

Spectra, dynamical systems, and RG flows

Based on:

S.G. arXiv: 1503.01474
arXiv: 1608.06638

see also:

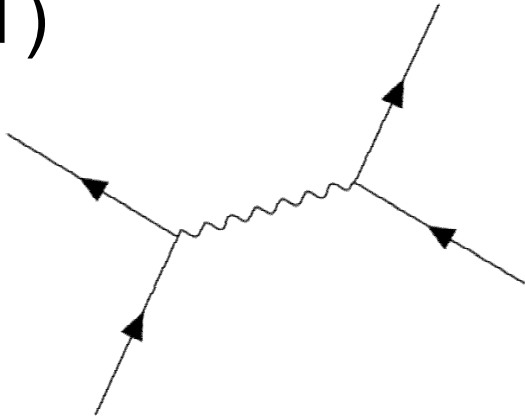
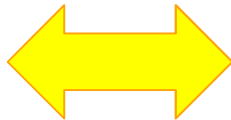
D.Gepner (1987)
C.Vafa, N.P.Warner (1989)
E.J.Martinec (1989)
W.Lerche, C.Vafa, N.P.Warner (1989)
B.R.Greene, S.-S.Roan, S.-T.Yau (1991)
D.Xie, S.-T.Yau (2015)
F.Kuipers, U.Gursoy, Y.Kuznetsov (2018)
C.B.Jepsen, I.R.Klebanov, F.K.Popov (2020)
C.B.Jepsen, F.K.Popov (2021)
;

+ work in progress w/ L.Katzarkov, K.S.Lee, J.Svoboda, ...

f

singularity, its orbifold, ...

Conformal Field Theory
(CFT)

 $\{\Delta_i\}$  $sp(f) =$ $= \{\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots \leq \alpha_\mu\}$ 

Definition: *A conformal field theory*
is a table of integrals.


- *Brian Greene*



Philippe Di Francesco
Pierre Mathieu
David Sénéchal

Conformal Field Theory



 Springer

Scaling and Renormalization in Statistical Physics



CAMBRIDGE
LECTURE
NOTES IN
PHYSICS

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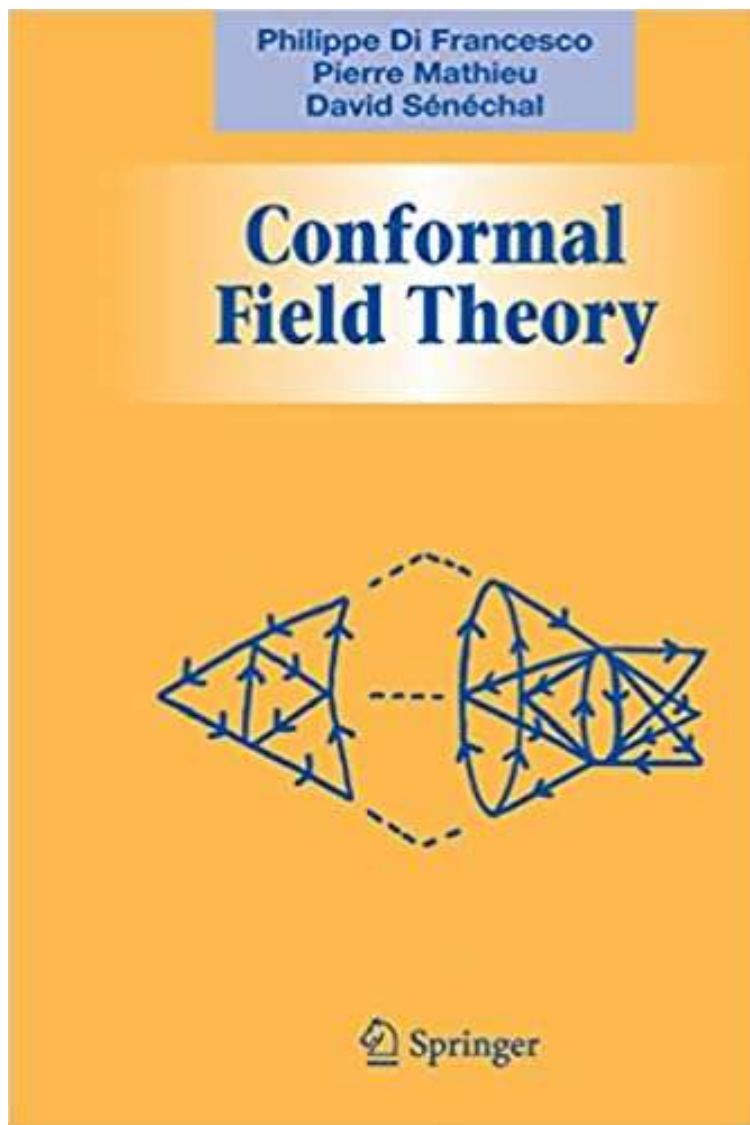
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A conformal field theory is a list of operators (states) and their correlation functions (many determined by scaling dimensions).

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12}}{r_{12}^{\Delta_1 + \Delta_2}}$$

- Spectrum
- Correlation functions
- Scaling dimensions


 $\{\Delta_i\}$

"The shell game that we play ... is technically called '**renormalization**'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate."

Richard Feynman (1985)





ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.

$$\frac{d\lambda_i}{dt} = (d - \Delta_i)\lambda_i - C_{ijk}\lambda_j\lambda_k + O(\lambda^3)$$

cf. Frobenius manifold

OPE coefficients

$$\varphi_i \star \varphi_j \sim \sum_k C_{ijk} \varphi_k$$

CFT: \mathbf{T}_{UV}

QFT

CFT: \mathbf{T}_{IR}

t

Exact Five-Loop Renormalization Group Functions of ϕ^4 -Theory with $O(N)$ -Symmetric and Cubic Interactions.

H. Kleinert and V. Schulte-Frohlinde

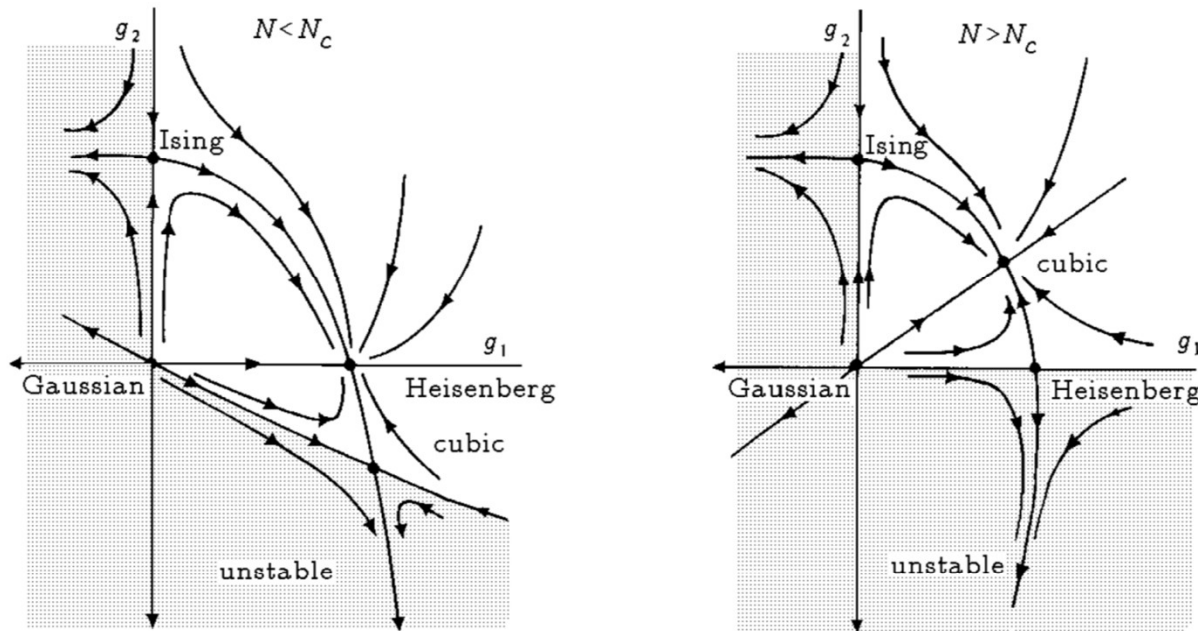
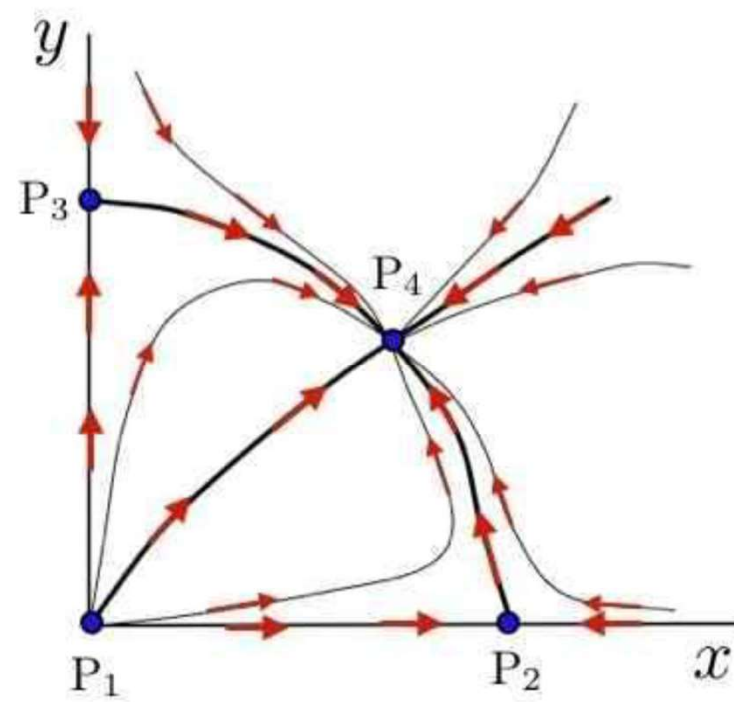
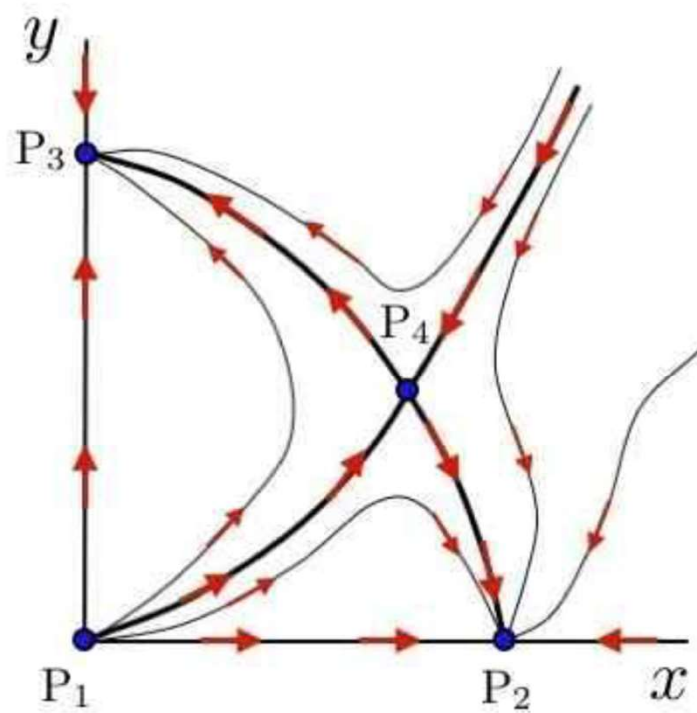


