Functions on commuting stacks via mirror symmetry

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Ingredients

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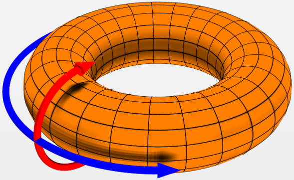
- G reductive complex group (Ex: $G = SL(n), PGL(n), GL(n), \ldots$)
- $G^\vee$ Langlands dual group (Ex: $G^\vee = PGL(n), SL(n), GL(n), \ldots$)
- X smooth projective complex curve.

  Today's focus: $X = E$ genus one curve.



A-side (automorphic) target

Pair $(T^*\mathrm{Bun}_G(X), \mathcal{N})$.

- $T^*\mathrm{Bun}_G(X)$ exact symplectic derived stack.
- $\mathcal{N} \subset T^*\mathrm{Bun}_G(X)$ closed conic Lagrangian.

Construction:

- $\mathrm{Bun}_G(X)$ moduli of G-bundles on X (smooth stack).
- $T^*\mathrm{Bun}_G(X)$ Higgs moduli (symplectic derived stack).
- $\mathcal{N} = \chi^{-1}(0)$ nilpotent cone (closed conic Lagrangian).
  $\chi : T^*\mathrm{Bun}_G(X) \to \mathfrak{c}_G(X)$ Hitchin system induced by characteristic polynomial $\mathfrak{g}^* \to \mathfrak{h}^*/\!/W$.

Informally:

- $T^*\mathrm{Bun}_G(X)$ is a "Weinstein manifold" with core $\mathrm{Bun}_G(X)$.
- $\mathcal{N}$ defines a "stop" at infinity.

Question

Does pair $(T^*\mathrm{Bun}_G(X), \mathcal{N})$ come from a "'Weinstein pair"?

- *Extra credit: Functorial in X?*

A-side (automorphic) branes

A-branes: $A_G(X) = Sh_{\mathcal{N}}(\mathrm{Bun}_G(X))$

dg derived category of complexes of sheaves on $\mathrm{Bun}_G(X)$,

singular support (Kashiwara-Schapira) lying in $\mathcal{N} \subset T^*\mathrm{Bun}_G(X)$.

Informally: partially wrapped branes for pair $(T^*\mathrm{Bun}_G(X), \mathcal{N})$.

Example ($G = GL(1)$)

- *Pair* $(T^*\mathrm{Pic}(X)$, *0-section*).
- $A_G(X)$ *dg derived category of local systems on* $\mathrm{Pic}(X)$.

B-side (spectral/Galois) target

Pair $(\mathrm{Loc}_{G^\vee}(X), \mathcal{N}^\vee)$.

- $\mathrm{Loc}_{G^\vee}(X)$ quasi-smooth derived stack.
- $\mathcal{N}^\vee \subset T^*[-1]\mathrm{Loc}_{G^\vee}(X)$ closed conic support condition.

Construction:

- $\mathrm{Loc}_{G^\vee}(X)$ moduli of $G^\vee$-local systems on X (affine scheme/linear group).
- $T^*[-1]\mathrm{Loc}_{G^\vee}(X)$ shifted cotangent bundle (classical stack).
- $\mathcal{N}^\vee = \chi^{-1}(0)$ nilpotent cone (closed conic subset).

  $\chi : T^*[-1]\mathrm{Loc}_{G^\vee}(X) \to (\mathfrak{g}^\vee)^*/G^\vee \simeq \mathfrak{h}/\!/W$ induced by
  Chevalley isomorphism $T^*[1]BG^\vee \simeq (\mathfrak{g}^\vee)^*/G^\vee \simeq \mathfrak{h}/\!/W$.

Informally:

- $\mathrm{Loc}_{G^\vee}(X)$ is a "singular variety".
- $\mathcal{N}^\vee$ support condition for "singularities/matrix factorizations".

B-side (spectral/Galois) branes

B-branes: $B_{G^\vee}(X) = \mathrm{IndCoh}_{\mathcal{N}^\vee}(\mathrm{Loc}_{G^\vee}(X))$

dg derived category of ind-coherent sheaves on $\mathrm{Loc}_{G^\vee}(X)$,

singular support (Arinkin-Gaitsgory) lying in $\mathcal{N}^\vee \subset T^*[-1]\mathrm{Loc}_{G^\vee}(X)$.

