Spectra, dynamical systems, and RG flows

Based on:

S.G. arXiv: 1503.01474
arXiv: 1608.06638

see also:

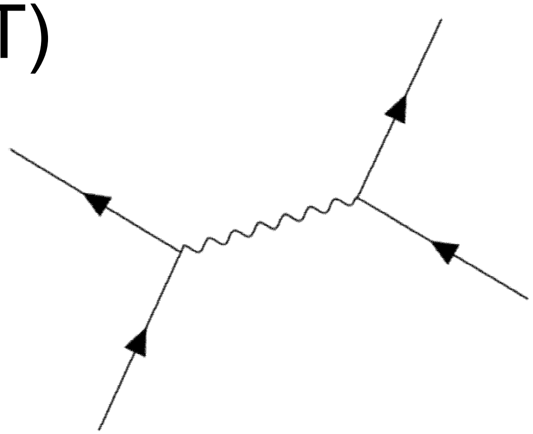
D.Gepner (1987)
C.Vafa, N.P.Warner (1989)
E.J.Martinec (1989)
W.Lerche, C.Vafa, N.P.Warner (1989)
B.R.Greene, S.-S.Roan, S.-T.Yau (1991)
D.Xie, S.-T.Yau (2015)
F.Kuipers, U.Gursoy, Y.Kuznetsov (2018)
C.B.Jepsen, I.R.Klebanov, F.K.Popov (2020)
C.B.Jepsen, F.K.Popov (2021)
:

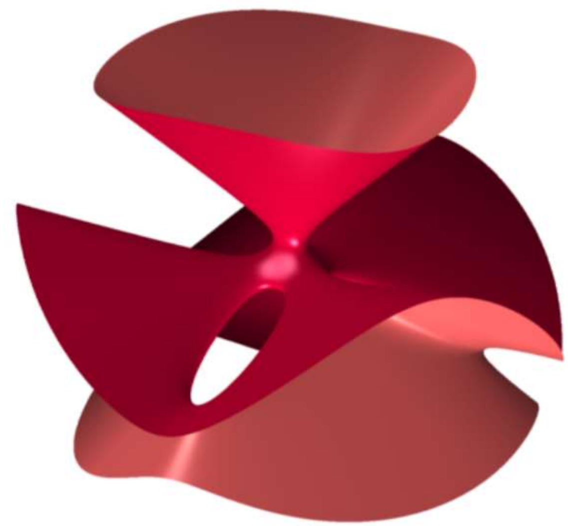
+ work in progress w/ L.Katzarkov, K.S.Lee, J.Svoboda, ...

f

singularity, its orbifold, ...

Conformal Field Theory (CFT)