Figure 1: The Stability of the fixed points in the ϕ^4 -theory with $O(N)$ -symmetric and cubic coupling for $N < N_c$ and $N > N_c$. Our results are compatible with $N_c = 3$.



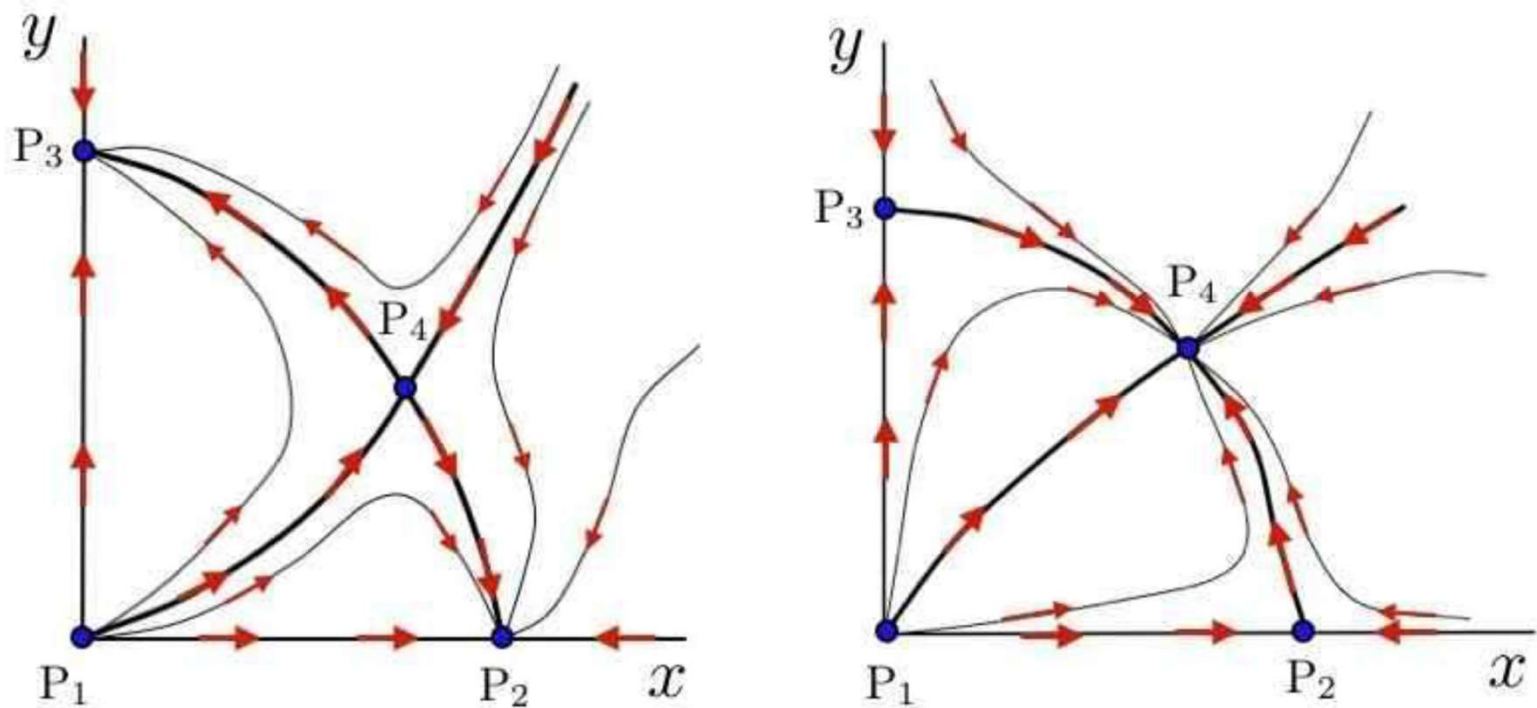
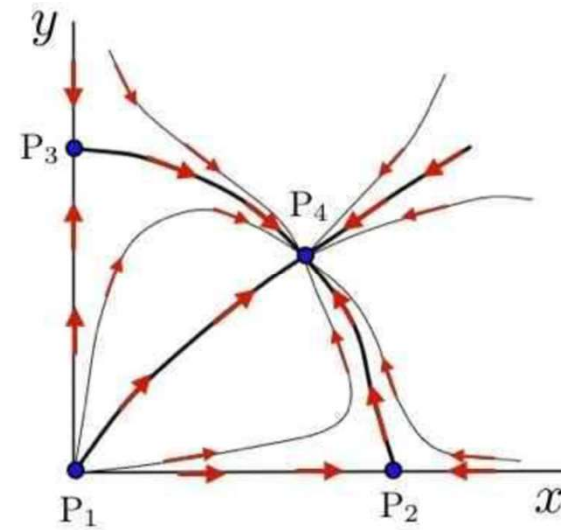
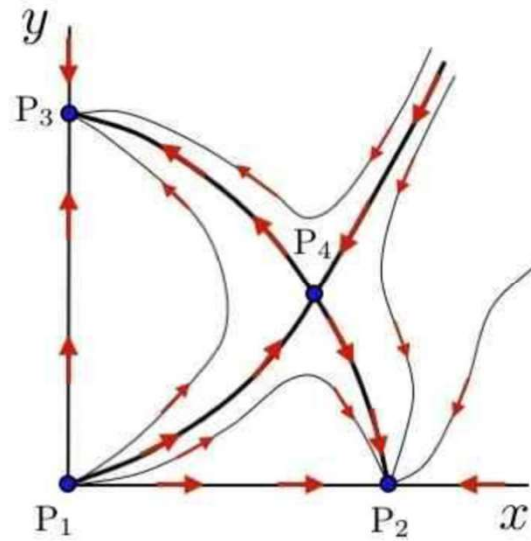


Figure 3.12: Two possible phase flows for the rabbits *vs.* sheep model of eqs. 3.61, Left panel: $k > r > k'^{-1}$. Right panel: $k < r < k'^{-1}$.

