Moduli in Quantum Toric Geometry HMS IMSA 2022

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Quantum (non-commutative) toric geometry: Foundations

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https://arxiv.org/pdf/2002.03876.pdf

First, I will review the paper (2021).

Classical toric geometry

The classical theory of toric geometry has found multiple applications in the resolution of problems in various fields of mathematics going from combinatorics to differential geometry.

Tori (both real and complex) are the building blocks of the classical theory; indeed, a classical n-complex dimensional compact, projective Kähler toric manifold X can be defined as an equivariant, projective compactification of the n-complex dimensional torus $\mathbb{T}^d_{\mathbb{C}} := \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$).

The classical moment map.

Real tori also play an important role in the classical theory: the real torus $\mathbb{T}^d_{\mathbb{R}} = S^1 \times \cdots \times S^1 \subset \mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{T}^d_{\mathbb{C}}$ acts holomorphically on the whole of $X \supset \mathbb{T}^d_{\mathbb{C}}$. Thinking of the Kähler manifold (X, g, J, ω) as a symplectic manifold, the action of the real compact Lie group $\mathbb{T}^d_{\mathbb{R}}$ on X is Hamiltonian, implying thus the existence of a continuous equivariant moment map μ with convex image P:

$$\mu \colon X \longrightarrow P \subseteq \mathbb{R}^d \cong \mathrm{Lie}(\mathbb{T}^d_{\mathbb{R}})^*.$$

A priori, whenever X is compact, P is a compact, convex set but, for a toric variety X, P turns out to be a convex, rational, Delzant polytope: that is, the combinatorial dual of P is a triangulation of the sphere S^{d-1}, and all the slopes of all the edges of P are rational.

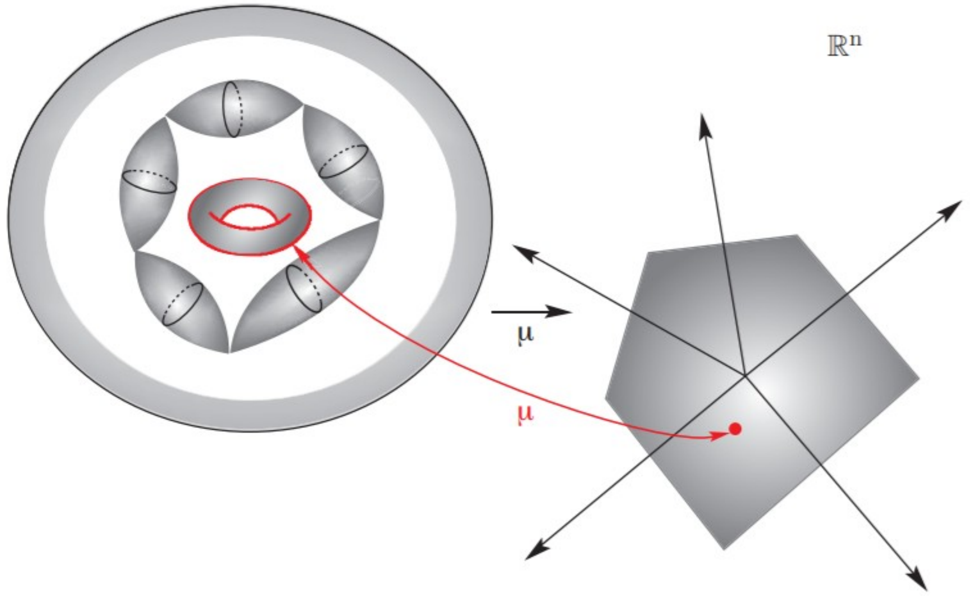


FIGURE 2. The moment map for a toric manifold: the inverse image of every point is a real torus of dimension equal to the dimension of the stratum of P where the point lands. The inverse image of edges are spheres made up of 1-tori (circles). In non.commutative toric geometry all tori and circles are replaced by their non-commutative counterparts

The classical moment map (figure taken from the Notices of the AMS, January 2021).

Fans

It is natural to consider P as a stratified space $P = P_0 \amalg P_1 \amalg \cdots \amalg P_d$ where P_i is the disjoint union of all facets of P of dimension i; this stratification is inherited by X via the moment map. To wit, the map $\mu|_{P_i} \colon X_i := \mu^{-1}(P_i) \longrightarrow P_i$ is a trivial real-torus bundle over P_i identifying X_i with the product $P_i \times \mathbb{T}^i_{\mathbb{R}}$, and so, $X = (P_0 \times \mathbb{T}^0_{\mathbb{R}}) \amalg \ldots \amalg (P_d \times \mathbb{T}^d_{\mathbb{R}})$, reconstructing then X as the disjoint union of real Lagrangian tori.

For more general toric varieties, fans (which can be thought in the polytopal case as the cone with vertex at the origin of the dual of the polytope) are used rather than polytopes, but still, an ubiquitous use of complex and real tori (often appearing in the theory in the guise of lattices on vector spaces), and their partial compactifications, are the basis of the classical theory.

The basic idea.

The basic idea behind the field of *Quantum Toric Geometry* is to replace all the tori appearing in classical toric geometry by quantum tori (also known as non-commutative tori): just in the same manner in which toric manifolds can be thought of as integrable systems, our quantum toric manifolds can, in turn, be interpreted as quantum integrable systems. And likewise, for the same essential reason that a version of mirror symmetry for toric varieties can be construed as a parametrized version of T-duality for tori, analogously, quantum toric manifolds will have a version of mirror symmetry in which the basic component is non-commutative T-duality.

Deformation

From a slightly different point of view, quantum toric geometry can be thought of as a deformation (with deformation parameter $\hbar$) of the whole field of toric geometry (say, as presented in [19]): while very many results from the classical theory have their counterparts in our quantum generalization, the proofs of such results are not entirely obvious. On the other hand, the flavor of our theory is familiar, for we encounter the usual suspects: quantum fans, quantum lattices, and the like. Furthermore, of course, the classical theory is a particular case of the quantum theory, as it should be.

[19] David A Cox, John B Little, and Henry K Schenck. *Toric varieties*. American Mathematical Soc., 2011.

The basic block

Thus, the basic building block of our theory is a non-commutative deformation of the classical tori $\mathbb{T}^i_{\mathbb{R}}$ known as the *quantum torus* $\mathscr{T}^i_{\mathbb{R},\hbar}$ (depending on a '*real* deformation parameter $\hbar$): it is one of the most important and basic spaces in the field of non-commutative geometry [17].

[17] Alain Connes. Non-commutative differential geometry. *Publications Mathematiques de l'IHES*, 62:41–144, 1985.

The real quantum 2-torus.

The quantum 2-torus $\mathcal{T}^2_{\mathbb{R},\hbar} \in \mathbf{NCSpaces} \cong \mathbf{NCAlgebras}/\sim_M$ (non-commutative algebras *up to Morita equivalence*) is a good starting example, its algebra $A_\hbar$ of smooth functions (in **NCAlgebras**) has two (periodic) generators X, Y that don't quite commute but rather satisfy the relation:

$$XY = e^{2\pi i\hbar} YX.$$

The algebra $A_\hbar$ can be realized as an operator algebra first appearing in quantum mechanics[3]. When we specialize the parameter $\hbar$ to be zero, we obtain a commutative algebra and, in fact, $\mathcal{T}^2_{\mathbb{R},\hbar=0} \cong \mathbb{T}^2_{\mathbb{R}}$, recovering the usual torus.

[3]In fact, this equation is precisely the classical Born-Heisenberg-Jordan commutation relation [11], [10] ,[23] in Weyl exponential form [56].

[10] Max Born, Werner Heisenberg, and Pascual Jordan. Zur quantenmechanik. ii. *Zeitschrift für Physik*, 35(8-9):557–615, 1926.

[11] Max Born and Pascual Jordan. Zur quantenmechanik. *Zeitschrift für Physik*, 34(1):858–888, 1925.

[56] Hermann Weyl. Quantenmechanik und gruppentheorie. *Zeitschrift für Physik*, 46(1-2):1–46, 1927.

The arithmetic condition.

There is an important dichotomy for the parameter $\hbar$; the space $\mathcal{T}^2_{\mathbb{R},\hbar}$ is truly non-commutative only when $\hbar$ is irrational; when $\hbar$ is rational, its algebra of functions is Morita equivalent to a commutative algebra.

The Kronecker foliation.

