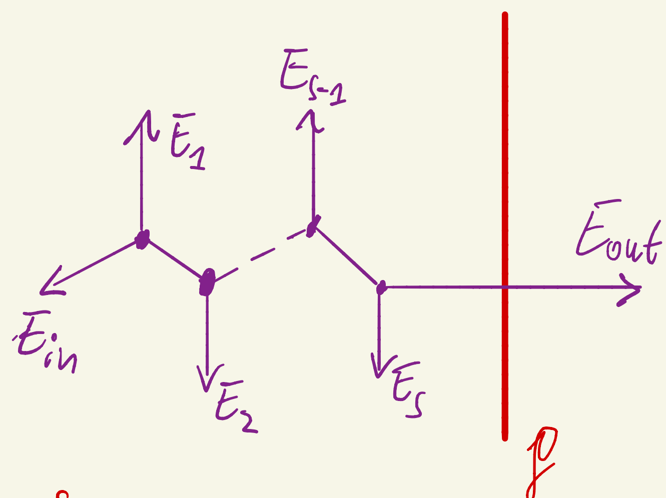
$$\tau \xrightarrow[E_{in}, E_1, \dots, E_s]{\text{split at}} \begin{cases} \tau_{in} \\ \tau_1, \dots, \tau_s \\ \tau_0 \end{cases}$$



Lemma 1: $\exists\, w$ s.th. $\# \Delta(w) = 1$

$\Delta(w) = \{\rho\}, \quad \rho = (\rho_{in} = \tau_{in}, \rho_1 = \tau_1, \dots, \rho_s = \tau_s, \rho_0)$

ρ_0: $\left(\begin{array}{l} \text{drawn for } n = 2 \\ \dim \sigma = n \end{array}\right)$



Lemma 2: $m(\rho) = \kappa_{out}^{-1} \kappa_{in} \cdot \prod_{i=1}^{s} (d \kappa_i)$

$d = |\pi(u_{in})|, \quad \pi : \Lambda_\sigma \longrightarrow \Lambda_\sigma / \Lambda_\rho \simeq \mathbb{Z}$

$\kappa_{out}, \kappa_{in}, \kappa_i : |\mathrm{color}(ev)_{tor}|$

Result:

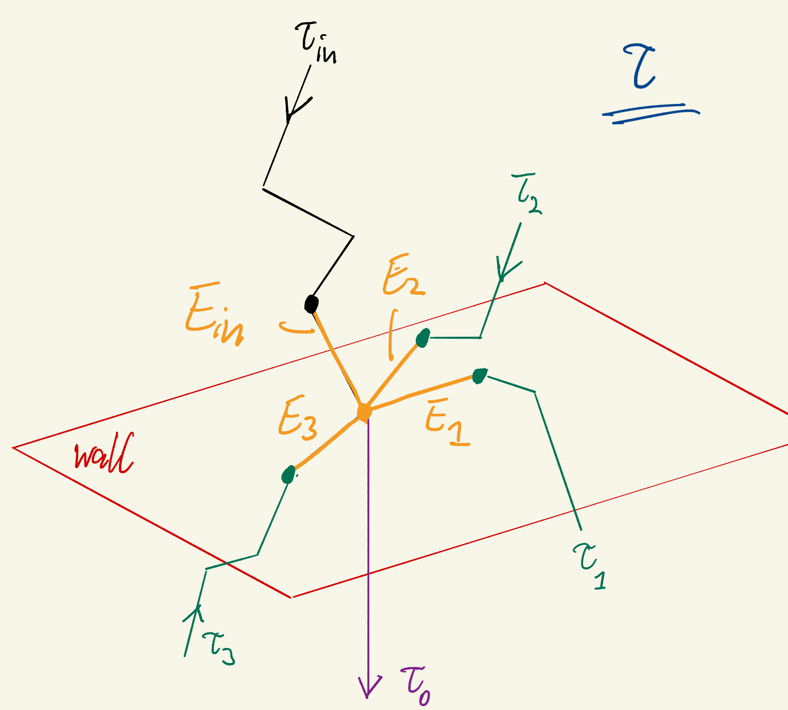
$$\delta_{\mathfrak{X}} [\mathcal{M}(X,\tau)]^{virt} = \frac{\mathcal{K}_{in}}{\mathcal{K}_{out}} (d\mathcal{K}_i) \cdot [\mathcal{M}(X,\tau_{in})]^{virt} \times [\mathcal{M}(X,s_0)]^{virt} \times \prod_{i=1}^{s} [\mathcal{M}(X,\tau_i)]^{virt}$$

in $A_{\mathfrak{X}}\left(\mathcal{M}(X,\tau_{in}) \times \mathcal{M}(X,\tau_0) \times \prod_i \mathcal{M}(X,\tau_i)\right)$

For broken line scattering rule:

LHS: result of interaction of broken line with wall $\mathfrak{p}$

RHS:



$\mathcal{M}(X,\tau_{in}) \leftrightarrow$ broken line before wall

$\mathcal{M}(X,\tau_i) \leftrightarrow$ 1-pt invariants carried by $\mathfrak{p}$

$\mathcal{M}(X,s_0) \leftrightarrow$ interaction term

Result:

$$\delta_* [\mathcal{M}(X,\tau)]^{virt} = \frac{\mathcal{K}_{in}}{\mathcal{K}_{out}} (d\mathcal{K}_i) \cdot [\mathcal{M}(X,\tau_{in})]^{virt} \times [\mathcal{M}(X, s_0)]^{virt} \times \prod_{i=1}^{s} [\mathcal{M}(X,\tau_i)]^{virt}$$

in $A_*\left(\mathcal{M}(X,\tau_{in}) \times \mathcal{M}(X,\tau_0) \times \prod_i \mathcal{M}(X,\tau_i)\right)$

Lemma 3: $\deg [\mathcal{M}(X,s_0)]^{virt} = \begin{cases} 1 & , \dim \sigma = n \\ 1/d & , \dim \sigma = n-1 \end{cases}$

$\dim \sigma = n-1 \Rightarrow$ only contribution from curve class $d \cdot [C]$ $\left(C \overset{P}{\Leftrightarrow} \sigma\right)$

$$\boxed{\text{Rest of } \vartheta^{log}_{p,Q} = \vartheta^{GHS}_{p,Q} : \text{ bookkeeping!}}$$