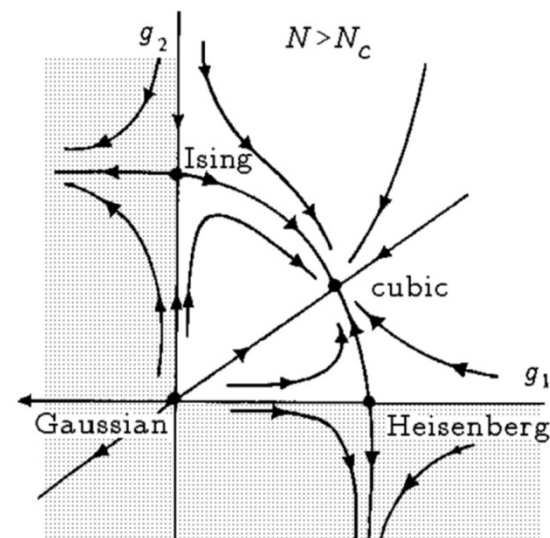
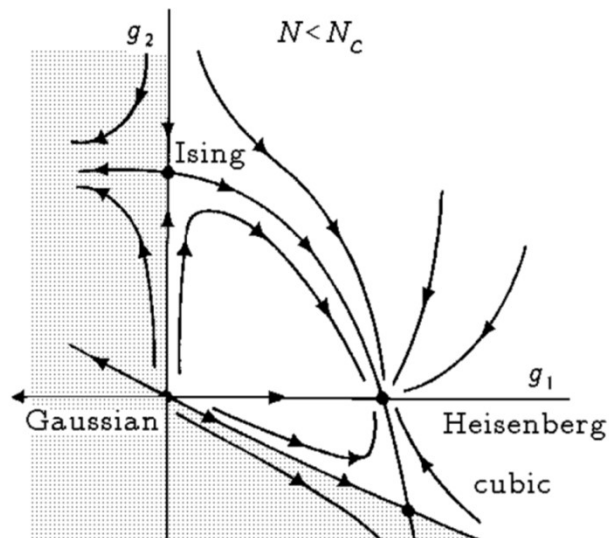


$$\begin{aligned}\dot{x} &= x(r - x - ky) \\ \dot{y} &= y(1 - y - k'x)\end{aligned}$$





RG Flow = Dynamical System



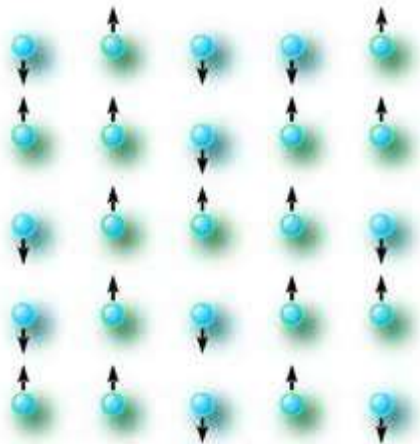
- New techniques:

Conley index

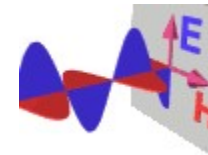
Bifurcations

- New predictions:

3d $O(N)$ model



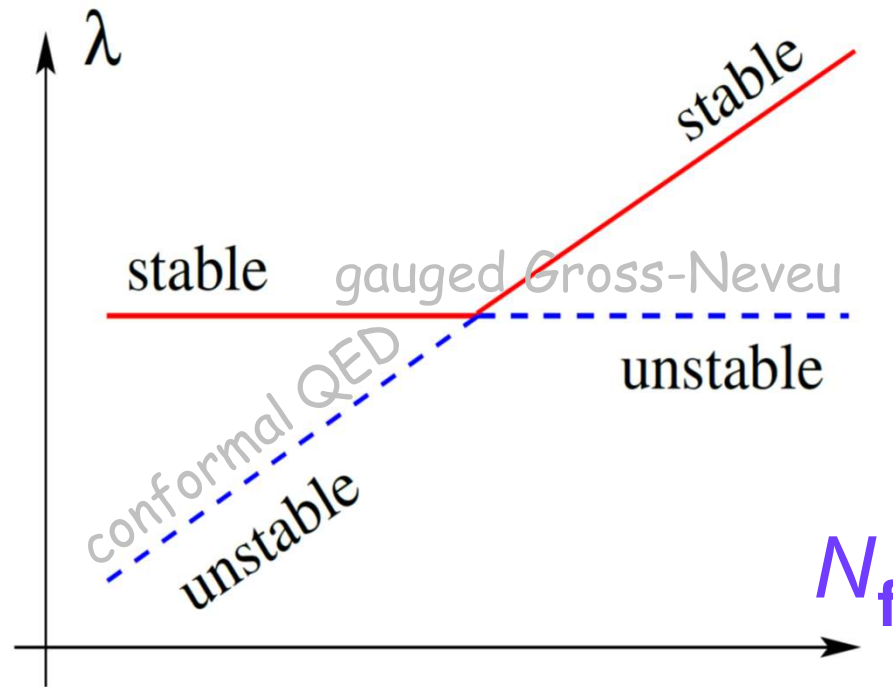
3d QED



4d QCD

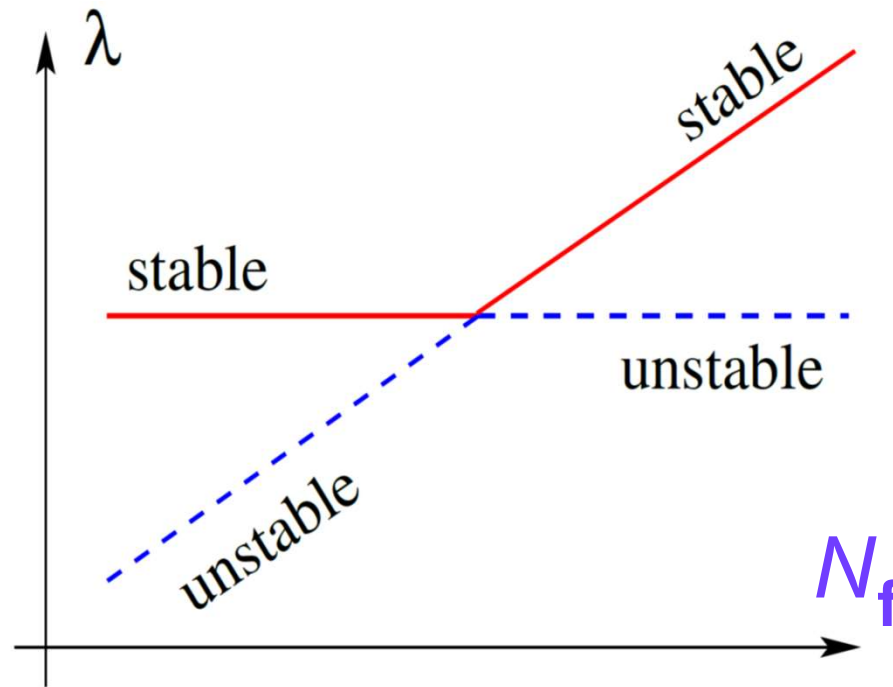


Transcritical Bifurcation



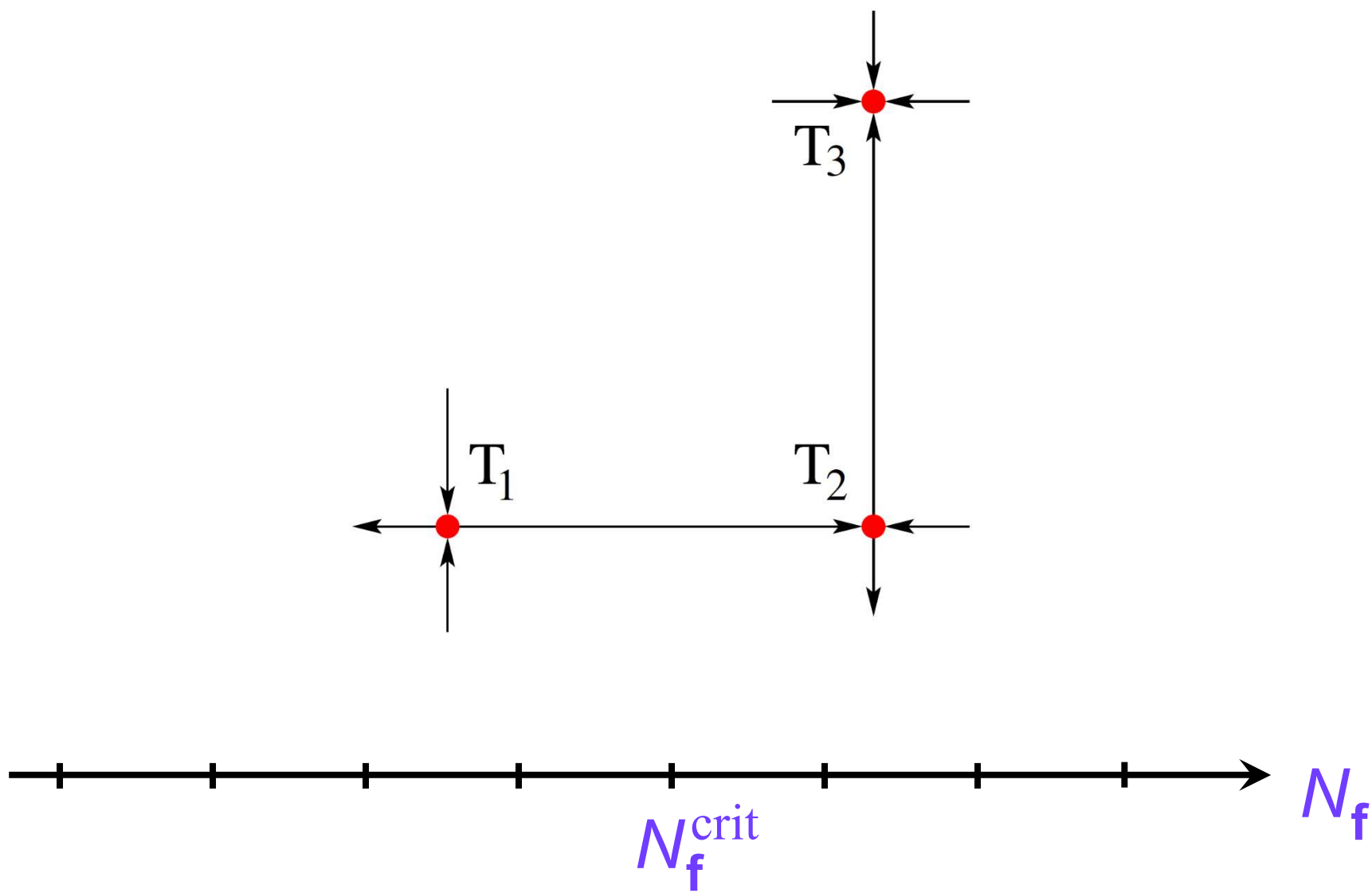
$$\begin{cases} \dot{\lambda}_1 &= (N_f - N_f^{\text{crit}}) \lambda_1 - \lambda_1^2 \\ \dot{\lambda}_2 &= -\lambda_2 \end{cases}$$