A. Connes has pointed out a beautiful geometric interpretation for the non-commutative space $\mathscr{T}^2_{\mathbb{R},\hbar}$; it can be thought of as the space of leaves of a foliation (see Section 6 of [18]). The Kronecker foliation of slope $\hbar$ on $\mathbb{T}^2_{\mathbb{R}}$ (depicted in Fig. 1) consists on taking the foliation of the Euclidean plane $\mathbb{R}^2$ and projecting it up by the translation action of the integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. This is the same as considering the image of $\mathbb{R}^2$ (and its foliation) into $\mathbb{C}^2$ given by the map E, defined as:

$$E : (x, y) \mapsto (\exp(2\pi i x), \exp(2\pi i y)),$$

and it is because of this that the exponential will play a fundamental role in our theory.

[18] Alain Connes and Matilde Marcolli. A walk in the noncommutative garden. *An invitation to noncommutative geometry*, pages 1–128, 2008.

The Kronecker foliation and its holonomy.

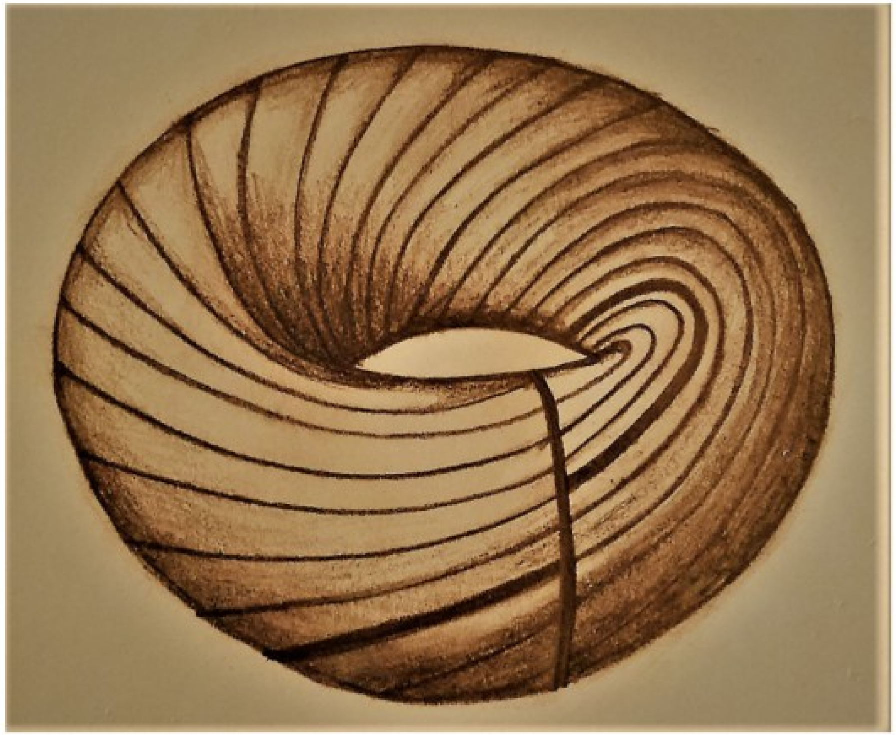


Fig. 1. The Kronecker foliation and its holonomy: the thick line represents an interval of a leaf in the Kronecker foliation going around once and thus defining a rotation from the transversal vertical circle into itself. We will denote the angle of rotation by $\hbar$.

The holonomy groupoid.

Whenever $\hbar = p/q$ is rational, this is a foliation of the real torus $\mathbb{T}^2_{\mathbb{R}}$ by circles (actually (p,q)-torus knots) but, otherwise, each leaf winds densely inside $\mathbb{T}^2_{\mathbb{R}}$.

As a first approximation, we think of the leaf space of the Kronecker foliation as the quotient topological space $\mathfrak{T}(\hbar) := \mathbb{T}^2_{\mathbb{R}}/E(\hbar)$ where $E(\hbar) := \{E(x, \hbar x) : x \in \mathbb{R}\})$ is the (possibly dense) leaf of the torus passing through the origin; it is also a normal subgroup of $\mathbb{T}^2_{\mathbb{R}}$, and the quotient is taken in the group sense. We could obtain the same quotient by considering only the transversal circle (the vertical circle in Fig. 1 above). If $\rho_\hbar \colon S^1 \to S^1$ is the holonomy map that rotates the circle by an angle $\hbar$, and $\langle \rho_\hbar \rangle$ is the discrete group of rotations of the circle it generates (we have an infinite cyclic group $\langle \rho_\hbar \rangle \cong \mathbb{Z}$ whenever $\hbar$ is irrational, and a finite cyclic group otherwise), then we have:

$$\mathfrak{T}(\hbar) := \mathbb{T}^2_{\mathbb{R}}/E(\hbar) \cong S^1/\langle \rho_\hbar \rangle,$$

again, a dichotomy ensues: either $\hbar$ is rational and $\mathfrak{T}(\hbar)$ is a circle (the quotient of a torus by an embedded torus knot) or $\hbar$ is irrational and $\mathfrak{T}(\hbar)$ is a non-Hausdorff topological space.

Non-commutative spaces as stacks.

When, in general, $\mathfrak{T} = T/\sim$ is a non-Hausdorff topological space obtained as the quotient of a manifold T divided by the action of a (possibly non-compact) group (really, any equivalence relation $\sim$ defined by a Lie groupoid action on T), there is, at least, two very fertile ways to enrich $\mathfrak{T}$ preserving some of the information of the geometric groupoid action on T (and landing in nicer categories than that of possibly non-Hausdorff topological spaces); (1) by using non-commutative algebras (taking the non-commutative quotient as in section 4 of [18]), and (2) by using *stacks* (sheafs of groupoids [22]): from $(T, \sim)$ (thought of as a topological groupoid), we can obtain three related objects; (a) a non-Hausdorff topological space, (b) a non-commutative algebra $A_{\mathcal{T}}$, and (c) a stack $\mathscr{T}$. From these, $\mathscr{T}$ is the richer, it has more information about the groupoid $(T, \sim)$ than the other two objects; then, by applying the Connes convolution algebra mapping ([16] page 5):

[22] Dan Edidin. Communications-what is a stack? *Notices of the American Mathematical Society*, 50(4):458–459, 2003.

[16] Pierre Cartier. A mad days work: from Grothendieck to Connes and Kontsevich the evolution of concepts of space and symmetry. *Bulletin of the American Mathematical Society*, 38(4):389–408, 2001.

The stack for the quantum torus.

$$\begin{tikzcd}
\mathbf{Groupoids} \arrow[r, "C"] \arrow[d, two heads] & \mathbf{NCAlgebras} \arrow[d, two heads] \\
\mathbf{Stacks} \arrow[r, "C"] & \mathbf{NCSpaces}
\end{tikzcd}$$

here, it is useful to remember that $\mathbf{Stacks} \cong \mathbf{Groupoids}/{\sim_M}$ and that $\mathbf{NCSpaces} \cong \mathbf{NCAlgebras}/{\sim_M}$, moreover, the descending arrows consists in both cases in quotienting out Morita equivalences.

Let us consider the example of the quantum torus. Here we have:

$$\begin{tikzcd}
(\mathbb{T}^2_{\mathbb{R}}, E(\hbar)) \arrow[r, mapsto, "C"] \arrow[d, two heads] & A_\hbar \arrow[d, two heads] \\
\mathscr{T}^2_{\mathbb{R},\hbar} \arrow[r, mapsto, "C"] & \mathcal{T}^2_{\mathbb{R},\hbar}
\end{tikzcd}$$

Avatars for the quantum torus.

The *dramatis personae* of this commutative diagram are as follows:

(i) The (translation) Lie groupoid $(\mathbb{T}^2_{\mathbb{R}}, E(\hbar))$ whose manifold of objects is the torus $\mathbb{T}^2_{\mathbb{R}}$ (which happens to be a Lie group), and whose arrows $(t, s) : t \mapsto t \cdot s$ are pairs of elements in $\mathbb{T}^2 \times E(\hbar)$.

(ii) The non-commutative algebra $A_\hbar$ (whose two generators satisfy $XY = e^{2\pi i\hbar} Y X$).

(iii) The non-commutative space[4] $\mathcal{T}^2_{\mathbb{R},\hbar}$, namely, the Morita equivalence class $[A_\hbar]_{\sim M}$ of the algebra $A_\hbar$. We will call this *the non-commutative torus* $\mathcal{T}^2_{\mathbb{R},\hbar}$.