Smooth B-branes: $B^\circ_{G^\vee}(X) = \mathrm{QCoh}(\mathrm{Loc}_{G^\vee}(X))$

- dg derived category of quasi-coherent sheaves on $\mathrm{Loc}_{G^\vee}(X)$.
- $B^\circ_{G^\vee}(X)$ tensor action on $B_{G^\vee}(X)$.
- $B_{G^\vee}(X)$ inverse image of branes supported on $\mathcal{N}^\vee$ within

$$\mathrm{Sing}(\mathrm{Loc}_{G^\vee}(X)) = (\mathrm{IndCoh}/\mathrm{QCoh})(\mathrm{Loc}_{G^\vee}(X))$$

Example ($G = GL(1)$)

- *Pair* $(\mathrm{Loc}_1(X), \textit{0-section})$.
- $B_G(X) = B^\circ_G(X) = \mathrm{QCoh}(\mathrm{Loc}_1(X))$.

B-side (spectral/Galois) is topological

Observation

Pair $(\mathrm{Loc}_{G^\vee}(X), \mathcal{N}^\vee)$ *only depends on topology of* X.

$$\begin{aligned} \mathrm{Loc}_{G^\vee} &\simeq \mathrm{Hom}(\pi_1(X, x_0), G^\vee)/G^\vee \\ &\simeq ((G^\vee)^{2g} \times_{G^\vee} \{1\})/G^\vee \end{aligned}$$

Consequence

B-branes $B_{G^\vee}(X), B^\circ_{G^\vee}(X)$ *only depend on topology of* X.

Spectral action

Chiral integration of spherical Hecke operators (modifications of G-bundles at points of X) provides:

Theorem (Spectral action, N-Yun)

Tensor action of smooth B-branes $B^{\circ}_{G}(X)$ on all A-branes $A_G(X)$.

Consequence

fix an A-brane $L_{\mathcal{O}}$ to obtain action functor:

$$\alpha : B^{\circ}_{G}(X) \to A_G(X) \qquad \alpha(V) = V \star L_{\mathcal{O}}$$

By construction: $\alpha(\mathcal{O}) = L_{\mathcal{O}}$.

Question

What should we take for $L_{\mathcal{O}}$?

Whittaker object

Recall:

- Hitchin system: $\chi : T^*\mathrm{Bun}_G(X) \to \mathfrak{c}_G(X)$ induced by characteristic polynomial $\mathfrak{g}^* \to \mathfrak{h}^*/\!/W$.

Hitchin-Kostant section:

- $\kappa : \mathfrak{c}_G(X) \to T^*\mathrm{Bun}_G(X)$ induced by Kostant section $\mathfrak{h}^*/\!/W \to \mathfrak{g}^*$ ("rational normal form").

Image Lagrangian:

- $L_{\mathcal{O}} = \kappa(\mathfrak{c}_G(X))$.
- Intersects $\mathcal{N} \subset T^*\mathrm{Bun}_G(X)$ transversely at smooth point

$$\xi = L_{\mathcal{O}} \cap \mathcal{N}$$

Definition

Whittaker object: $\mathrm{Wh} \in A_G(X)$ *sheaf corepresenting:*

Microstalk at $\xi = L_{\mathcal{O}} \cap \mathcal{N}$.

Informally: "Floer pairing $CF^*(L_{\mathcal{O}}, -)$*".*

Betti Langlands conjecture (unramified)

Conjecture (Betti Langlands conjecture)

Spectral action on Wh *extends to* $B_G^\circ(X)$*-equivariant equivalence*

$$B_{G^\vee}(X) \xrightarrow{\sim} A_G(X)$$

Conjecture (Reality check...)

A-side (automorphic) category $A_G(X)$ *only depends on topology of* X.

Remark

Work of N-Shende implies invariance of A-branes on stable Higgs bundles when smooth Weinstein manifold (e.g. ramified with generic parameters).

Betti Langlands conjecture (unramified) status

Theorem (Derived Satake, Bezrukavnikov-Finkelberg)

$$B_{G^\vee}(\mathbb{P}^1) \xrightarrow{\sim} A_G(\mathbb{P}^1)$$

Remark

Bezrukavnikov has also proved fundamental tamely ramified cases:

- *"disk" ($\mathbb{P}^1$ with 1 marked point)*
- *"cylinder" ($\mathbb{P}^1$ with 2 marked points).*

Main result of today:

Theorem (Genus one, Li-N-Yun)

For E of genus one, Betti Langlands holds:

$$B_{G^\vee}(E) \xrightarrow{\sim} A_G(E)$$

Application to commuting stack

Fix $X = E$ genus one, find stack of commuting pairs

$$\mathrm{Loc}_{G^\vee}(E) = \{(g_1, g_2) \in G^\vee \times G^\vee \mid [g_1, g_2] = e\}/G^\vee$$

- (Affine scheme $\subset G^\vee \times G^\vee$)/ (linear group $G^\vee$).
- Affine is quasi-smooth but derived: $(G^\vee \times G^\vee) \times_{G^\vee} \{e\}$.

Question

What is ring of functions $\mathcal{O}(\mathrm{Loc}_{G^\vee}(E))$?