Transcritical Bifurcation

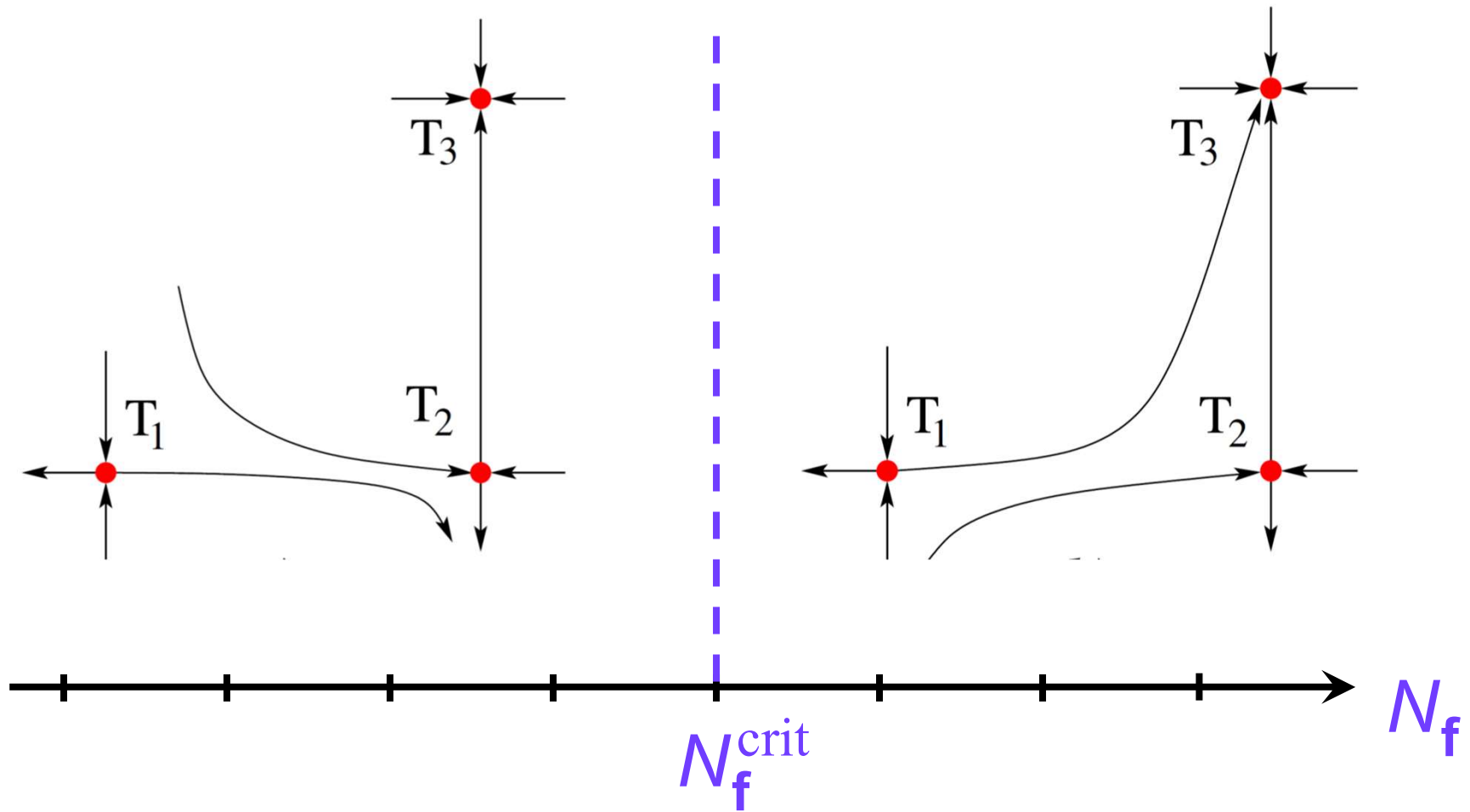


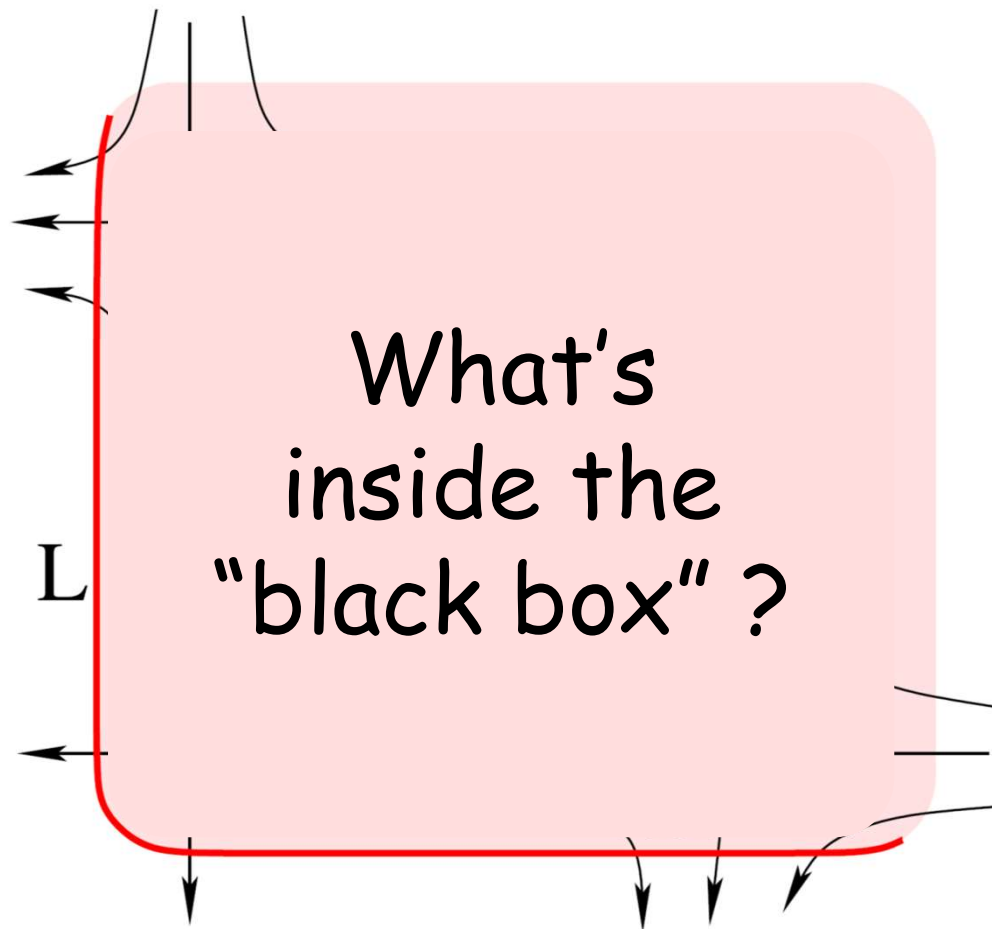
- Codimension-2
- Structurally unstable

$$\Rightarrow \Delta - d \sim \sqrt{N_f - N_f^{\text{crit}}}$$



Heteroclinic Bifurcation

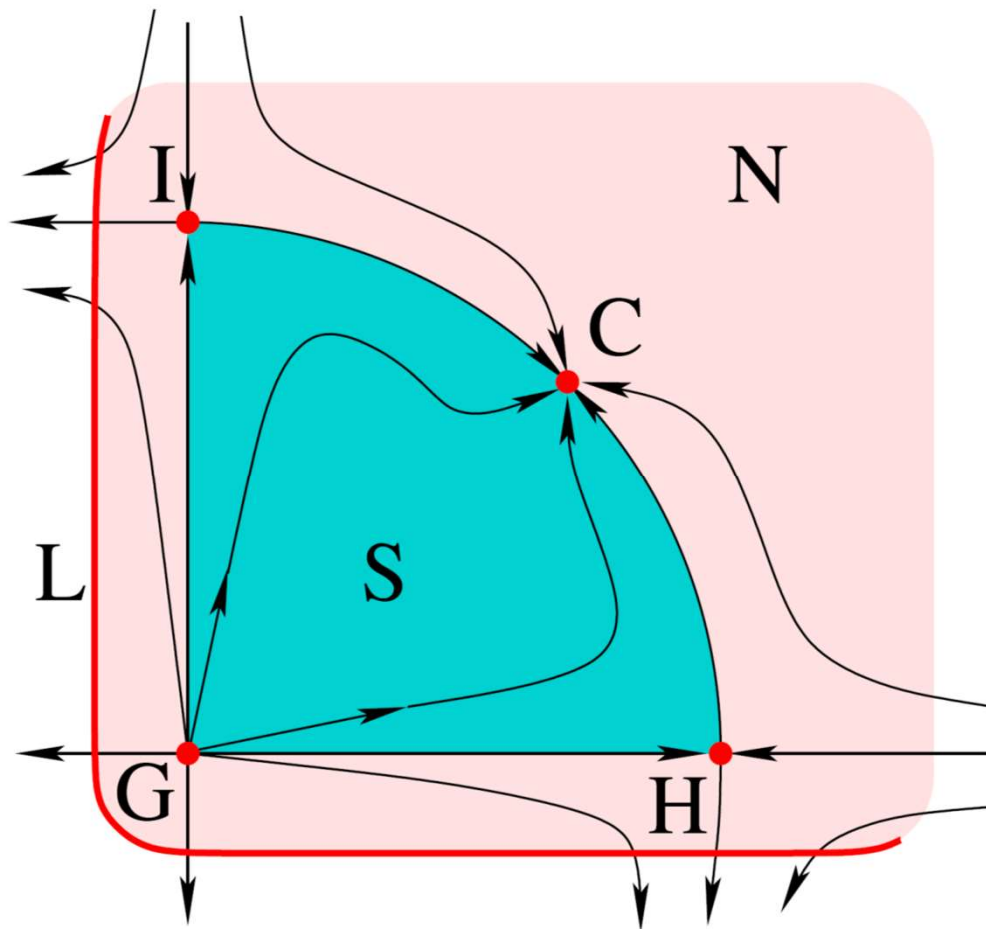




Charles C. Conley
1933-1984

$$\frac{\ker \Delta}{\operatorname{im} \Delta} \cong CH_*(S)$$

$$\Delta \circ \Delta = 0$$



$$CH_*(T_C) = (\mathbb{Z}, 0, 0, 0, \dots)$$