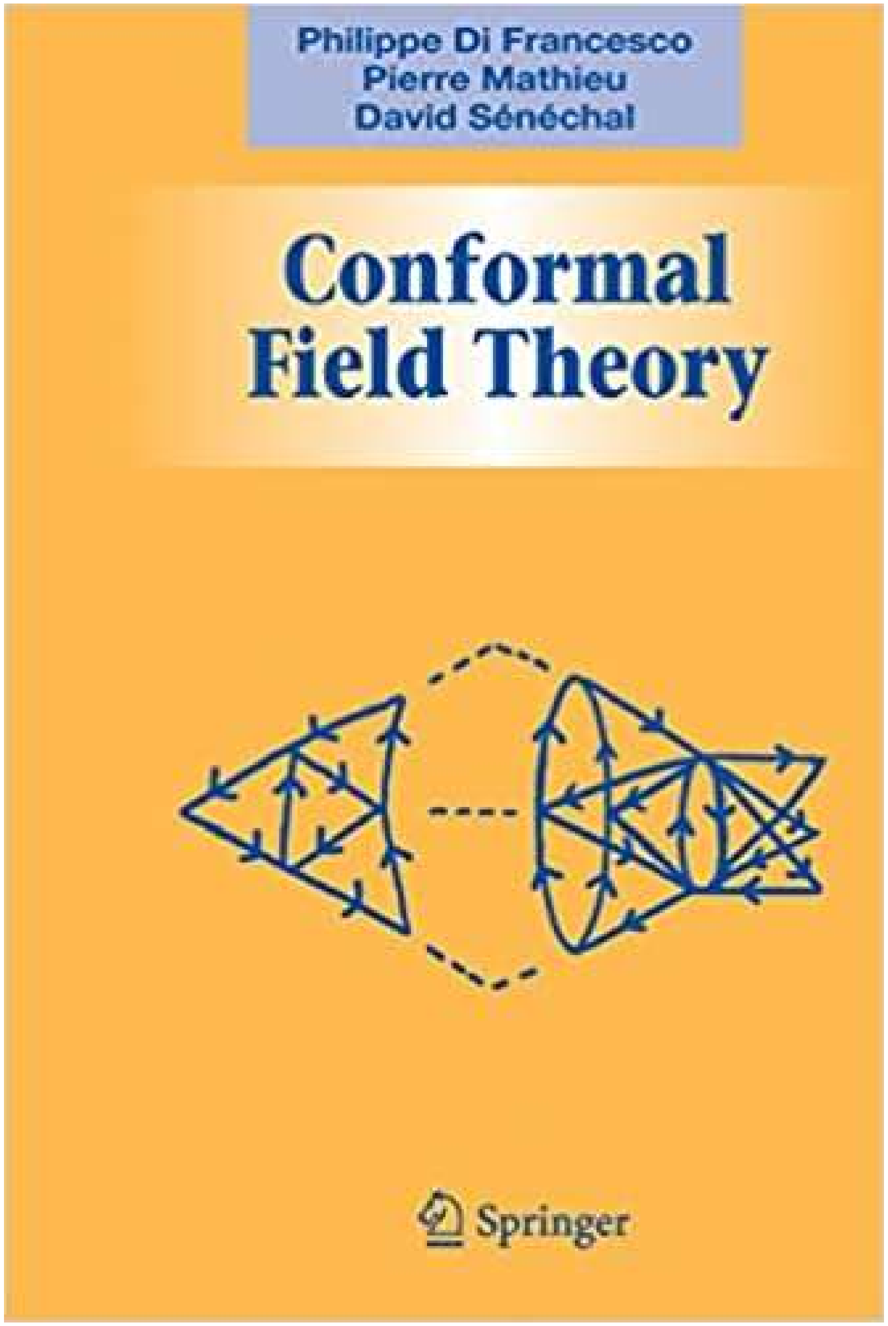


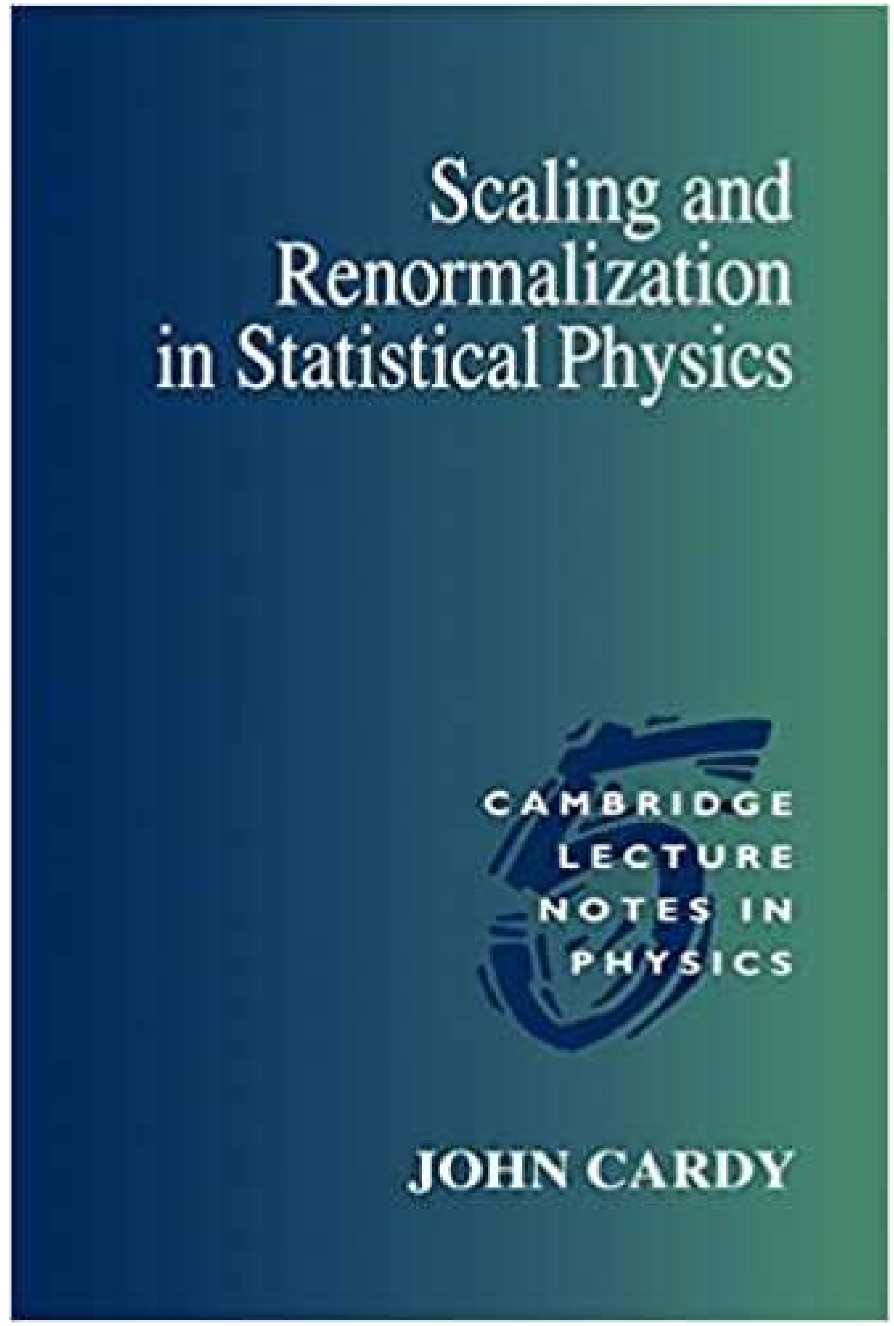
$\{\Delta_i\}$ ⟺ $sp(f) =$

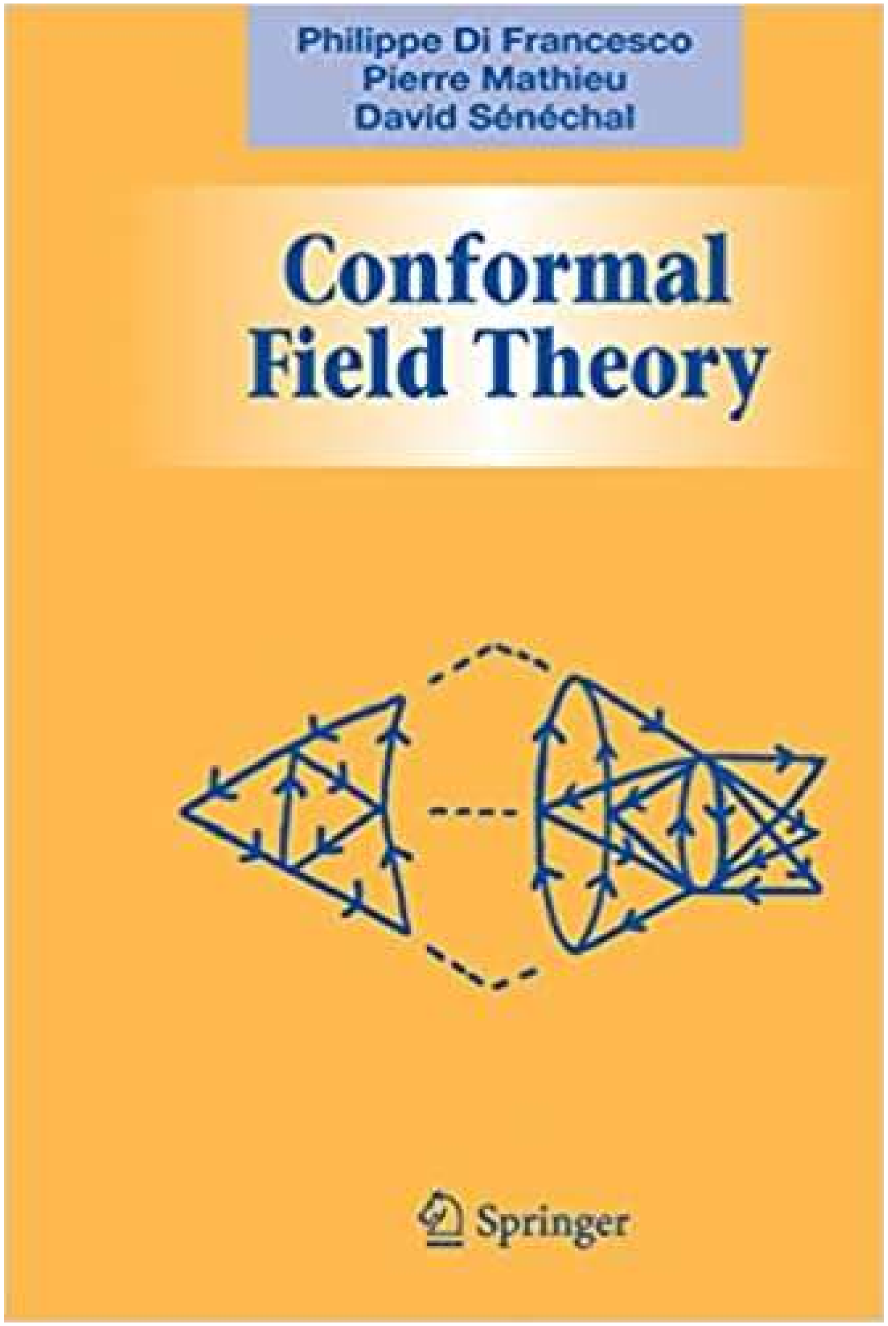
$$= \{\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots \leq \alpha_\mu\}$$

Definition: *conformal field theory* is a table of integrals.

- Brian Greene


Philippe Di Francesco
Pierre Mathieu
David Sénéchal
Conformal Field Theory
Springer


Scaling and Renormalization in Statistical Physics
CAMBRIDGE LECTURE NOTES IN PHYSICS
5
JOHN CARDY


Philippe Di Francesco
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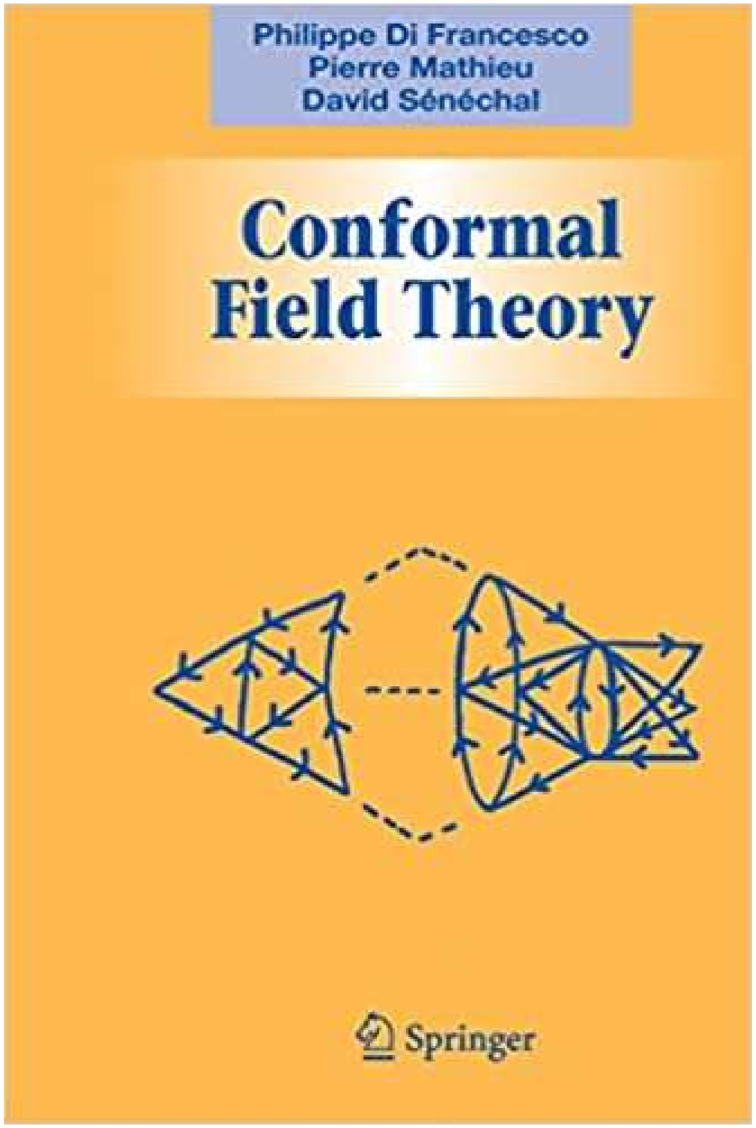