(iv) The (non-separated, non-algebraic, smooth) stack $\mathscr{T}^2_{\mathbb{R},\hbar}$ obtained by stackification of $(\mathbb{T}^2_{\mathbb{R}}, E(\hbar))$. We will call this the *quantum torus* $\mathscr{T}^2_{\mathbb{R},\hbar}$.

The exponential isomorphism.

Our notation for stacky quotients uses brackets so that, for example, we have:

$$\mathscr{T}^2_{\mathbb{R},\hbar} := [\mathbb{T}^2_{\mathbb{R}}/E(\hbar)] \cong [S^1/\langle\rho_\hbar\rangle],$$

and, from now on, we will always use the presentation $\mathscr{T}^2_{\mathbb{R},\hbar} := [S^1/\langle\rho_\hbar\rangle]$ for the quantum torus. Actually, we will need to pass to the Lie algebra by taking logarithms. Indeed, we will find convenient to use the exponential group homomorphism (with kernel $\mathbb{Z} = \langle 1\rangle$): $E : x \in \mathbb{R} \mapsto E(X) := \exp(2\pi i x) \in S^1$, which, in turn, induces a map

$$E : [\mathbb{R}/\langle 1, \hbar\rangle] \to [S^1/\langle\rho_\hbar = E(\hbar)\rangle] = \mathscr{T}^2_{\mathbb{R},\hbar}$$

The quantum lattice.

$$\Gamma := \langle 1, \hbar \rangle \subset \mathbb{R}.$$

Sometimes Γ is called a *quasi-lattice* but, given our motivation, we will call it a *quantum lattice* or, simply, a *q-lattice*. Clearly, Γ behaves quite differently whether $\hbar$ is rational or not: in the former case, Γ really is a lattice in $\mathbb{R}$, for it is always the case that $\Gamma \cong \mathbb{Z}$. In any case, Γ plays the role of the 'Lie algebra' of the rotation group $\langle \rho_\hbar \rangle$:

$$E : \Gamma \to \langle \rho_\hbar \rangle.$$

With this, we arrive at the *logarithmic representation of the quantum torus*:

$$\mathscr{T}^2_{\mathbb{R},\hbar} \cong [\mathbb{R}/\Gamma].$$

The complex quantum d-dim torus.

There are two variations to the previous setting that we will need in our theory. First, we will work mostly with complex quantum tori rather than with real quantum tori (although Lagrangian tori will still be real):

$$\mathscr{T}^2_{\mathbb{C},\hbar} := [\mathbb{T}^2_{\mathbb{C}}/E(\hbar)] \cong [(\mathbb{C}^*)/\langle \rho_\hbar \rangle] \cong [\mathbb{C}/\Gamma].$$

The second important variation arises from the fact that we will need to work with tori of arbitrary integer dimension $d+1$, so that, in general, we define:

$$\mathscr{T}_{\mathbb{C},d,\Gamma} := \mathscr{T}^{d+1}_{\mathbb{C},\Gamma} := [\mathbb{T}^d_{\mathbb{C}}/E(\Gamma)] \cong [\mathbb{C}^d/\Gamma].$$

where Γ is a *q-lattice* (namely, a finitely generated additive subgroup of some $\mathbb{R}^d$ spanning it over the real number field). We are to think of Γ as the holonomy of a linear foliation on $\mathbb{T}^{d+1}_{\mathbb{C}}$ analogous to that of Figure 1 (where $d=1$ and $\Gamma = \langle 1, \hbar \rangle$), of $\mathbb{T}^d_{\mathbb{C}}$ as a transversal to the foliation, and of $\mathbb{C}^d$ as the universal cover to such transversal.

Quantum P1

The simplest example of a quantum toric variety is probably a quantum projective line; just a projective line is an equivariant compactification of a one dimensional complex torus:

$$\mathbb{C}P^1 = \mathbb{C}^* \cup \{0\} \cup \{\infty\},$$

the analogous statement is true for a quantum projective line (which is then, in turn, a compactification of a quantum torus):

$$\mathbb{C}\mathscr{P}^1_\hbar = \mathscr{T}_{\mathbb{C},1,\hbar} \cup \{0\} \cup \{\infty\}.$$

Quantum P1

below. Enough is to say here that we construct $\mathbb{C}\mathscr{P}^1_\hbar$ with two charts, both of the form $[\mathbb{C}/\exp(2i\pi\hbar\mathbb{Z})]$ (which is a partial compactification of $[\mathbb{C}^*/\exp(2i\pi\hbar\mathbb{Z})]$), glued by the attaching map:

$$[z] \in [\mathbb{C}^*/\exp(2i\pi\Gamma)] \longmapsto [z^{-1}] \in [\mathbb{C}^*/\exp(2i\pi(-\Gamma))].$$

Notice that a quantum projective line $\mathbb{C}\mathscr{P}^1_\hbar$ is a compactification of $\mathscr{T}^2_{\mathbb{C},\hbar}$. You may want to imaginatively think that $\mathbb{C}$ (resp. $\mathbb{R}$) is both the Lie algebra and the universal covering of $\mathscr{T}^2_{\mathbb{C},\hbar}$ (resp. $\mathscr{T}^2_{\mathbb{R},\hbar}$), and that $\pi_1(\mathscr{T}^2_{\mathbb{R},\hbar}) \cong \Gamma$, but this would be off by one dimension ($2 \neq 1$) for the case $\hbar = 0$. The dimensions may, at first, look confusing to the reader. To clarify this possible confusion let us mention that:

Dimension counting.

i) The 'naive dimension' of $\mathscr{T}^2_{\mathbb{R},\hbar}$ seems to be $\dim \mathbb{T}^2 - \dim E(\hbar) = 2 - 1 = 1$. This is why we shift to the notation $\mathscr{T}_{1,\mathbb{R},\hbar} := \mathscr{T}^2_{\mathbb{R},\hbar}$ in the body of the paper.

ii) The 'homotopy type' of $\mathscr{T}^2_{\mathbb{R},\hbar}$ is given by the homotopy quotient $\mathbb{T}^2 \times_{E(\hbar)} E\mathbb{R}$ which in turn is homotopy equivalent to $\mathbb{T}^2$ (for $E(\hbar) \cong \mathbb{R}$ is contractible), and hence has 'homotopic-dimension' two. The same holds for $\mathscr{T}^2_{\mathbb{C},\hbar}$. This will be reflected in the periodic cyclic homology of $\mathcal{T}^2_{\mathbb{R},\hbar}$: from the homological point of view, it will look like a two-dimensional space.

LVM manifolds appear…

iii) As mentioned above, a quantum projective line $\mathbb{C}\mathscr{P}^1_\hbar$ is a compactification of $\mathscr{T}^2_{\mathbb{C},\hbar}$, and, indeed, it will also have a 'naive complex dimension' of 1 and a 'homotopic dimension' of 2. Moreover, we will describe a complex manifold $N_\hbar$ (known as a LVM-manifold cf. Section 8 below) together with a foliation $\mathcal{F}_\hbar$ (defined in Subsection 8.5) so that the groupoid $(N_\hbar, \mathcal{F}_\hbar)$ compactifies the Kronecker groupoid $(\mathbb{T}^2_{\mathbb{C}}, E(\hbar))$ and the stack $\mathbb{C}\mathscr{P}^1_\hbar \cong [N_\hbar/\mathcal{F}_\hbar]$ equivariantly compactifies the stack $\mathscr{T}^2_{\mathbb{C},\hbar} := [\mathbb{T}^2_{\mathbb{C}}/E(\hbar)]$. The point here is that the complex dimension of $N_\hbar$ is two (in fact $N_\hbar \cong S^1 \times S^3$ is a complex non-symplectic Hopf surface[6]), and (in the irrational case) the leaves of the foliation $\mathscr{F}_\hbar$ are all isomorphic to $\mathbb{C}$ and hence, are contractible. In this situation, the fact that $\mathbb{C}\mathscr{P}^1_\hbar \cong [N_\hbar/\mathbb{C}]$ explains both the naive and homotopic dimension countings for this quantum projective line.

Classical torics as LVM foliations.