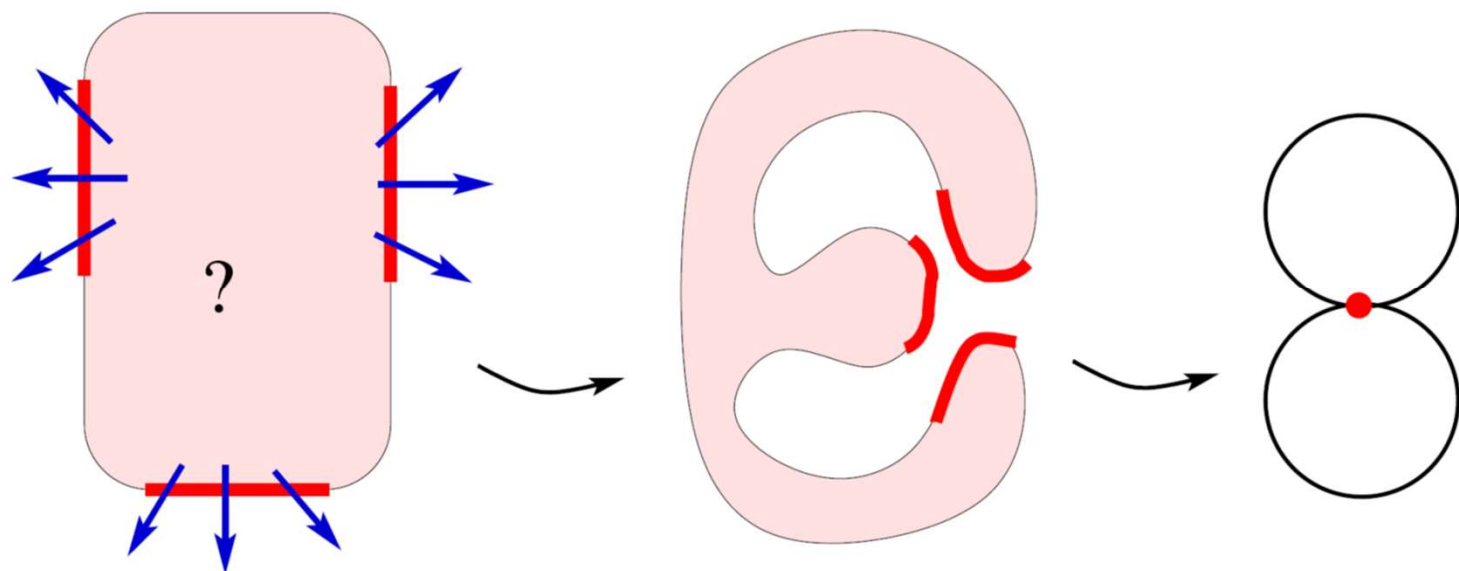
$$CH_*(T_H) = (0, \mathbb{Z}, 0, 0, \dots)$$

$$CH_*(T_I) = (0, \mathbb{Z}, 0, 0, \dots)$$

$$CH_*(T_G) = (0, 0, \mathbb{Z}, 0, \dots)$$



Charles C. Conley
1933-1984

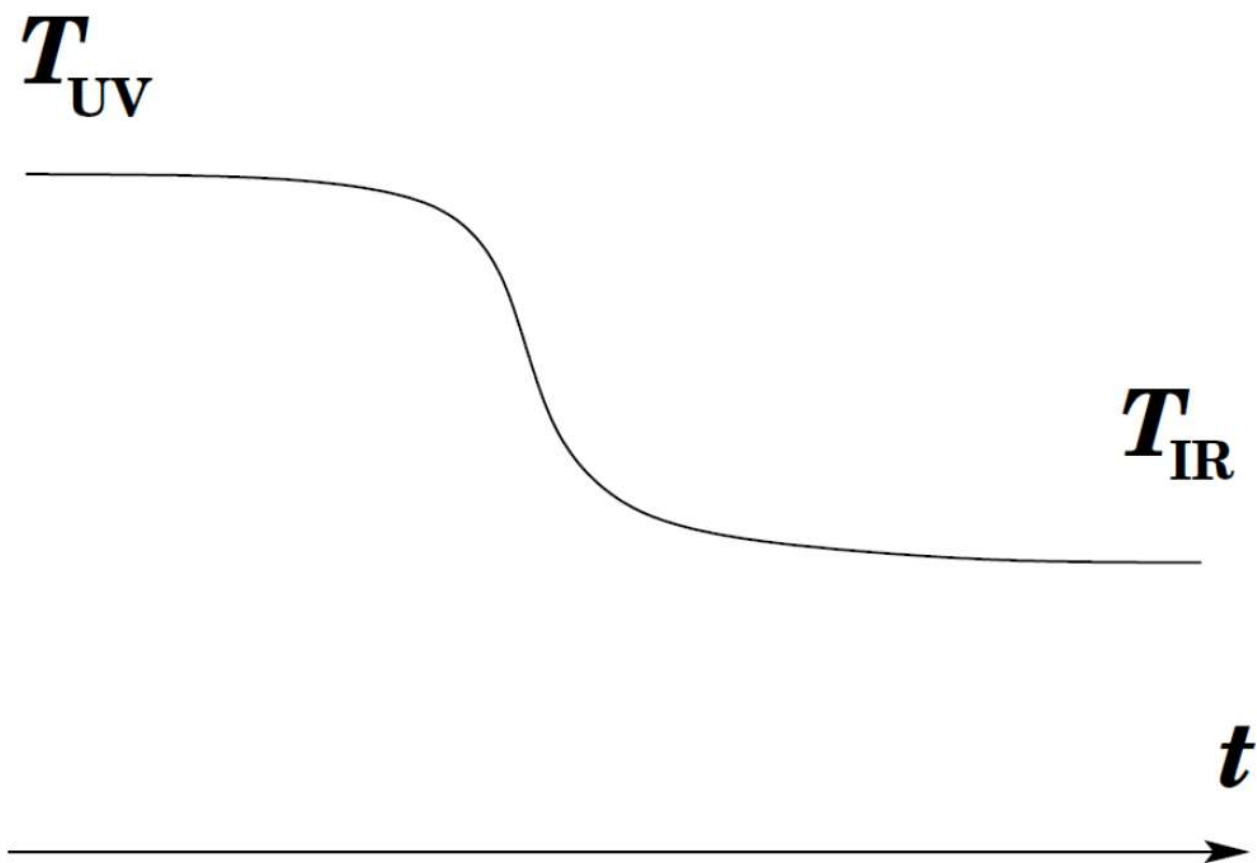


exit set L	N/L	$CH_*(S)$
\emptyset	$D^2 \sqcup \{\text{pt}\}$	$\mathbb{Z}[0]$
S^1	S^2	$\mathbb{Z}[2]$
I	D^2	0
$I \sqcup I$	S^1	$\mathbb{Z}[1]$
$I \sqcup I \sqcup I$	$S^1 \vee S^1$	$\mathbb{Z}[1] \oplus \mathbb{Z}[1]$