A conformal field theory is a list of operators (states) and their correlation functions (many determined by scaling dimensions).

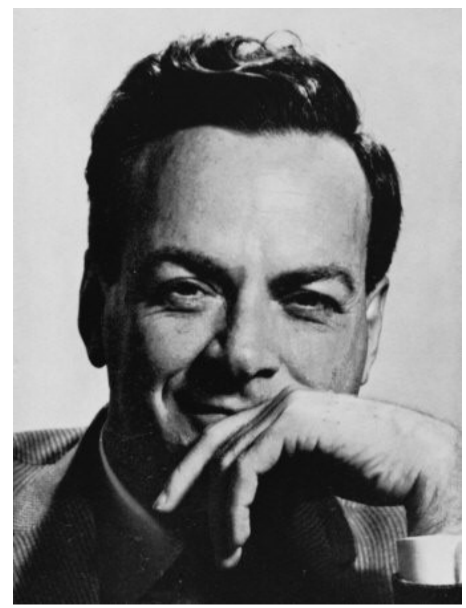
$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}}$$

- Spectrum
- Correlation functions
- Scaling dimensions

$$\{\Delta_i\}$$

"The shell game that we play ... is technically called 'renormalization'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate."

Richard Feynman (1985)



JOAN CARTIER

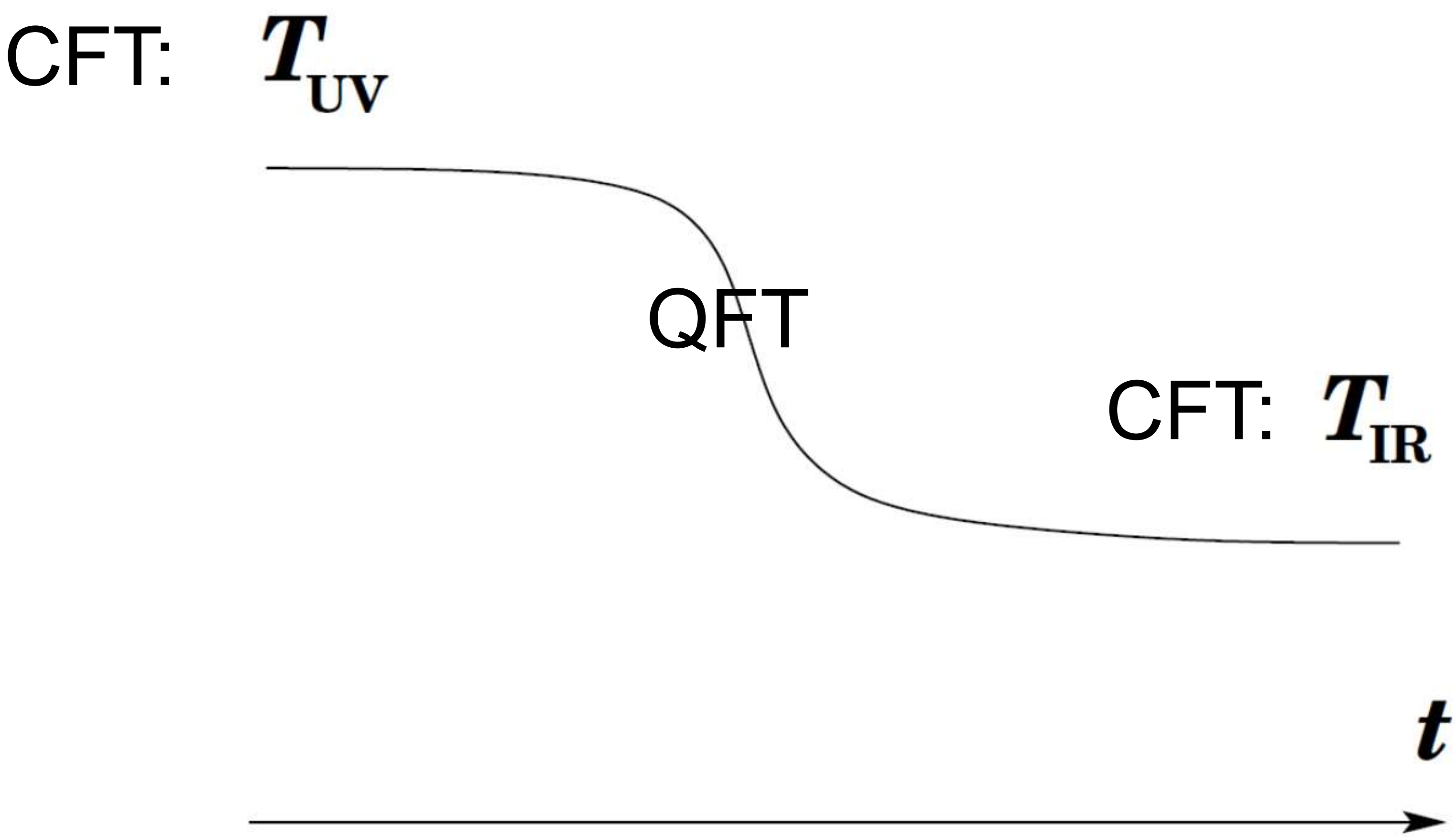
ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.

$$\frac{d\lambda_i}{dt} = (d - \Delta_i)\lambda_i - C_{ijk}\lambda_j\lambda_k + O(\lambda^3)$$

OPE coefficients

cf. Frobenius manifold

$$\varphi_i \star \varphi_j \sim \sum_k C_{ijk}\varphi_k$$



Exact Five-Loop Renormalization Group Functions of ϕ^4-Theory with $O(N)$-Symmetric and Cubic Interactions.