[6]Hopf manifolds and the more generally, Calabi-Eckmann manifolds, [15] are non-Kähler manifolds. Topologically they are of the form $S^{2n-1} \times S^{2m-1}$ and they are deformations of an elliptic, holomorphic fibration $E \to S^{2n-1} \times S^{2m-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$. In this example we are interested in the Hopf case $n = 1$, $m = 2$. Of course, $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is a toric variety. The generalization of a Calabi-Eckmann manifold corresponding to a general toric variety X are the *LVM-manifolds* as it is proved in [44].

[44] Laurent Meersseman and Alberto Verjovsky. Holomorphic principal bundles over projective toric varieties. *Journal für die Reine und Angewandte*, 572:57–96, 2004.

Gerbes and Calibrations.

The case of rational $\hbar$ (say $\hbar = 0$) requires more care, for here the leaves of the foliation $\mathscr{F}_\hbar$ wind up on themselves and rather than being copies of $\mathbb{C}$, they become elliptic curves of the form $S^1 \times S^1$ (indeed, the map $N_\hbar \cong S^1 \times S^3 \to [N_0/\mathcal{F}_0] \cong [N_0/S^1 \times S^1] \cong \mathbb{P}^1 \cong S^2$ is the trivial constant map crossed with the Hopf fibration, with fiber $S^1 \times S^1$), and then the homotopical dimension of $\mathbb{P}^1$ and the naive complex geometric dimension coincide and are both equal to 1 (of course). Notice here that the stack $\mathbb{C}\mathscr{P}^1_0 \cong [N_0/\mathbb{C}](\ncong [N_\hbar/\mathcal{F}_\hbar] \cong \mathbb{P}^1)$ still has homotopical dimension equal to two, and we have a fibration

$$\mathbb{C}^* \simeq \mathscr{B}\mathbb{Z} \to \mathbb{C}\mathscr{P}^1_0 \to \mathbb{P}^1,$$

that is to say $\mathbb{C}\mathscr{P}^1_0$ *is a gerbe over* $\mathbb{P}^1$ *with abelian band* $\mathbb{Z}$, which explains the difference of dimensions by 1. We refer to this process as *calibrating* $\mathbb{P}^1$ to obtain $\mathbb{C}\mathscr{P}^1_0$, the choice of calibration is by no means unique (cf. Definition 4.9, Subsection 6.1 and Subsection 6.3. Here we are using a standard calibration as in Example 4.18). In any case, it is easy to recover $\mathbb{P}^1$ from $\mathbb{C}\mathscr{P}^1_0$ (by forgetting the gerbe) and vice-versa, $\mathbb{C}\mathscr{P}^1_0$ can be naturally constructed from is the complex version of the '2-fold homotopy cover' of $\mathbb{P}^1$ (the complex 2-fold homotopy cover of S^2 is $S^1 \times S^3$).

A simple quantum fan.

Recall that all the information to reconstruct $\mathbb{C}P^1$ can be combinatorially encoded by a fan in $\mathbb{R}^1$ with three cones: $\{0\}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ together with the integral lattice $\mathbb{Z} \subset \mathbb{R}^1$. Likewise, all we need to reconstruct $\mathbb{C}\mathscr{P}^1_\hbar$ is the *quantum fan* consisting of three cones $\{0\}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ together with the q-lattice $\Gamma \subset \mathbb{R}^1$.

The toric site

First, let us recall that an *affine toric variety* is an irreducible affine variety T containing a complex torus $\mathbb{T}^d := \mathbb{T}^d_{\mathbb{C}}$ as a Zariski open subset such that the action of $\mathbb{T}^d$ on itself extends to an action of $\mathbb{T}^d$ on T given by an algebraic morphism $\mathbb{T}^d \times T \to T$. A toric morphism $T \to T'$ between toric varieties is a morphism of varieties that is equivariant with repect to the corresponding torus actions $\mathbb{T}^d \times T \to T$, and $\mathbb{T}^{d'} \times T' \to T'$. We will say that a Zariski open subset of T is *toric* if it is itself a toric subvariety of T.

We take as base category the category $\mathfrak{A}$ of affine toric varieties and toric morphisms. We take for covering of an affine toric variety T a decomposition $T = T_1 \cup \cdots \cup T_n$ into toric Zariski open subsets of T. With these coverings, $\mathfrak{A}$ is a site.

The equivariant analytic site

We will also need the category $\mathfrak{G}$ of complex analytic spaces $X = \overline{G} \supset G$ endowed with a holomorphic action of a complex abelian Lie group G with a Zariski open orbit isomorphic to G. Morphisms $(X, G) \to (X', G')$ are equivariant holomorphic mappings $X \to X'$ that restrict to Lie group homomorphisms $G \to G'$. Observe that there is a natural fully faithful functor from $\mathfrak{A}$ to $\mathfrak{G}$.

By a *cover* $p\colon T' \to T$ of an object T of $\mathfrak{A}$, we mean an object $\tilde{T}$ of $\mathfrak{G}$ endowed with a free and proper holomorphic action of a discrete abelian group H whose quotient is T together with a quotient map $p : \tilde{T} \to T$ which is an unramified analytic equivariant cover of T.

Quantum toric stacks

Quantum Toric Stacks are of the form $[X/H]$ with $X \in \mathfrak{G}$ and H a discrete abelian group or a complex abelian Lie group, or are descent data of such $[X/H]$, cf. Section 5. To be more precise, we consider the category $[X/H]$ whose objects $(\tilde{T}, T, m)$ are covers $\tilde{T}$ of a space $T \in \mathfrak{A}$ with an equivariant holomorphic map m

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{m} & X \\ \downarrow & & \\ T & & \end{array} \tag{2.1}$$

Here m is assumed to be equivariant with respect to both the H-action and the G-action.

Morphisms $(\tilde{T}, T, m) \to (\tilde{S}, S, n)$ are commutative diagrams of the form:

$$\begin{tikzcd} & & X \\ \tilde{T} \arrow[urr, bend left, "m"] \arrow[r] \arrow[d] & \tilde{S} \arrow[ur, "n"'] \arrow[d] & \\ T \arrow[r] & S & \end{tikzcd} \tag{2.2}$$

Example

Example 3.2. Let $d = 1$ and define Γ as the subgroup of $(\mathbb{R}, +)$ generated by 1 and $\sqrt{2}$. Since $\sqrt{2}$ is irrational, it is dense in $\mathbb{R}$. It acts freely on $\mathbb{R}$ by translation but this action is not proper and the topological quotient is not Hausdorff. Indeed, for any pair of real numbers x, y, since $y - x$ is an accumulation point of Γ, the images of x and y are not separated.

It follows that the associated quantum torus $[\mathbb{C}/\Gamma]$ is not a manifold and stack language is needed to handle it as a complex "space".

Hence $\mathscr{T}_{d,\Gamma} = [\mathbb{C}^d/\Gamma]$ has to be understood more formally as a category fibered in groupoids $\mathscr{T}_{d,\Gamma} \to \mathfrak{A}$, as in Section 2.

Objects of $\mathscr{T}_{d,\Gamma}$ are Γ-covers T over a space $A \in \mathfrak{A}$ with an equivariant holomorphic map m in $\mathbb{C}^d$

$$\begin{tikzcd} T \arrow[r, "m"] \arrow[d] & \mathbb{C}^d \\ A & \end{tikzcd} \tag{3.1}$$

Example

Example 3.6. Let $d = 1$ and Γ be generated by 1 and $\sqrt{2}$ as in Example 3.2. Consider the linear map

$$z \in \mathbb{C} \longmapsto \sqrt{2} \cdot z \in \mathbb{C}$$

It sends 1 onto $\sqrt{2}$ and $\sqrt{2}$ onto 2 hence it preserves Γ. So it descends as the torus morphism

$$[z] \in [\mathbb{C}/\Gamma] \longmapsto [\sqrt{2}z] \in [\mathbb{C}/\Gamma]$$

or, in multiplicative form,

$$[w] \in [\mathbb{T}/E(\Gamma)] \longmapsto [w^{\sqrt{2}}] \in [\mathbb{T}/E(\Gamma)] \tag{3.13}$$

Classical P1 is obtained...

by gluing two copies of the affine toric variety $\mathbb{C}$ along the Zariski open set $\mathbb{C}^* \subset \mathbb{C}$ via the map $z \to 1/z$. As a stack on $\mathfrak{A}$, it may be presented as the following descent data of affine toric varieties.