μ -theorem:

$$\mu(T_{\text{UV}}) > \mu(T_{\text{IR}})$$

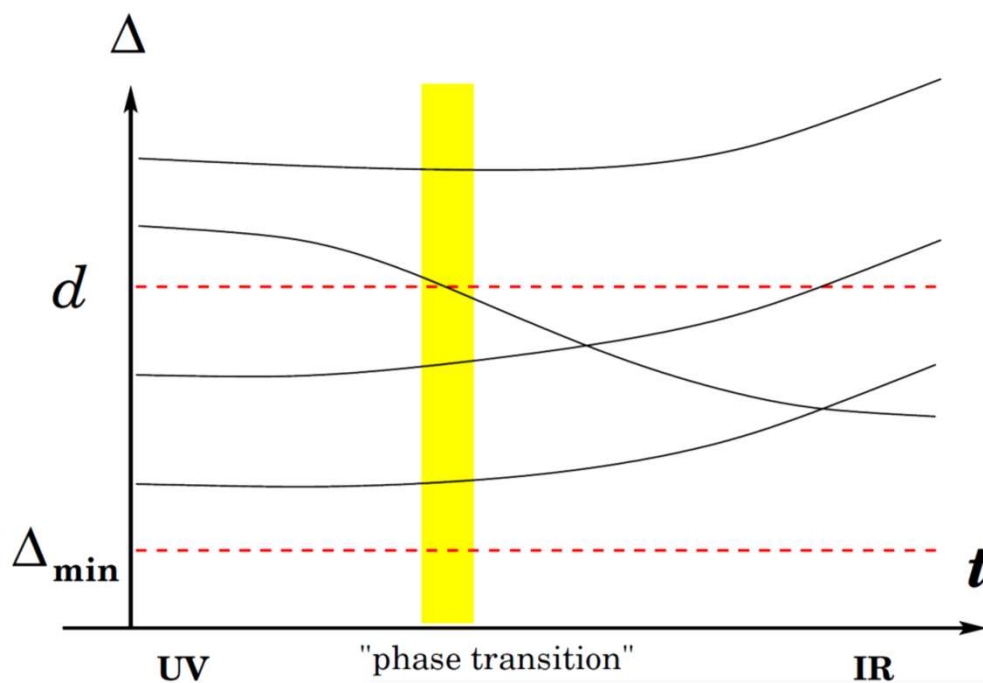
where $\mu(T) := \#\left\{\frac{d-2}{2} \leq \Delta_i < d\right\}$



μ -theorem:

$$\mu(T_{\text{UV}}) > \mu(T_{\text{IR}})$$

where $\mu(T) := \#\left\{\frac{d-2}{2} \leq \Delta_i < d\right\}$



cf. \mathcal{C} -theorem

$$\mathcal{C}(T_{\text{UV}}) > \mathcal{C}(T_{\text{IR}})$$

$$\mu(T) := \#\left\{\frac{d-2}{2} \leq \Delta_i < d\right\}$$

3-state Potts model
 $\mu=3, c=4/5$

Tricritical Ising model
 $\mu=2, c=7/10$

Ising model
 $\mu=1, c=1/2$



$$d = 2 : \quad 0 \leq \Delta_i < 2$$

Spectrum of 2d $\mathcal{N}=2$ superconformal field theory:

$$\{\Delta_i\} \quad 0 \leq \Delta_i < 2$$

$$\Delta_i = R_i$$

“scaling dimensions”
a.k.a. “conformal dimensions”

“R-charges”

Central charge of $\mathcal{N}=2$ Landau-Ginzburg model:

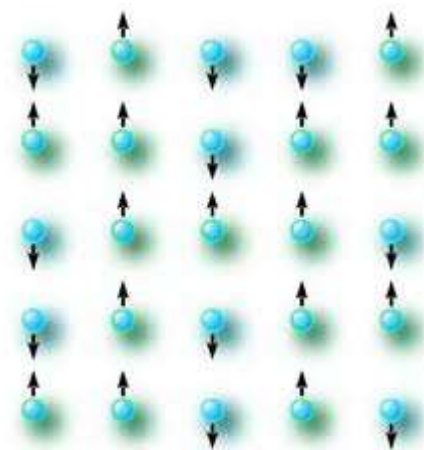
$$c = 3 \sum_i (1 - R_i)$$

$$W(\lambda^{R_i} \Phi_i) = \lambda^{(2)} W(\Phi_i)$$

A_N $\mathcal{N}=2$ minimal model

$$c = 3 - \frac{6}{N+1}$$

$$\Delta_i = \frac{2i}{N+1} \quad i = 1, \dots, N$$



can be described as a LG model with the superpotential

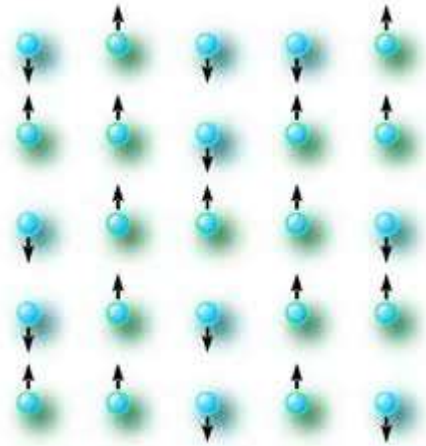
$$W = x^{N+1}$$

$$\text{cf. } \text{sp}(x^{N+1}) = \left\{ \frac{1}{N+1}, \frac{2}{N+1}, \dots, \frac{N}{N+1} \right\}$$

$$\text{chiral ring: } \mathbb{C}[x]/dW = H^*(\mathbb{CP}^{N-1})$$

$$\text{is the classical cohomology ring of } \mathbb{CP}^{N-1} = \frac{SU(N)}{U(N-1)}$$

A_N $\mathcal{N}=2$ minimal model = $\mathcal{N}=2$ Kazama-Suzuki coset model



$$\frac{SU(N)}{SU(N-1) \times U(1)} \text{ at level 1}$$

$\mathcal{N}=2$ Kazama-Suzuki model

$$\frac{SU(N)}{SU(N-k) \times SU(k) \times U(1)} \text{ at level 1}$$

$$c = \frac{3k(N-k)}{N+1}$$

$\mathcal{N}=2$ Kazama-Suzuki model

$$\frac{SU(N)}{SU(N-k) \times SU(k) \times U(1)}$$

$$W = x_1^{N+1} + x_2^{N+1} + \dots + x_k^{N+1}$$

expressed as a polynomial in the elementary symmetric functions

$$z_i = \sigma_i(x_1, \dots, x_k)$$

e.g.

$$W(x_1, x_2) = x_1^2 + x_2^2 \quad \longrightarrow \quad W(z_1, z_2) = z_1^2 - 2z_2$$

$$z_1 = x_1 + x_2$$

$$z_2 = x_1 x_2$$

chiral ring (Jacobi ring / Milnor algebra):

$$\frac{\mathbb{C}[z_i]}{\{\partial_i W\}} = H^*(Gr(k, N))$$

is the classical cohomology ring of the Grassmannian

cf. Thom-Sebastiani sum:

1	2	3	2	1
$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$

$$sp(x^4 + y^4) = \left\{ \frac{2}{4'}, \frac{3}{4'}, \frac{3}{4'}, \frac{4}{4'}, \frac{4}{4'}, \frac{4}{4'}, \frac{5}{4'}, \frac{5}{4'}, \frac{6}{4'} \right\}$$

and similarly for other Brieskorn-Pham singularities.

chiral ring (Jacobi ring / Milnor algebra):

$$\frac{\mathbb{C}[z_i]}{\{\partial_i W\}} = H^*(Gr(k, N))$$

is the classical cohomology ring of the Grassmannian

cf. Thom-Sebastiani sum:

1	2	3	2	1
$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$

$$sp(x^4 + y^4) = \left\{ \frac{2}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{4}, \frac{4}{4}, \frac{4}{4}, \frac{5}{4}, \frac{5}{4}, \frac{6}{4} \right\}$$

Note: $sp(W) \subset (0, k)$ while $0 \leq \Delta_i < 2$

Definition: An invariant I of a singularity is *semi-continuous* if for each adjacency $f \rightsquigarrow g_1 + g_2 + \dots + g_N$ one has

$$I(f) \geq \sum_{i=1}^N I(g_i)$$

Definition: A subset $S \subset \mathbb{R}$ is called a semi-continuity set if $\#S \cap sp(f)$ is semi-continuous.

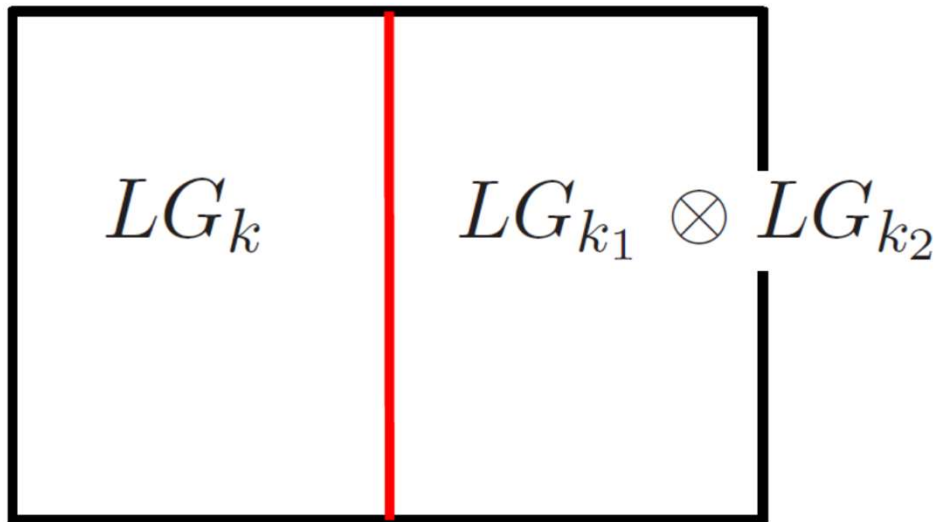


Theorem:

(A. Varchenko, 1983): if f is quasi-homogeneous then each open interval $(a, a + 1)$ is a semi-continuity set.