Find answer on A-side (automorphic)

Fix $X = E$ genus one. Calculation of "wrapped Floer cochains" of Wh_G:

Theorem (Li-N-Yun)

Assume G is adjoint type. Then

$$\mathrm{End}(\mathrm{Wh}_G) \simeq \mathcal{O}(T^\vee \times T^\vee \times \mathfrak{t}^\vee[-1])^W$$

Corollary

Assume $G^\vee$ is simply-connected. Then

$$\mathcal{O}(\mathrm{Loc}_{G^\vee}(E)) \simeq \mathcal{O}(T^\vee \times T^\vee \times \mathfrak{t}^\vee[-1])^W$$

How to calculate $\mathrm{End}(\mathrm{Wh}_G)$ in genus one?

Rest of talk will be on A-side.
Work with tamely ramified A-branes $A_G(X, S)$, for marked points $S \subset G$.
First, reduce from genus one curve to genus zero with two marked points.
Universal affine Hecke category $\mathcal{H}_{LG} = Sh_{\mathcal{N}}(I^\circ \backslash LG / I^\circ)$.

Theorem (N-Yun, Li-N-Yun)

- *Bubbling degeneration $\mathbb{P}^1 \rightsquigarrow \mathbb{P}^1 \vee \mathbb{P}^1$ equips $A_G(\mathbb{P}^1, \{0, \infty\})$ with monoidal structure equivalent to universal affine Hecke category*

$$\mathcal{H}_{LG} \simeq A_G(\mathbb{P}^1, \{0, \infty\})$$

- *Tate nodal degeneration $E \rightsquigarrow E_0$ provides equivalence from cocenter, ie Hochschild homology category, compatible with Whittaker objects*

$$hh(\mathcal{H}_{LG}) \xrightarrow{\sim} A_G(E)$$

How to calculate $\mathrm{End}(\mathrm{Wh}_G)$ in cocenter?

Next, must make calculations in cocenter $hh(\mathcal{H}_{LG})$.
Strategy: replace

$$\text{algebra of cocenter } hh \rightsquigarrow \text{geometry of conjugacy classes in } LG$$

Assume for simplicity G is simply-connected.
I^a simple roots of loop group LG.
$P_J \subset LG$ standard parahoric, $G_J \subset LG$ standard Levihoric, $J \subsetneq I^a$.
$\mathcal{H}_{G_J} = Sh_{\mathcal{N}}(U \backslash G_J / U)$ universal finite Hecke category.

Theorem (universal version of Tao-Travkin result)

There is a natural equivalence in monoidal categories

$$\mathrm{colim}_{J \subset I^a} \mathcal{H}_{G,J} \xrightarrow{\sim} \mathcal{H}_{LG}$$

$X_{LG,J} = LG/P_J$ Lusztig's parahoric character sheaf space.

Corollary

$$hh(\mathcal{H}_{LG}) = \mathrm{colim}_{J \subset I^a} hh(\mathcal{H}_{G_J}, \mathcal{H}_{LG}) \simeq \mathrm{colim}_{J \subset I^a} Sh_{\mathcal{N}}(X_{LG,J})$$

How to calculate $\mathrm{End}(\mathrm{Wh}_G)$ in parahoric character sheaves?

Next, must make calculations in $\mathrm{colim}_{J \subset I^a} Sh_{\mathcal{N}}(X_{LG,J})$.
Strategy: replace

$$\text{parahoric character sheaves } hh(\mathcal{H}_{G_J}, \mathcal{H}_{LG}) \simeq Sh_{\mathcal{N}}(X_{LG,J})$$

$$\rightsquigarrow \text{traditional character sheaves } hh(\mathcal{H}_{G_J}, \mathcal{H}_{G_J}) \simeq Sh_{\mathcal{N}}(G_J/G_J)$$

Arrive at main technical result: prove conjecture of Li-N.

Theorem (Li-N-Yun)

The natural map is fully faithful

$$\mathrm{colim}_{J \subset I^a} Sh_{\mathcal{N}}(G_J/G_J) \simeq \mathrm{colim}_{J \subset I^a} hh(\mathcal{H}_{G,J}) \hookrightarrow hh(\mathcal{H}_{LG})$$

Idea: Morse theory on Bruhat-Tits building of LG following He-Nie. Lowest critical energy.

How to calculate $\mathrm{End}(\mathrm{Wh}_G)$ in colimit of character sheaves?

Finally, must make calculations in $\mathrm{colim}_{J \subset I^a} Sh_{\mathcal{N}}(G_J/G_J)$.
Strategy: use orthogonality results of Li to reduce

all character sheaves

$\rightsquigarrow$ Springer block of character sheaves induced from torus

When G adjoint type, arrive at:

$$\mathrm{End}(\mathrm{Wh}_G) \simeq \mathcal{O}(T^\vee \times T^\vee \times \mathfrak{t}^\vee[-1])^W$$

Acknowledgements

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