An object over $T \in \mathfrak{A}$ is a pair $\left(T_1 \xrightarrow{m_1} \mathbb{C} \;,\; T_2 \xrightarrow{m_2} \mathbb{C} \right)$ such that

i) $T = T_1 \cup T_2$ is a covering of T.

ii) We have $m_1(T_1 \cap T_2) = m_2(T_1 \cap T_2) = \mathbb{C}^*$.

iii) The following diagram commutes

$$\begin{tikzcd}
T_1 \cap T_2 \arrow[r, "m_1"] \arrow[dr, "m_2"'] & \mathbb{C}^* \arrow[d, "z \mapsto 1/z"] \\
& \mathbb{C}^*
\end{tikzcd}$$

Classical P1

And a morphism above $T \xrightarrow{f} S$ between $(\ T_1 \xrightarrow{m_1} \mathbb{C}\ ,\ T_2 \xrightarrow{m_2} \mathbb{C}\ )$ and $(\ S_1 \xrightarrow{n_1} \mathbb{C}\ ,\ S_2 \xrightarrow{n_2} \mathbb{C}\ )$ must satisfy $n_i \circ f_i = m_i$ for $i = 1, 2$. Of course, this is isomorphic to the stack $\underline{\mathbb{P}^1}$ and is a complicated way of describing it. But the point here is that, once given the category of affine toric varieties, general toric varieties can be defined directly and functorially through this descent data procedure.

In the Quantum case, we will first define affine simplicial quantum toric varieties as discrete quotient stacks; and then general simplicial quantum toric varieties through descent.

Quantum Fans.

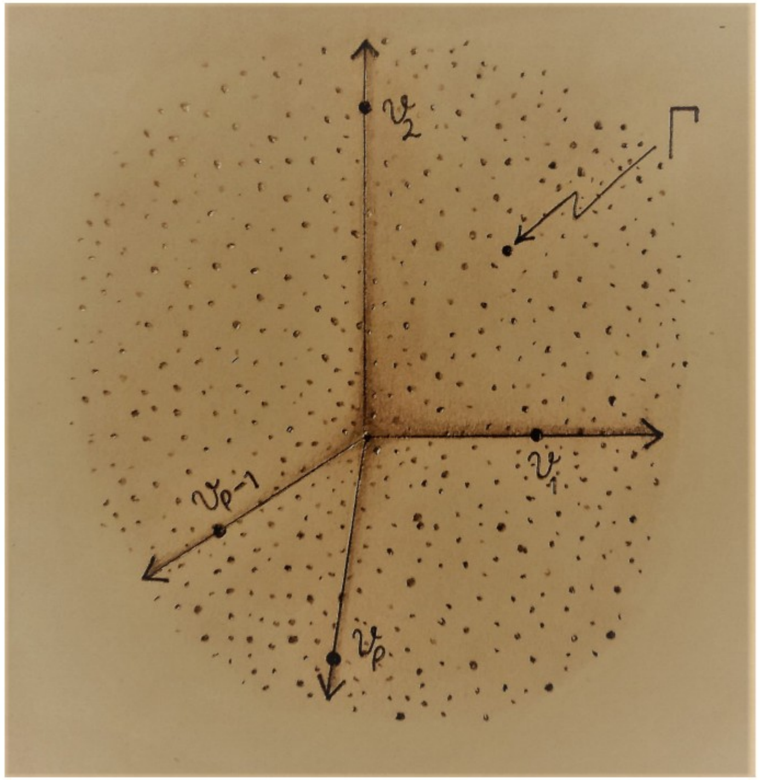


FIGURE 2. A quantum fan (Δ, v) in Γ is very similar to a classical fan Δ in toric geometry, but instead of an integral lattice, it is equipped with a q-lattice Γ (cf. Definition 4.1). Notice that we must mark the (non-canonical) 'primitive vectors' $(v_1, \ldots, v_p)$ (all in Γ) on every ray of the 1-skeleton of the fan. The fan Δ no longer needs to be rational.

A general quantum toric stack can be constructed starting from a general (not necessarily rational) *quantum fan* (see Figure 2 and Definition 4.1): such a q-fan carried a q-lattice $\Gamma \subset \mathbb{R}^d$, and therefore defines a q-torus $\mathscr{T}_{\mathbb{C},d,\Gamma} \cong [\mathbb{C}^d/\Gamma]$. The quantum toric stack $\mathscr{X}_{\Delta,\Gamma,v}$ (cf. Definition 5.8) is an equivariant compactification of $\mathscr{T}_{\mathbb{C},d,\Gamma}$ given by the data of the quantum fan (Δ, v).

Calibrated quantum toric stacks

As explained before, we really want to consider the calibrated case (adding gerbe degrees of freedom). At the level of fans, this is achieved by the definition of a *calibrated quantum fan*. The precise description of a calibrated quantum fan (depicted in Figure 3) is found in Definition 4.9 below. For now, think of a calibration as a homomorphism $h : \mathbb{Z}^n \to \Gamma$. Given such a calibrated quantum fan, then we define a *Calibrated Quantum Toric Stack* in Definition 6.17 and denoted[8] by $\mathscr{X}^{cal}_{\Delta,h,J}$.

A calibrated quantum fan.

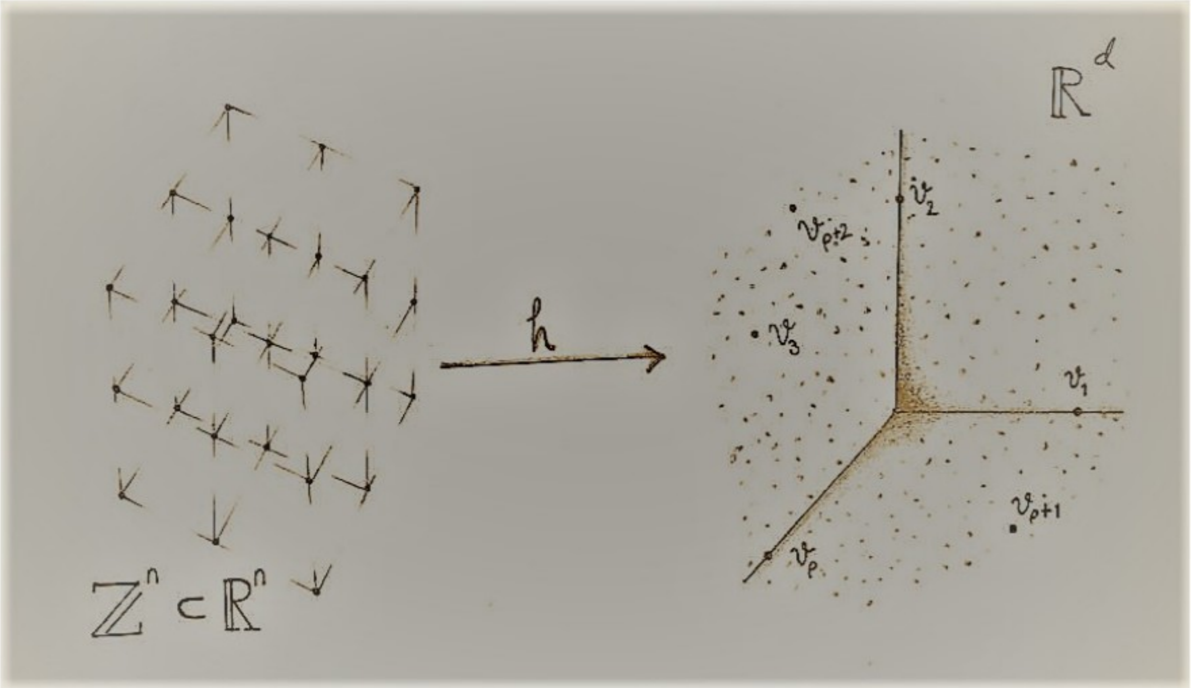


FIGURE 3. A calibrated quantum fan (Δ, h) is essentially a quantum fan plus a calibration, namely, a homomorphism $h : \mathbb{Z}^n \to \Gamma$ 'determining the various Planck lengths of the quantum system' (cf. Definition 4.9).

Calibrated = uncalibrated + gerbe.