H. Kleinert and V. Schulte-Frohlinde

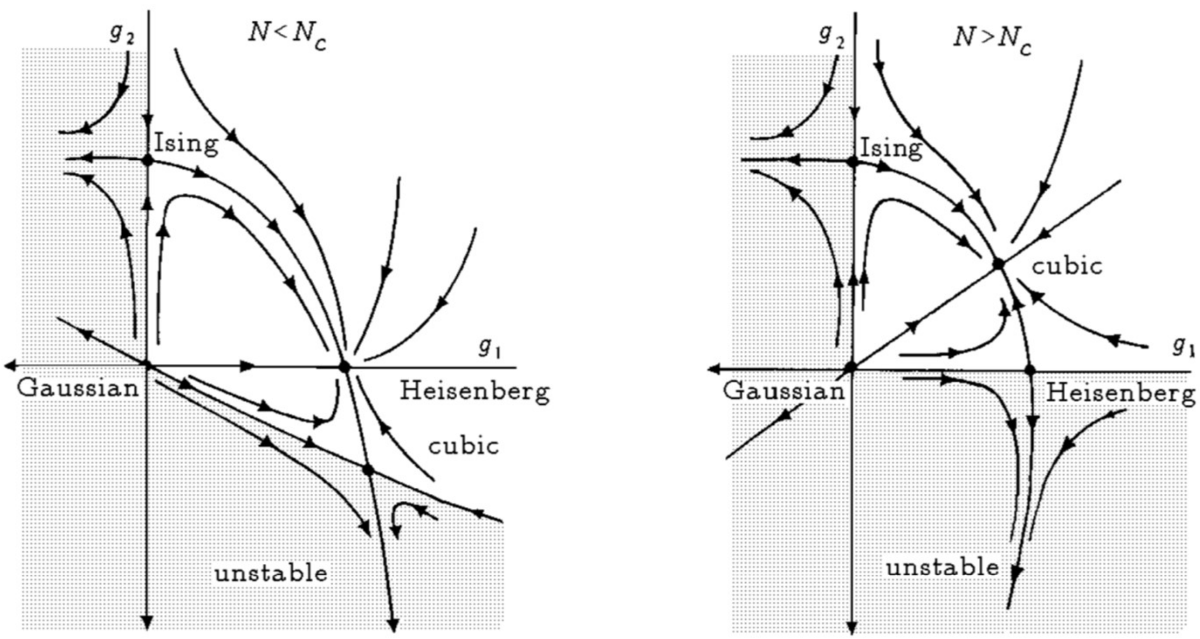
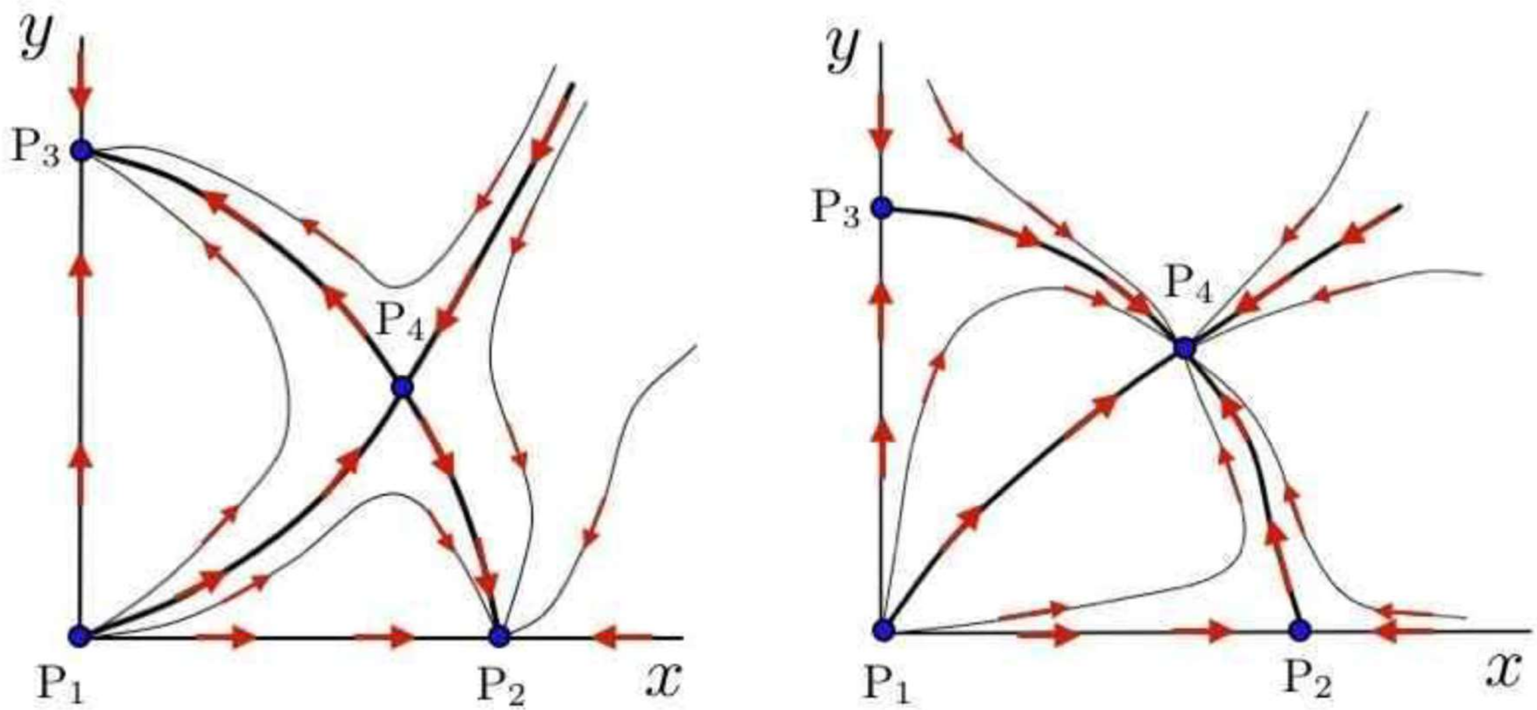


Figure 1: The Stability of the fixed points in the ϕ^4-theory with $O(N)$-symmetric and cubic coupling for $N < N_c$ and $N > N_c$. Our results are compatible with $N_c = 3$.