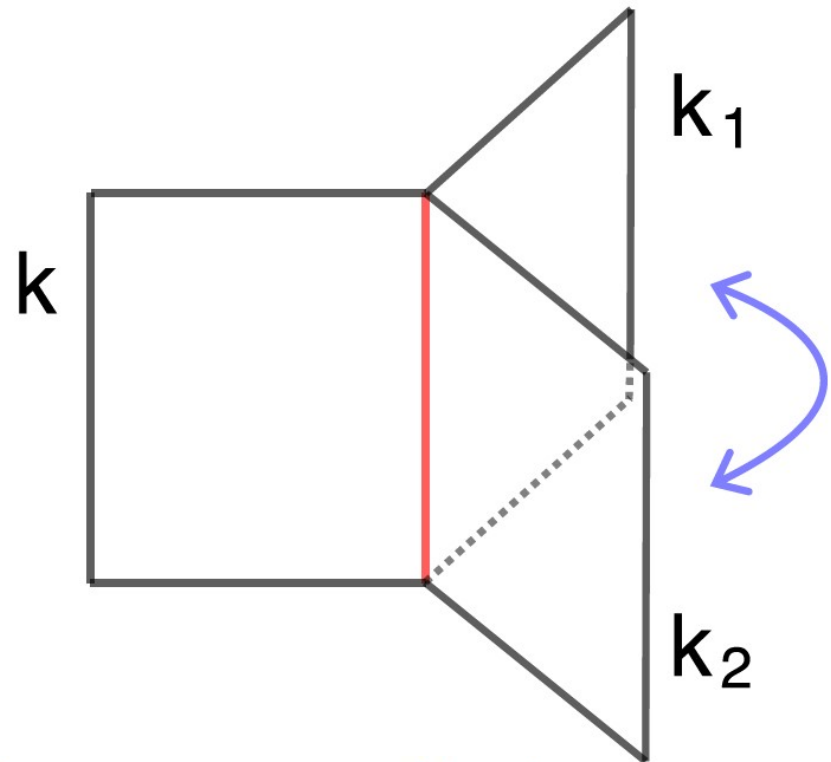
(J. Steenbrink, 1985): for general f each interval $(a, a + 1]$ is semi-continuity set.

Spectra of walls in KS models



$$\mathcal{I}_k^{k_1, k_2} =$$

Matrix factorization



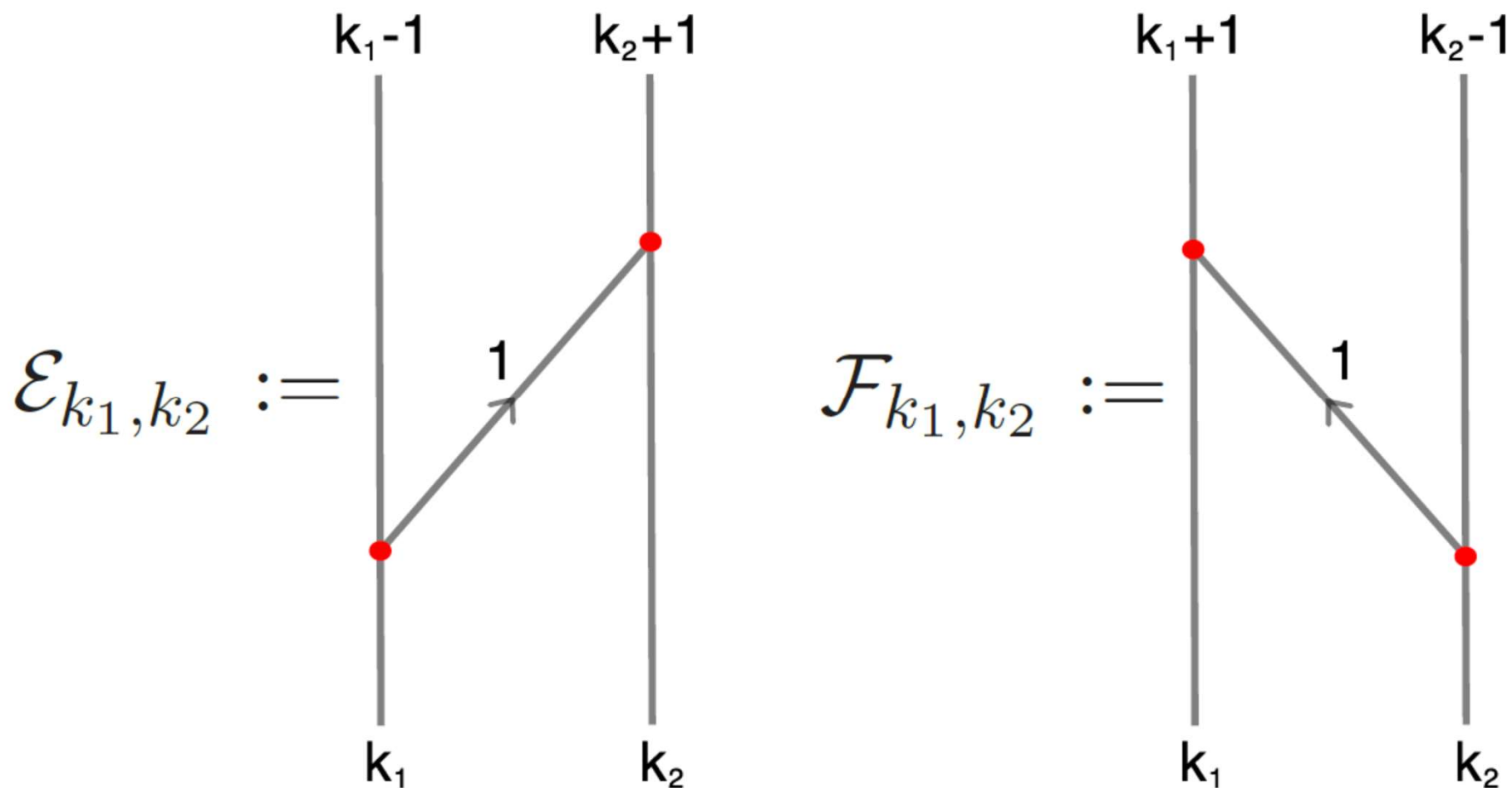
$$LG_k: \quad W = x_1^{N+1} + x_2^{N+1} + \dots + x_k^{N+1}$$

Quantum groups and KS models

$$\mathcal{I}_{k_1, k_2}^k * \mathcal{I}_k^{k_1, k_2} = \text{Diagram 1} = \text{Diagram 2} \left\{ \begin{bmatrix} k \\ k_1 \end{bmatrix} \right\}$$

The diagram on the left shows a circle with two vertical lines passing through it. The top line is labeled k and the bottom line is labeled k . The left side of the circle is labeled k_1 and the right side is labeled k_2 . The intersection points of the lines and the circle are marked with red dots.

The diagram on the right shows a single vertical line labeled k at both the top and bottom.



$$\mathcal{E}_{k_1, k_2} := \left(\text{Id}_{k_1-1} \otimes \mathcal{I}_{1, k_2}^{k_2+1} \right) * \left(\mathcal{I}_{k_1}^{k_1-1, 1} \otimes \text{Id}_{k_2} \right)$$

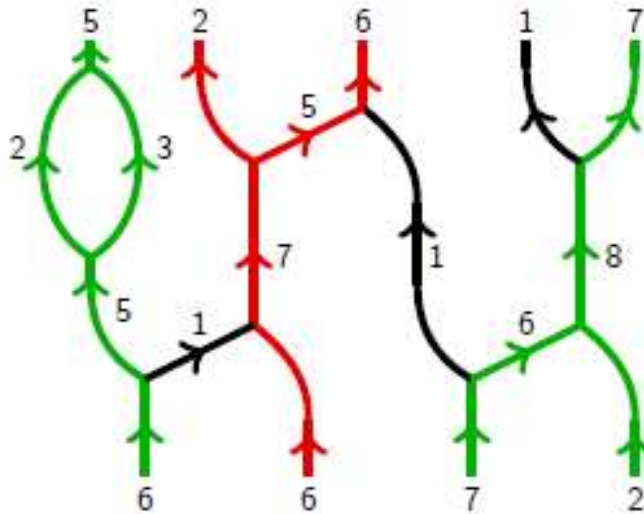
$$\mathcal{F}_{k_1, k_2} := \left(\mathcal{I}_{k_1, 1}^{k_1+1} \otimes \text{Id}_{k_2-1} \right) * \left(\text{Id}_{k_1} \otimes \mathcal{I}_{k_2}^{1, k_2-1} \right)$$