The relation between the calibrated Quantum Toric stack $\mathscr{X}^{cal}_{\Delta,h,J}$ and its un-calibrated version $\mathscr{X}_{\Delta,\Gamma,v}$ is explained in Proposition 6.20; $\mathscr{X}^{cal}_{\Delta,h,J}$ is a gerbe over $\mathscr{X}_{\Delta,\Gamma,v}$ with band $\mathbb{Z}^a$ (where $a := n - \mathrm{rank}_{\mathbb{Z}}(\Gamma)$): $\mathscr{X}^{cal}_{\Delta,h,J}$ is completely determined by a classifying map $\mathscr{X}_{\Delta,\Gamma,v} \to \mathscr{B}\mathscr{B}\mathbb{Z}$.

Proposition 6.20. *The calibrated Quantum Torics* $\mathscr{X}^{cal}_{\Delta,h,J}$ *is a gerbe over* $\mathscr{X}_{\Delta,\Gamma,v}$ *with band* $\mathbb{Z}^a$*. In particular, if* $a = 0$, $\mathscr{X}^{cal}_{\Delta,h,J}$ *and* $\mathscr{X}_{\Delta,\Gamma,v}$ *are isomorphic.*

Homotopically, such a gerbe is given by a map in the classifying space $\mathcal{B}\mathcal{B}(\mathbb{Z}^a)$, which is nothing else than $(\mathbb{P}^\infty)^a$. In other words, such a gerbe defines a principal $\mathbb{T}^a$ bundle over $\mathscr{X}_{\Delta,\Gamma,v}$ up to homotopy. We can describe this bundle as h.

Example

Example 5.13. As an example, we deal with the case of a quantum projective line. Here, we use the Quantum Fan of Example 4.18, so we assume (4.5), (4.6) and (4.7). Note that, if $a = 0$, this is the fan of the classical $\mathbb{P}^1$.

We have two charts. Both are modelled onto $Q_I = Q_J = [\mathbb{C}/\exp(2i\pi a\mathbb{Z})]$. The gluing is the mapping

$$[z] \in \mathbb{T}/\exp(2i\pi\Gamma) \longmapsto [z^{-1}] \in \mathbb{T}/\exp(2i\pi(-\Gamma)) \tag{5.39}$$

If $a \in \mathbb{Z}$, this is just the classical $\mathbb{P}^1$. If $a = p/q$ with p and q irreducible, then Γ is the lattice $1/q\mathbb{Z}$, so v_1 and v_2 are not primitive and we obtain a toric orbifold with 0 and ∞ having stabilizer $\mathbb{Z}_q$. Finally, if a is irrational, then we obtain a stack with two points having a stabilizer equal to $\mathbb{Z}$. This is not an orbifold.

Example

Example 4.8. Consider the Γ-complete and complete standard Quantum Fan of $\mathbb{R}^2$ generated by

$$v_1 = e_1 \qquad v_2 = e_2 \qquad v_3 = ae_1 + be_2$$

with $a < 0$ and $b < 0$. For $a = b = -1$, this is exactly the fan of the classical $\mathbb{P}^2$.

We call such a fan *a* Γ-*complete quantum deformation* of $\mathbb{P}^2$'s fan.

Example 5.17. We construct now the Γ-complete quantum deformations of $\mathbb{P}^2$ whose fan is given in Example 4.8.

The three maximal cones $\langle 1,2\rangle$, $\langle 2,3\rangle$ and $\langle 3,1\rangle$ give rise to the three matrices

$$A_{12}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad A_{23}=\frac{1}{a}\begin{pmatrix}-b&a\\1&0\end{pmatrix},\qquad A_{31}=\frac{1}{b}\begin{pmatrix}0&1\\b&-a\end{pmatrix}\tag{5.46}$$

and the corresponding three charts

$$Q_{12}=\left[\mathbb{C}^2\Big/E\left(\mathbb{Z}\begin{pmatrix}a\\b\end{pmatrix}\right)\right],\qquad Q_{23}=\left[\mathbb{C}^2\Big/E\left(\mathbb{Z}\begin{pmatrix}-b/a\\1/a\end{pmatrix}\right)\right]\tag{5.47}$$

and

$$Q_{31}=\left[\mathbb{C}^2\Big/E\left(\mathbb{Z}\begin{pmatrix}1/b\\-a/b\end{pmatrix}\right)\right]\tag{5.48}$$

Gluings are done using matrices (5.46). For example, the gluing between Q_{23} and Q_{12} is given by $A_{23}A_{12}^{-1}$, that is

$$[z,w]\in Q_{23}^*\longmapsto[z^{-b/a}w,z^{1/a}]\in Q_{12}^*\tag{5.49}$$

where the $*$ means that the common coordinate, here that corresponding to the cone 2, is not zero. The other two gluings are

$$\begin{aligned}[z,w]\in Q_{31}^*&\longmapsto[w^{1/b},zw^{-a/b}]\in Q_{12}^*\\ \text{and }[z,w]\in Q_{31}^*&\longmapsto[z^{1/b}w,z^{-a/b}]\in Q_{23}^*\end{aligned}\tag{5.50}$$

So if $a=b=-1$, we have three charts modelled on $\mathbb{C}^2$ and we recover the classical $\mathbb{P}^2$.

Quantum torics and quantum fans

Let $\mathscr{Q}^{cal}$ be the category of simplicial calibrated Quantum Toric Varieties. Then $\mathscr{Q}^{cal}$ is equivalent to the category of quantum toric fans Q^{cal}.

Quantum GIT

7.1. **Quantum calibrated GIT.** We first deal with the calibrated case. So start now from a simplicial calibrated Quantum Fan (Δ, h) in Γ. Set $v_i = h(e_i)$ for $i = 1, \dots, n$. We will show in this section how to construct the calibrated quantum toric variety $\mathscr{X}^{cal}_{\Delta,h,J}$ as a quotient stack.

To do this, we make use of Gale transforms, a classical tool in convex geometry. We thus perform a Gale transform of $v = (v_1, \dots, v_n)$, that is we choose some vectors $A = (A_1, \dots, A_n)$ of $\mathbb{R}^{n-d}$ such that

$$h(x) = \sum_{i=1}^{n} x_i v_i = 0 \iff \begin{cases} x_1 = \langle A_1, t\rangle \\ \quad\vdots \\ x_n = \langle A_n, t\rangle \end{cases} \quad \text{for some } t \in \mathbb{R}^{n-d} \tag{7.1}$$

This leads to a short exact sequence

$$0 \longrightarrow \mathbb{R}^{n-d} \xrightarrow{k} \mathbb{R}^n \xrightarrow{h} \mathbb{R}^d \longrightarrow 0 \tag{7.2}$$

for

$$k(t_1, \dots, t_{n-d}) = k(t) = (\langle A_1, t\rangle, \dots, \langle A_n, t\rangle) \tag{7.3}$$

Calibrated QGIT

Define

$$i \in I_z \iff z_i \neq 0. \tag{7.5}$$

and set

$$\mathscr{S} = \{z \in \mathbb{C}^n \mid \{1, \ldots, n+1\} \setminus I_z \text{ is a cone of } \Delta\} \tag{7.6}$$

Lemma 7.3. *The set $\mathscr{S}$ is an affine toric variety.*

We let now $\mathbb{C}^{n-d} \ni T$ act holomorphically on $\mathbb{C}^n \ni z$ through

$$T \cdot z := (z_i E(\langle A_i, T \rangle))_{i=1}^n \tag{7.9}$$

Note that

Lemma 7.5. *Action (7.9) preserves $\mathscr{S}$ and commutes with the action of $\mathbb{T}^n$ on $\mathscr{S}$.*

We denote by $\mathcal{A}$ this action and we form the global quotient $[\mathscr{S}/\mathcal{A}]$. We

Calibrated QGIT

Theorem 7.6. *The stacks* $[\mathscr{S}/\mathcal{A}]$ *and* $\mathscr{X}^{cal}_{\Delta,h,J}$ *are isomorphic.*

Uncalibrated QGIT

$$\mathscr{X}^{cal}_{\Delta,h,J} \cong [\mathscr{S}/\mathcal{A}],$$

where $\mathscr{S}$ is the complement in $\mathbb{C}^n$ of a union of coordinate vector subspaces and the classical torus $\mathbb{T}^n$ acts multiplicatively on it with a Zariski dense orbit (we denote by $\mathcal{A}$ this action). Moreover, $\mathcal{A}$ actually defines a foliation on $\mathscr{S}$ so that the stackification of the holonomy groupoid of said foliation is isomorphic to the un-calibrated toric stack $\mathscr{X}_{\Delta,\Gamma,v}$ (Theorem 7.10).