y
P3
P4
P1
P2
x
y
P3
P4
P1
P2
x

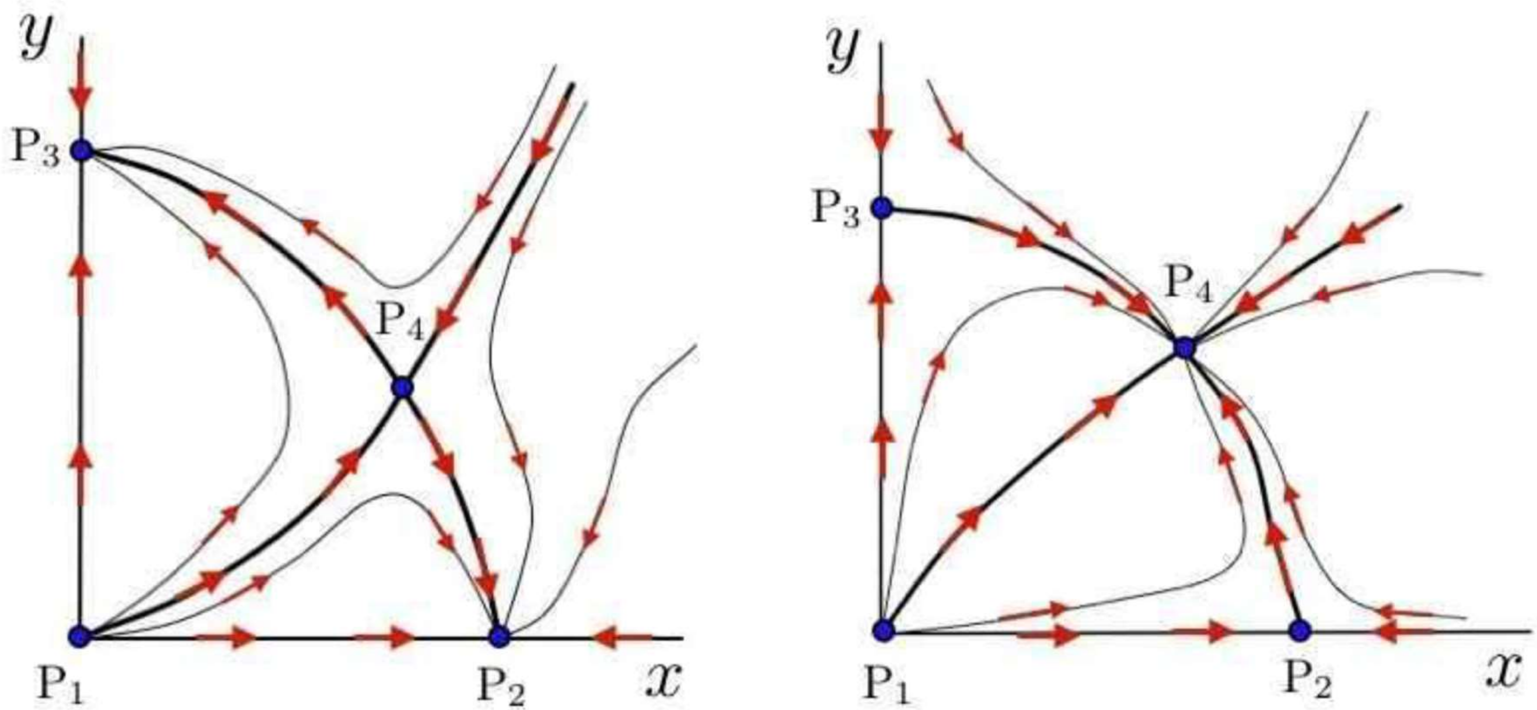
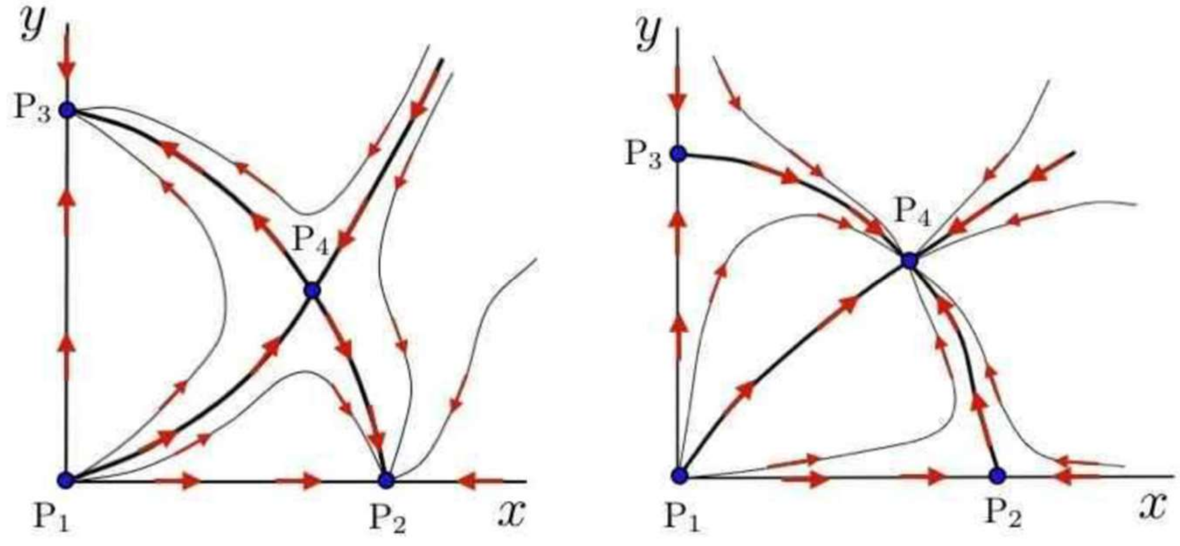
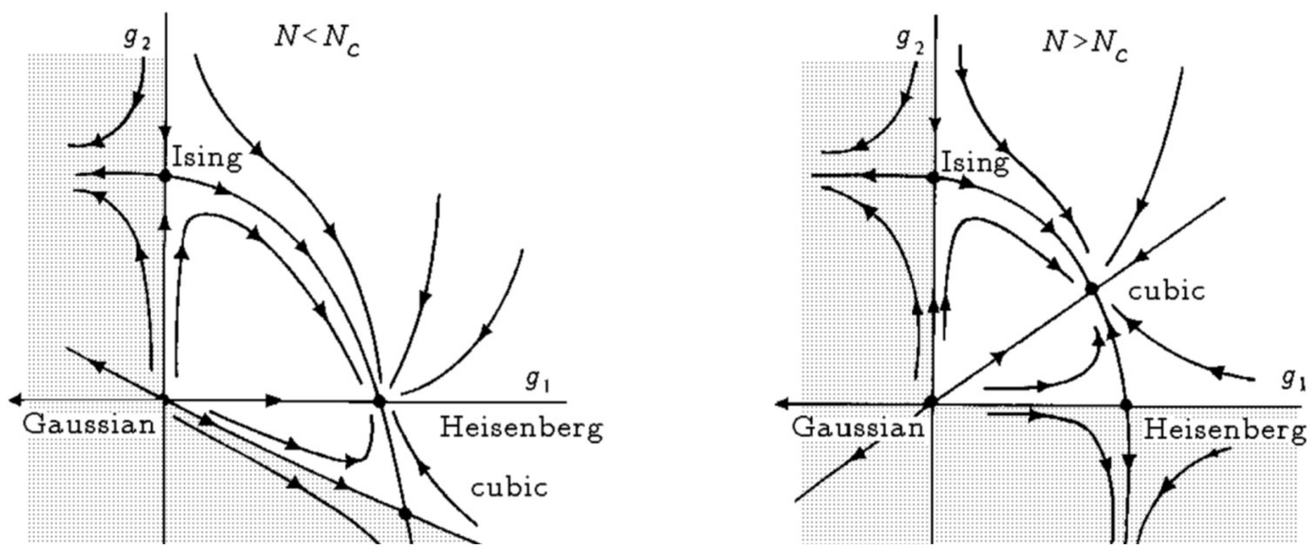


Figure 3.12: Two possible phase flows for the rabbits *vs.* sheep model of eqs. 3.61, Left panel: $k > r > k'^{-1}$. Right panel: $k < r < k'^{-1}$.

$$\dot{x} = x\,(r - x - ky)$$
$$\dot{y} = y\,(1 - y - k'x)$$



RG Flow = Dynamical System



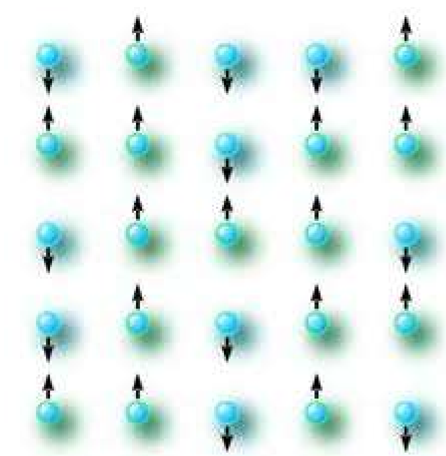
- New techniques:

Conley index

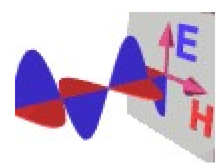
Bifurcations

- New predictions:

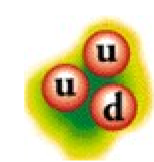
3d O(N) model



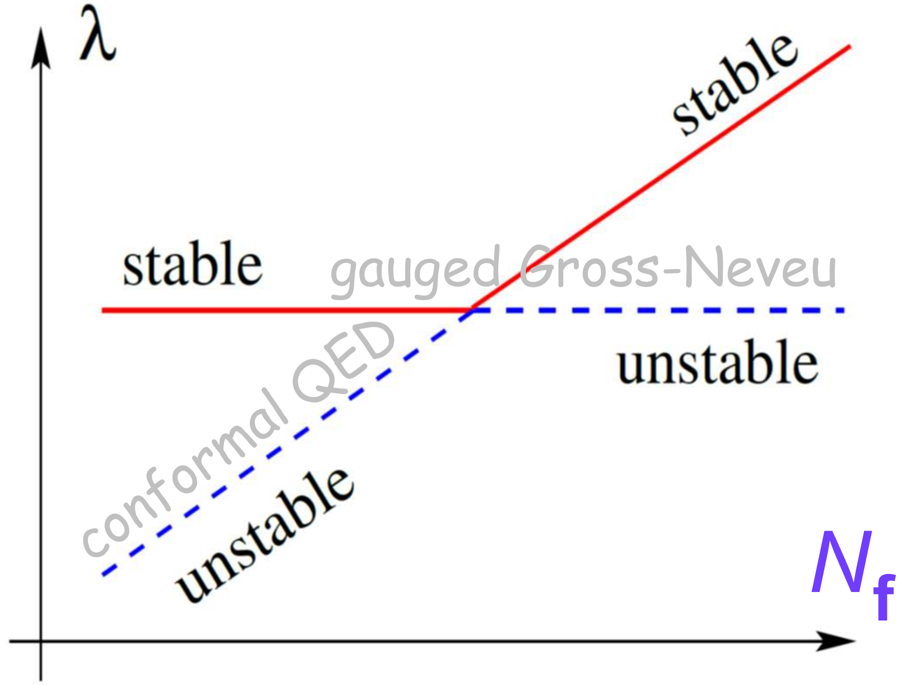
3d QED



4d QCD

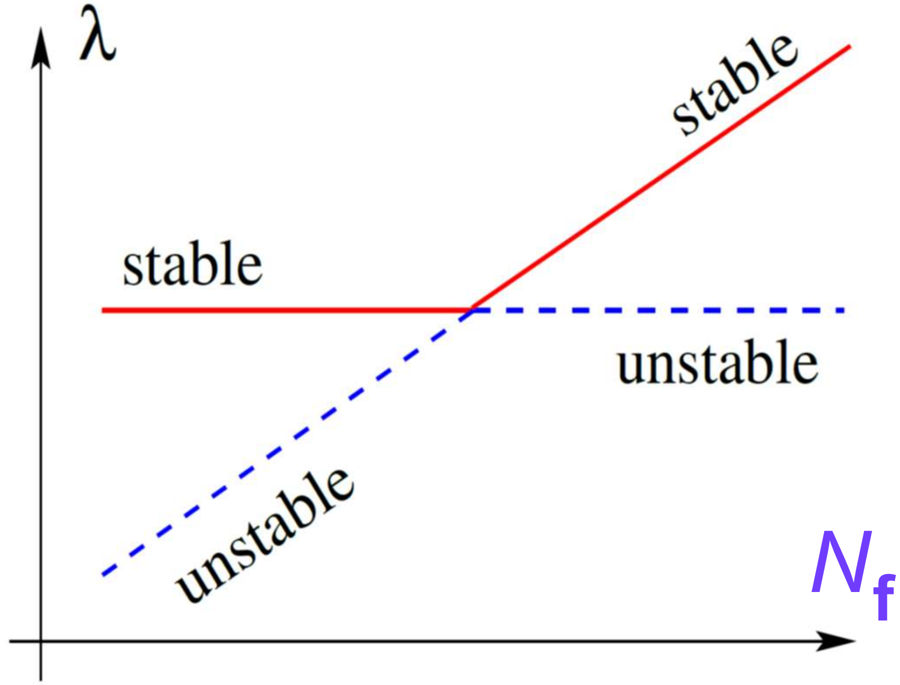


Transcritical Bifurcation



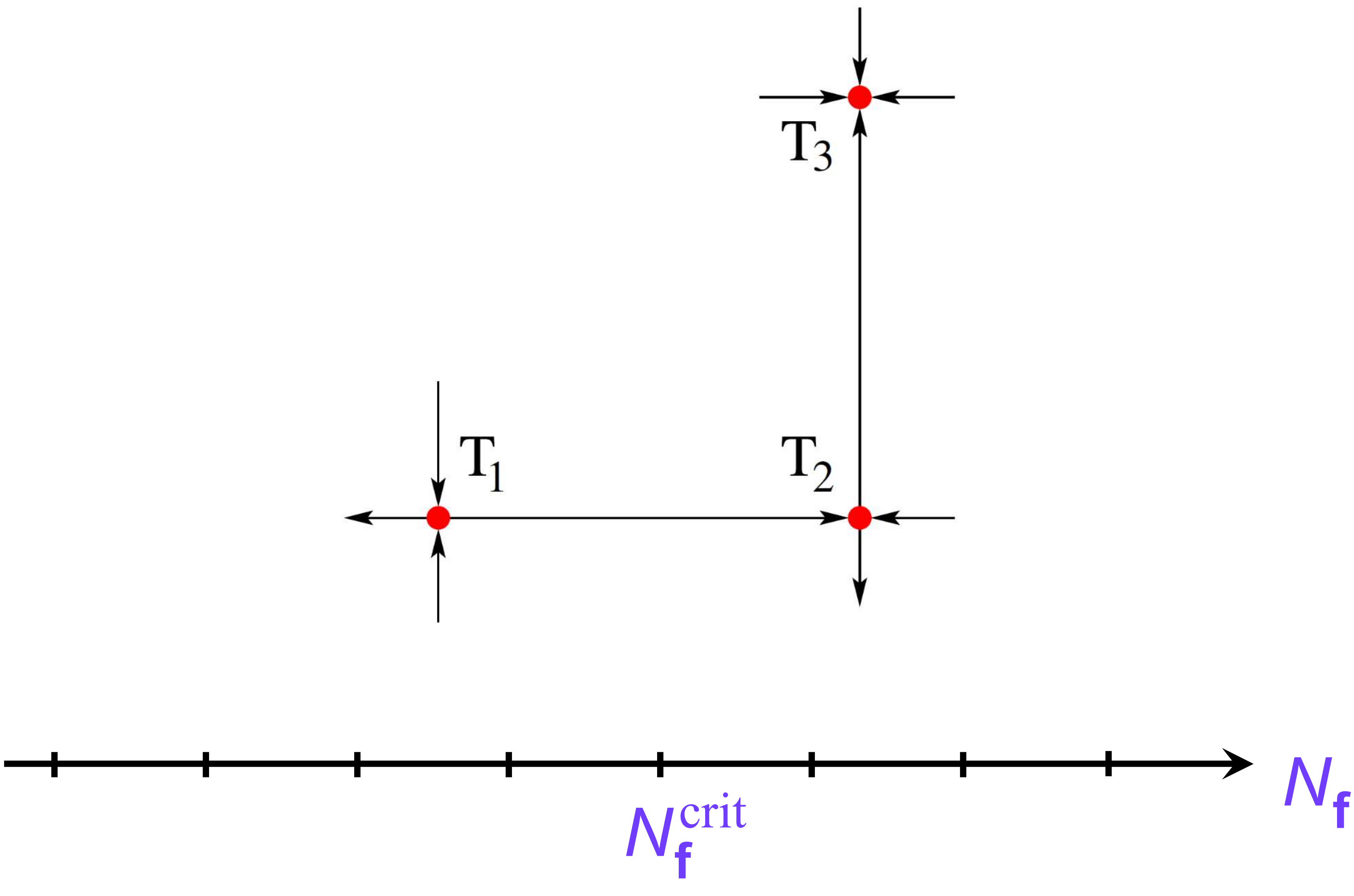
$$\begin{cases} \dot{\lambda}_1 = (N_f - N_f^{\text{crit}})\lambda_1 - \lambda_1^2 \\ \dot{\lambda}_2 = -\lambda_2 \end{cases}$$