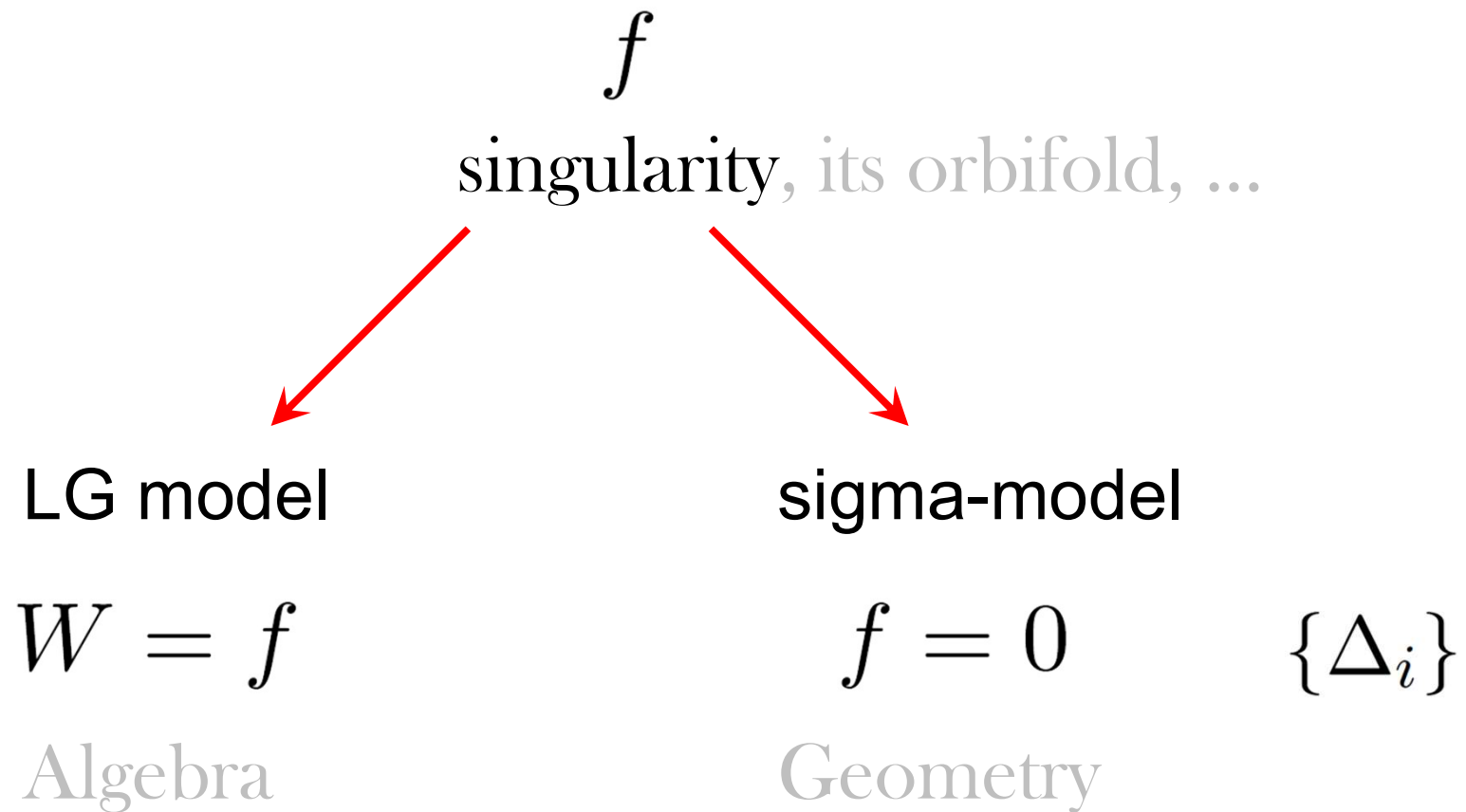
Categorification of Quantum Groups

$$\mathcal{E}_{k_1+1, k_2-1} * \mathcal{F}_{k_1, k_2} \cong \mathcal{F}_{k_1-1, k_2+1} * \mathcal{E}_{k_1, k_2} \oplus \text{Id}_{k_1, k_2} \{[k_2 - k_1]\}$$

and, similarly, for $k_1 \geq k_2$:

$$\mathcal{F}_{k_1-1, k_2+1} * \mathcal{E}_{k_1, k_2} \cong \mathcal{E}_{k_1+1, k_2-1} * \mathcal{F}_{k_1, k_2} \oplus \text{Id}_{k_1, k_2} \{[k_1 - k_2]\}$$



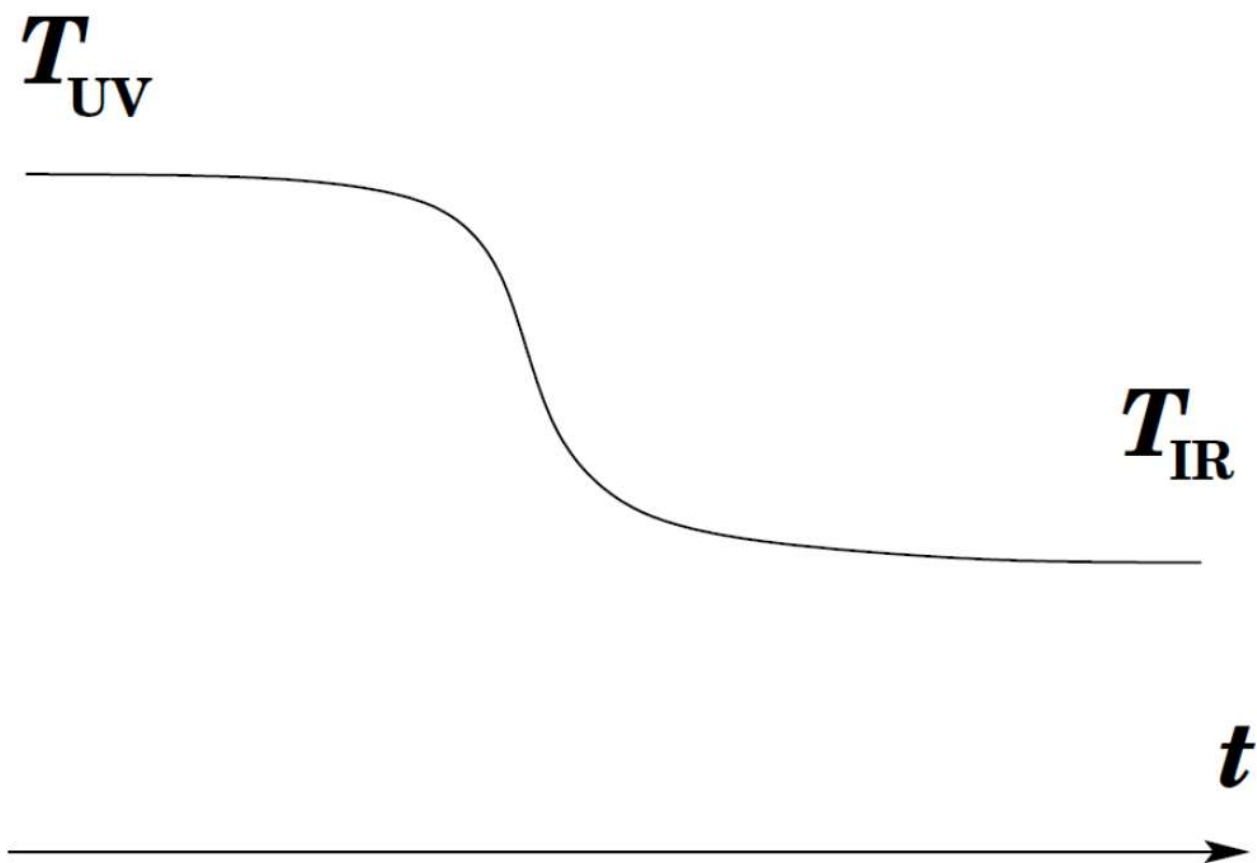


Also a product of $\mathcal{N}=2$
minimal models and
Kazama-Suzuki models!

$$\frac{SL(2)}{U(1)} \otimes \text{LG}(W = f)$$

$$\mathcal{H}_{\text{UV}}^{BPS} \xrightarrow[\text{spectral sequence}]{?} \mathcal{H}_{\text{IR}}^{BPS}$$

Categorification of semicontinuity?



Thanks for listening.

Questions?

Coset theory	c	Degrees of generators
$\frac{\mathrm{SU}(n+m)}{\mathrm{SU}(n) \otimes \mathrm{SU}(m) \otimes \mathrm{U}(1)}$	$\frac{3nm}{m+n+1}$	$1, 2, \dots, \min(n, m)$
$\frac{\mathrm{SO}(n+2)}{\mathrm{SO}(n) \otimes \mathrm{U}(1)}, (n \text{ even})$	$\frac{3n}{n+1}$	$1, n/2$
$\frac{\mathrm{SO}(2n)}{\mathrm{SU}(n) \otimes \mathrm{U}(1)}$	$\frac{3n(n-1)}{2(2n-2+1)}$	$4i-2, \begin{cases} i=1, \dots, n/2 & (n \text{ even}) \\ i=1, \dots, (n-1)/2 & (n \text{ odd}) \end{cases}$
$\frac{E_6}{\mathrm{SO}(10) \otimes \mathrm{U}(1)}$	$\frac{48}{13}$	$1, 4$
$\frac{E_7}{E_6 \otimes \mathrm{U}(1)}$	$\frac{81}{19}$	$1, 5, 9$

e.g. $\mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$:

W.Lerche, C.Vafa, N.P.Warner

$$W = x_1^{n+1} + x_1 x_2^2$$

D_{n+2} minimal model