Theorem 7.10. *The stackification of the holonomy groupoid* $G_1 \rightrightarrows G_0$ *over* $\mathfrak{A}$ *is* $\mathscr{X}_{\Delta,\Gamma,v}$.

QGIT and LVM-theory.

From the point of view of QGIT, there is a deep and beautiful relation beetween quantum toric stacks and *LVMB theory* (see section 8 for definitions). Such relation occurs only when $n - d$ is even (cf. Definition 9.2). The reader may want to think for now of a LVMB-manifold (together with a canonical foliation induced by a holomorphic C^m-action) $(N, \mathcal{F})$ as a generalization of the Calabi-Eckmann manifolds (and their elliptic foliation induced by a $\mathbb{C}$-action, where $\mathbb{C}$ covers the elliptic curve) so that (cf. Theorem 9.13):

$$\mathscr{X}_{\Delta,\Gamma,v} \cong [N/\mathcal{F}]$$

and

$$\mathscr{X}^{cal}_{\Delta,h,J} \cong [N/\mathbb{C}^m].$$

Quantum LVM = QLVM

Theorem 9.13. *Let* $(\mathcal{S}, \Lambda)$ *and* (Δ, h) *as above. Then,*

i) *The stack* $[N_\Lambda/\mathscr{F}_\Lambda]$ *is isomorphic to the Quantum Toric Variety* $\mathscr{X}_{\Delta,\Gamma,v}$.

ii) *The stack* $[N_\Lambda/\mathbb{C}^m]$ *is isomorphic to the calibrated Quantum Toric Variety* $\mathscr{X}^{cal}_{\Delta,h,J}$.

Kählerness (Uses Ishida's results).

Theorem 10.2. *A Quantum torics $\mathscr{X}_{\Delta,\Gamma,v}$ with Δ complete is Kähler if and only if Δ is polytopal.*

[30] Hiroaki Ishida. Torus invariant transverse Kähler foliations. *Transactions of the American Mathematical Society*, 369(7):5137–5155, 2017.

Unlike classical toric varieties which are rigid (as equivariant toric spaces), quantum toric stacks admit moduli. In Section 11, the final section of this paper, we study various moduli spaces, specially the moduli space $\mathscr{M}_{\text{toric}}^{D,d}$ of quantum toric stacks with fixed combinatorial type D (and Γ-complete), the moduli space $\hat{\mathscr{M}}_{\text{toric}}^{D,n,d}$ of calibrated Quantum Toric Stacks (of maximal length) and fixed combinatorial type D and the moduli space $\mathcal{M}_{m,n}^{\mathcal{S}}$ of G-biholomorphism classes of LVMB manifolds (see Figure 4). The main

Moduli spaces of quantum toric stacks.

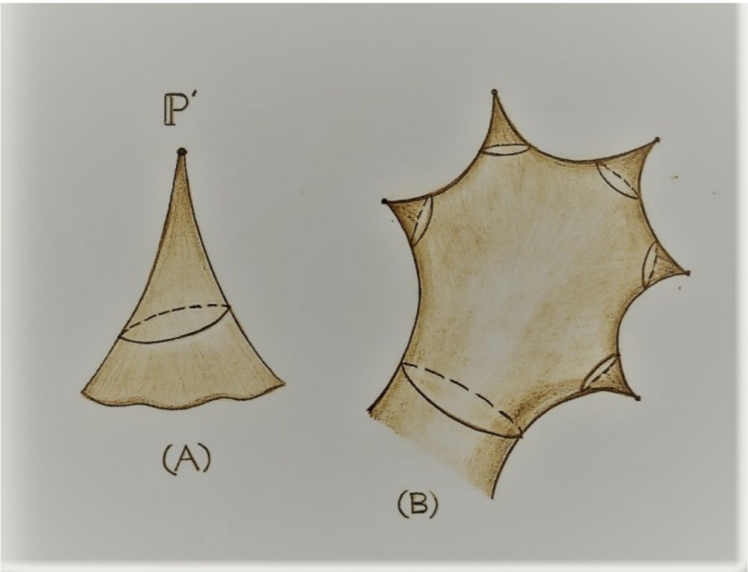


FIGURE 4. Moduli spaces of quantum toric stacks: in (A) we depict the moduli space of quantum projective lines: it has only one orbifold point with stabilizer $\mathbb{Z}_2$ corresponding to the classical $\mathbb{P}^1$ (or equivalently, to its non-commutative avatar $\mathbb{C}\mathscr{P}^1_0$); for the case of $\mathbb{P}^d$, see Subsection 11.4. In (B) we depict a more fanciful representation of the moduli space $\hat{\mathscr{M}}^{D,n,d}_{\text{toric}}$ of calibrated Quantum Toric Stacks with fixed combinatorial type D: the orbifold points occur whenever the fan suddenly has more symmetries (cf. Section 11). The classical toric varieties will land on the rational locus of some of these moduli spaces.

Moduli are orbifolds. Teichmuller.

Proposition 11.27. *The moduli space* $\mathcal{M}^{\mathcal{S}}_{m,n}$ *is the quotient of* $\mathcal{T}_{\mathcal{S}}$ *by the action of* $\mathrm{GL}_{n-1}(\mathbb{Z})$ *described in* (11.34).

We claim

Theorem 11.28. *If the number* k *of indispensable points is less than* $m+1$, *then the moduli space* $\mathcal{M}^{\mathcal{S}}_{m,n}$ *can be endowed with a structure of a complex orbifold.*

Twistor complexification.

$$\mathbb{R}^{(n-d)^2/2} \to \mathcal{M}^{\mathcal{S}}_{m,n} \to \hat{\mathscr{M}}^{D,n,d}_{\text{toric}},$$

namely, $\mathcal{M}^{\mathcal{S}}_{m,n}$ is a (sometimes complex) orbibundle of even rank $(n-d)^2/2$ over $\hat{\mathscr{M}}^{D,n,d}_{\text{toric}}$. This immediately implies the homotopy equivalence (a diffeomorphism when $n = d$):

$$\mathcal{M}^{\mathcal{S}}_{m,n} \simeq \hat{\mathscr{M}}^{D,n,d}_{\text{toric}},$$

namely, the moduli space $\hat{\mathscr{M}}^{D,n,d}_{\text{toric}}$ can be promoted (in the same homotopy class) to a complex orbifold $\mathcal{M}^{\mathcal{S}}_{m,n}$. Let us point out here that there is an interesting analogy with the classical case of the moduli space of curves $M(g,n)$ where g is the genus (combinatorial information) and n is a marking that makes the compactification of the moduli space nicer. Here too, we have combinatorial information (D,d), and marking information n, making the moduli space nicer, and even sometimes giving it a complex structure. We will explore this analogy elsewhere.

The basic moduli space of torics

Theorem

Let D be the combinatorial type of an simplicial fan of $\mathbb{R}^d$ and n be an integer. Then

$$\Omega(D) \coloneqq \{h \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d) \mid h_{|\mathbb{R}^d} = id_{\mathbb{R}^d}, (h(e_p))_{p \in D(1)} \text{ D-realizable}\}$$

is (isomorphic to) an open subset of $\mathbb{R}^{d(n-d)}$ and

$$\mathscr{M}(d,n,D) \simeq \left[\Omega(D)/(\mathrm{Aut}_{\mathbf{Poset}}(D) \times \mathfrak{S}_{D(1)^c})\right]$$

where the action of $\mathrm{Aut}_{\mathbf{Poset}}(D) \times \mathfrak{S}_{D(1)^c}$ is given as follows :

$$(\sigma,\tau)\cdot h = \left(h(e_{\sigma(1)}) \cdots h(e_{\sigma(d)})\right)^{-1} h \begin{pmatrix} P_\sigma & 0 \\ 0 & P_\tau \end{pmatrix}$$

Remember the classical M(g,n)