Transcritical Bifurcation



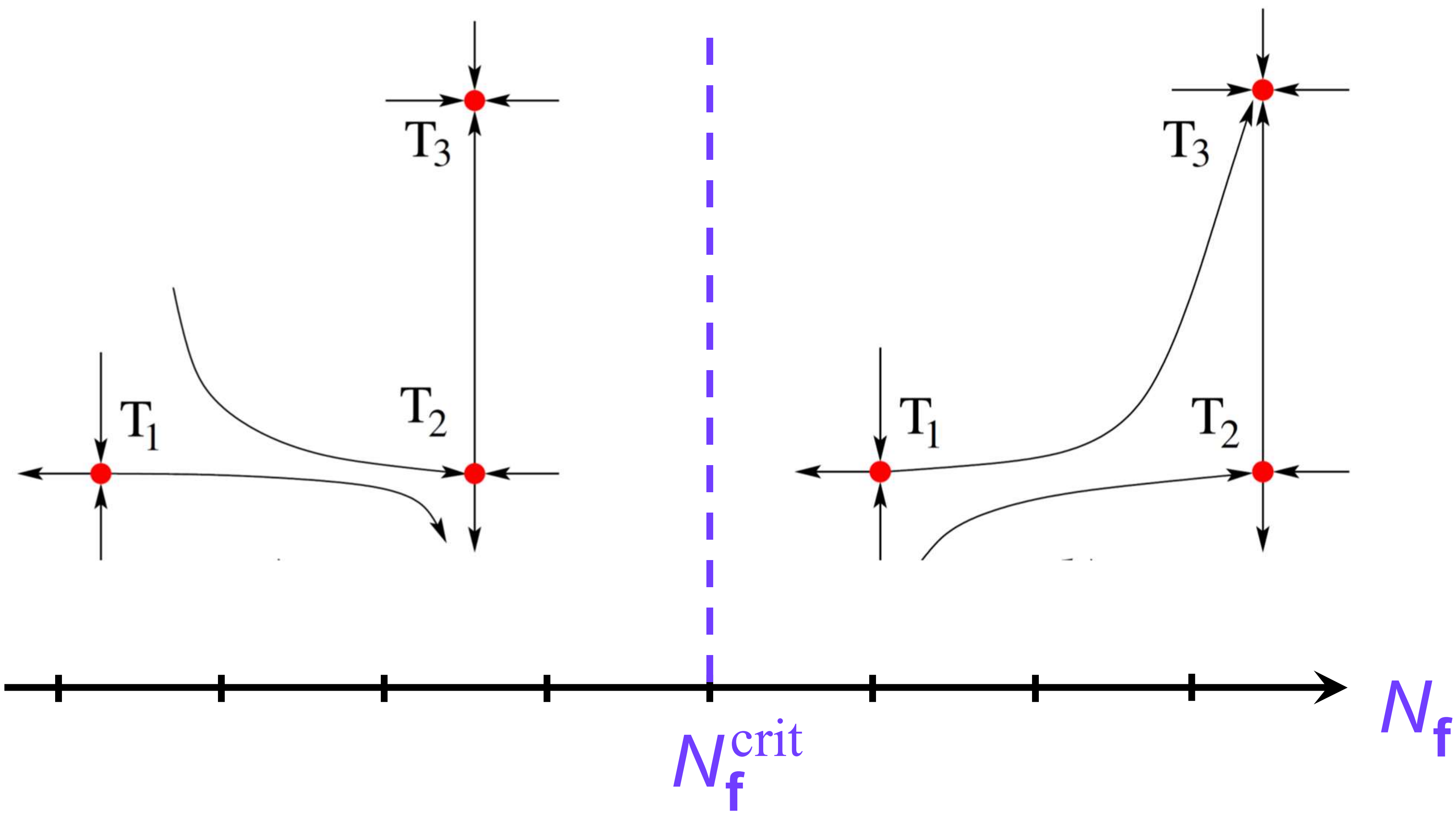
- Codimension-2
- Structurally unstable

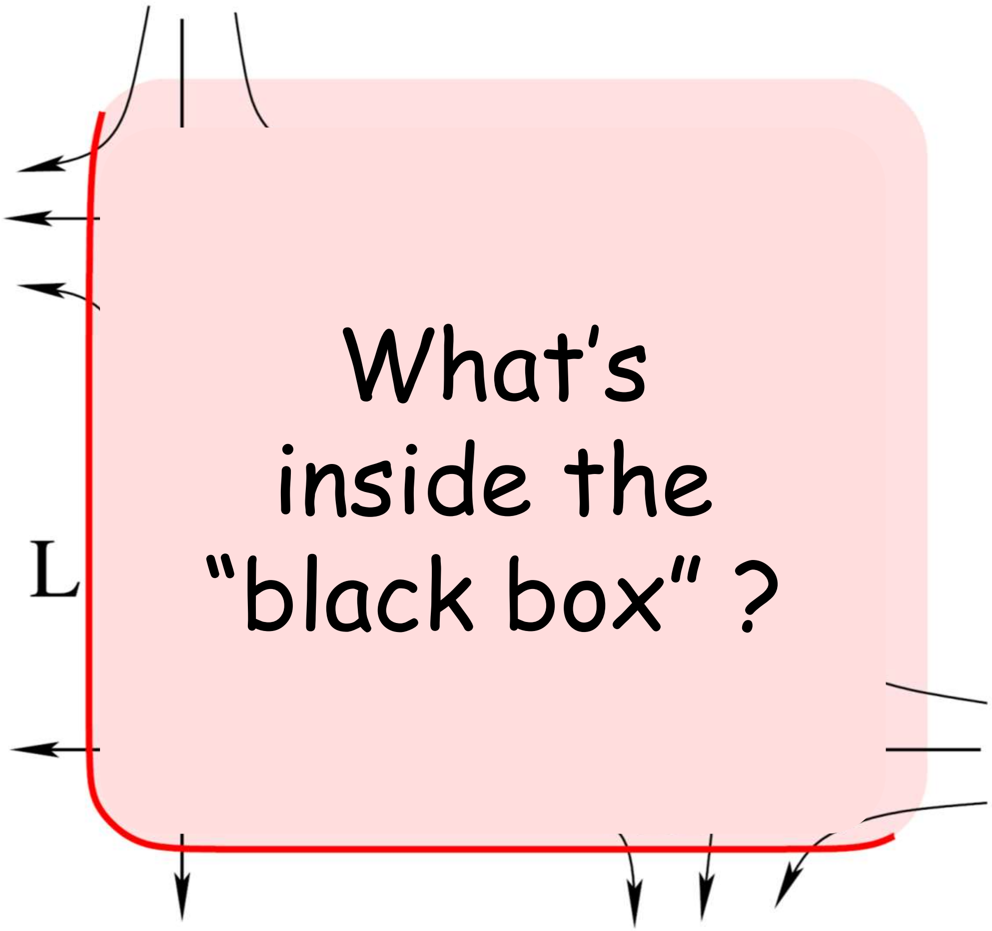
$$\Delta - d \sim \sqrt{N_f - N_f^{\text{crit}}}$$