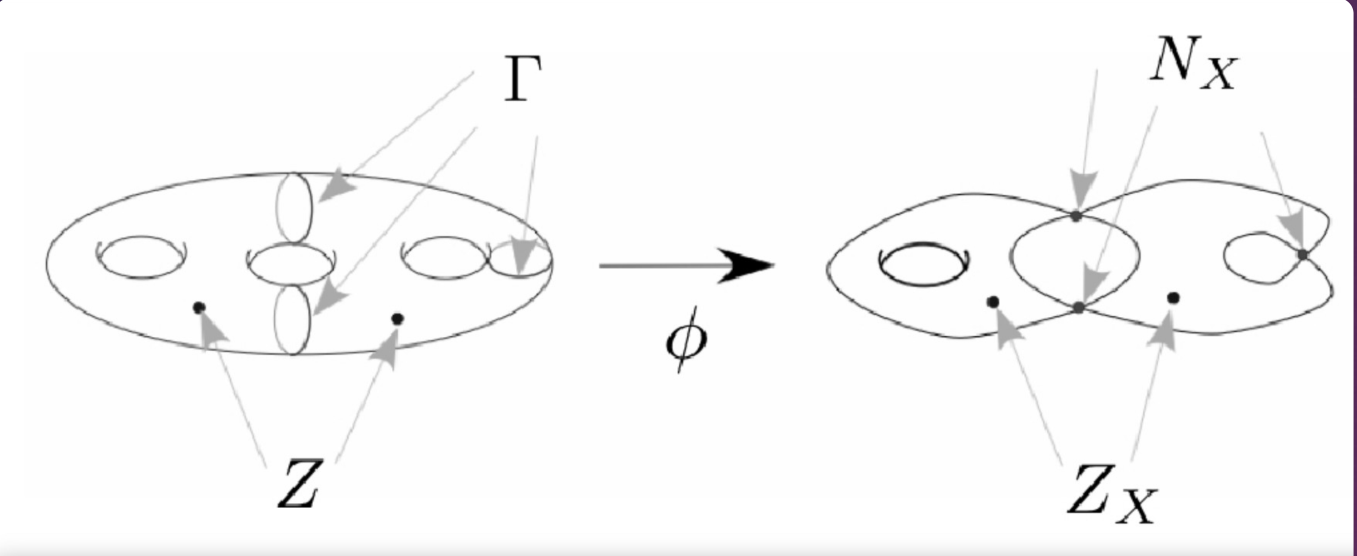


Figure 2. On the left is the topological surface S with two marked points in the set Z. There is a multicurve Γ drawn on S−Z. On the right is the stable curve X with two marked points in the set ZX , and three nodes in… Expand

Published in 2013

An analytic construction of the Deligne-Mumford compactification of the moduli space of curves

J. Hubbard, Sarah C. Koch

Compactification of the moduli of quantum projective spaces

Theorem 1. *Let*

$$S := \{[a_1; \ldots ; a_{d+1}] \in \mathbb{RP}^d \mid a_i > 0 \textit{ for all } i\} \tag{2}$$

Then, the moduli stack of Quantum $\mathbb{P}^d$*s is the real orbifold* $\mathscr{M}_d := [S/\mathfrak{S}_{d+1}]$ *for* $\mathfrak{S}_{d+1}$ *acting by permutation on the homogeneous coordinates of* S.

Universal family

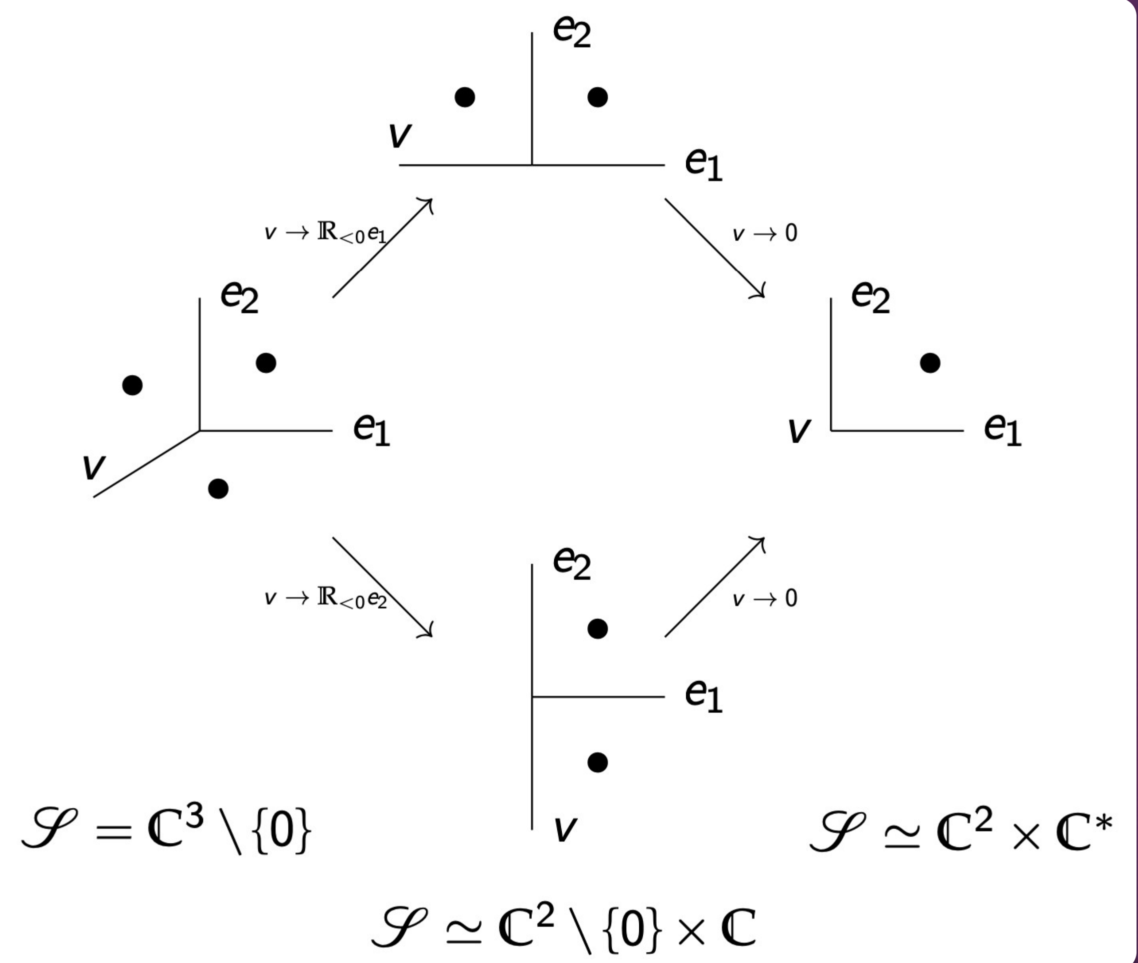
Theorem 2. *The moduli stack $\mathscr{M}_{\mathrm{d}}$ is the base space of a universal family $\mathscr{X}/\mathfrak{S}_{d+1} \to \mathscr{M}_{\mathrm{d}}$ of Quantum $\mathbb{P}^d$s.*

The compactification

Theorem 3. *The natural compactification $\overline{\mathscr{M}_{\mathrm{d}}}$ of the moduli stack $\mathscr{M}_{\mathrm{d}}$ is the base space of a family $\overline{\mathscr{X}}/\mathfrak{S}_{d+1} \to \overline{\mathscr{M}_{\mathrm{d}}}$ of Quantum projective spaces satisfying*

(i) *Its restriction to $\mathscr{M}_{\mathrm{d}}$ is the universal family of $\mathbb{P}^{\mathrm{d}}$s of Theorem 2.*

(ii) *The boundary $\overline{\mathscr{M}_{\mathrm{d}}} \setminus \mathscr{M}_{\mathrm{d}}$ is isomorphic to $\mathscr{M}_{\mathrm{d-1}}$ and the restriction of $\overline{\mathscr{X}}$ to it is isomorphic to the family $\overline{\mathscr{X}}/\mathfrak{S}_{\mathrm{d}} \to \overline{\mathscr{M}_{\mathrm{d-1}}}$*



e_2
v
e_1
$v \to \mathbb{R}_{<0}e_1$
$v \to 0$
e_2
e_1
v
e_2
v
e_1
$v \to \mathbb{R}_{<0}e_2$
$v \to 0$
e_2
e_1
v
$\mathscr{S} = \mathbb{C}^3 \setminus \{0\}$
$\mathscr{S} \simeq \mathbb{C}^2 \times \mathbb{C}^*$
$\mathscr{S} \simeq \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$

Case of P2

The general compactification (A. Boivin)

Theorem

The stack $\overline{\mathscr{M}}$ is a natural compactification of the moduli stack $\mathscr{M}$. Moreover, the fiber over $\overline{\Omega} \setminus \Omega$ are quantum toric stacks defined by degenerate types obtained from D.