T_3
T_1
T_2
N_f^{crit}
N_f

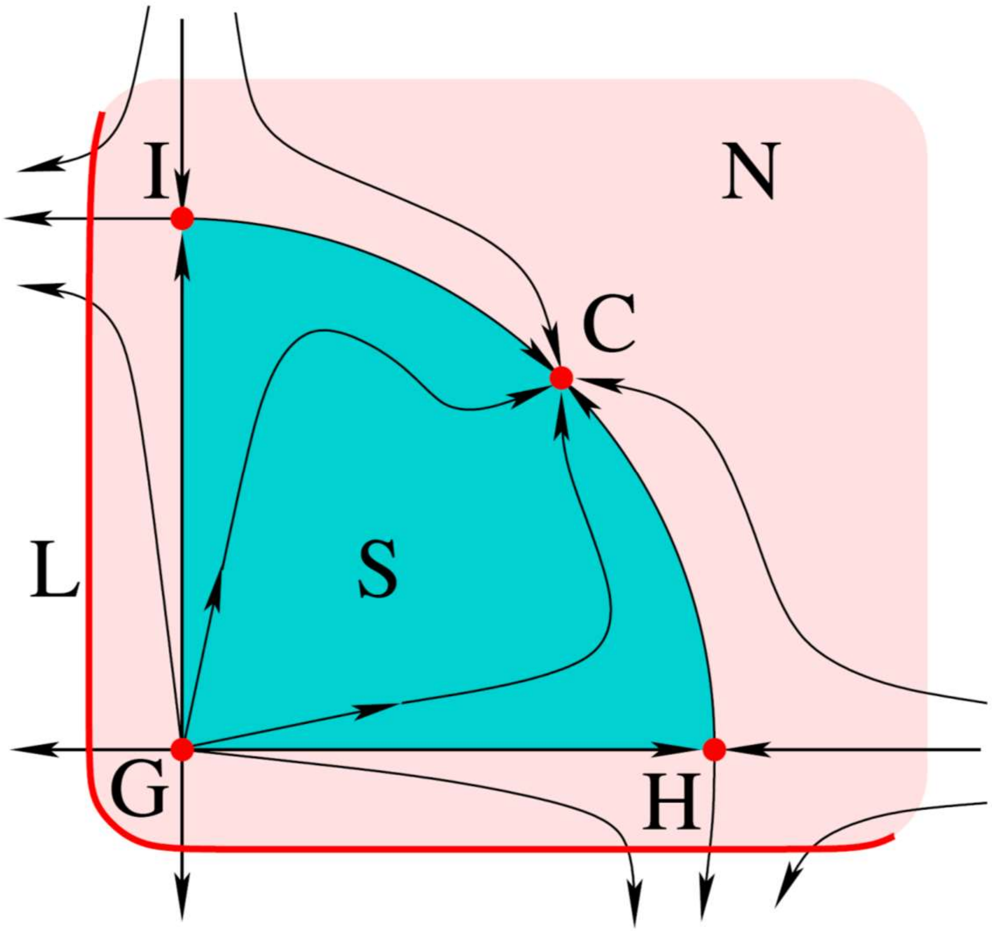
Heteroclinic Bifurcation





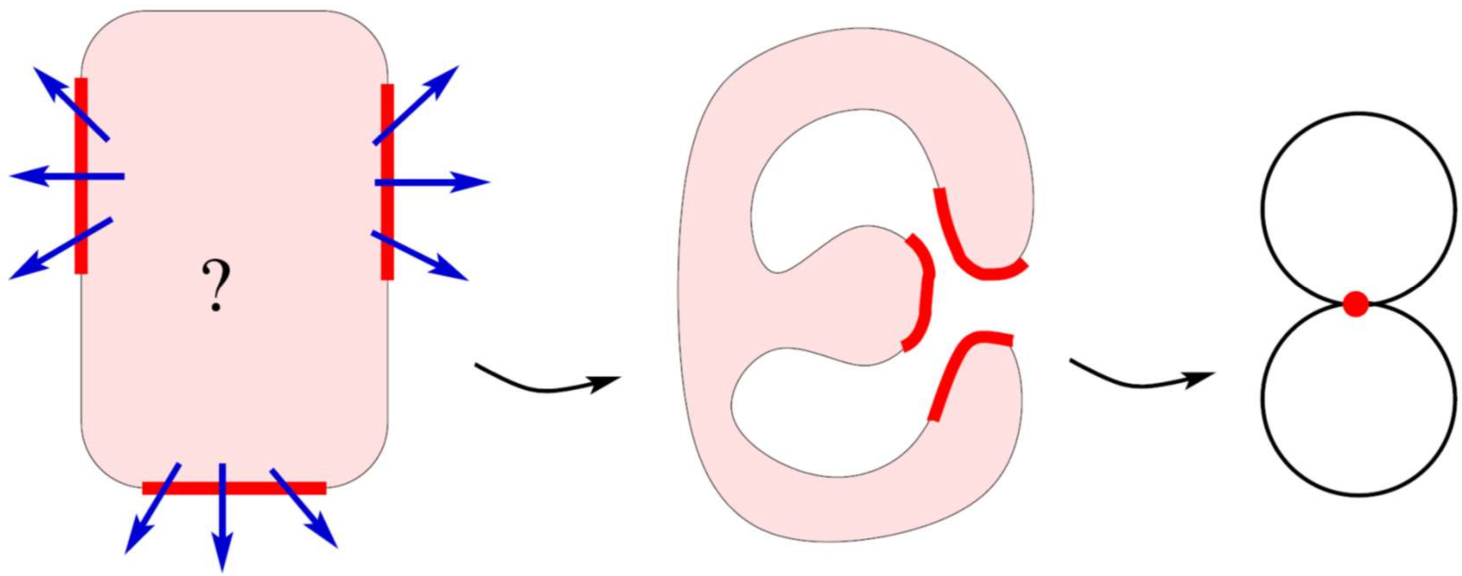
Charles C. Conley
1933-1984

$$\frac{\ker \Delta}{\operatorname{im} \Delta} \cong CH_*(S) \qquad\qquad \Delta \circ \Delta = 0$$



$$CH_*(T_C) = (\mathbb{Z}, 0, 0, 0, \ldots)$$
$$CH_*(T_H) = (0, \mathbb{Z}, 0, 0, \ldots)$$
$$CH_*(T_I) = (0, \mathbb{Z}, 0, 0, \ldots)$$
$$CH_*(T_G) = (0, 0, \mathbb{Z}, 0, \ldots)$$

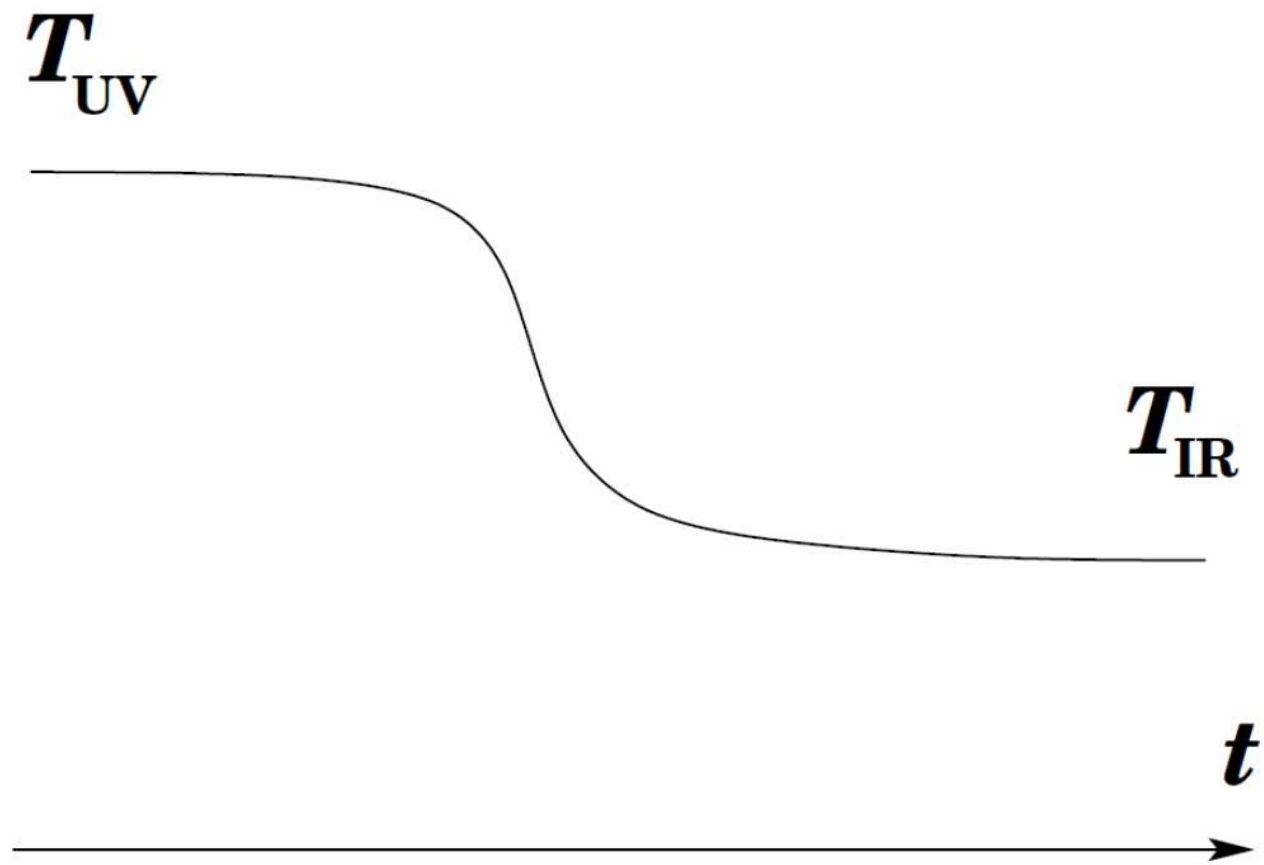
Charles C. Conley
1933-1984



exit set L	N/L	$CH_*(S)$
$\emptyset$	$D^2 \sqcup \{\mathrm{pt}\}$	$\mathbb{Z}[0]$
S^1	S^2	$\mathbb{Z}[2]$
I	D^2	0
$I \sqcup I$	S^1	$\mathbb{Z}[1]$
$I \sqcup I \sqcup I$	$S^1 \vee S^1$	$\mathbb{Z}[1] \oplus \mathbb{Z}[1]$

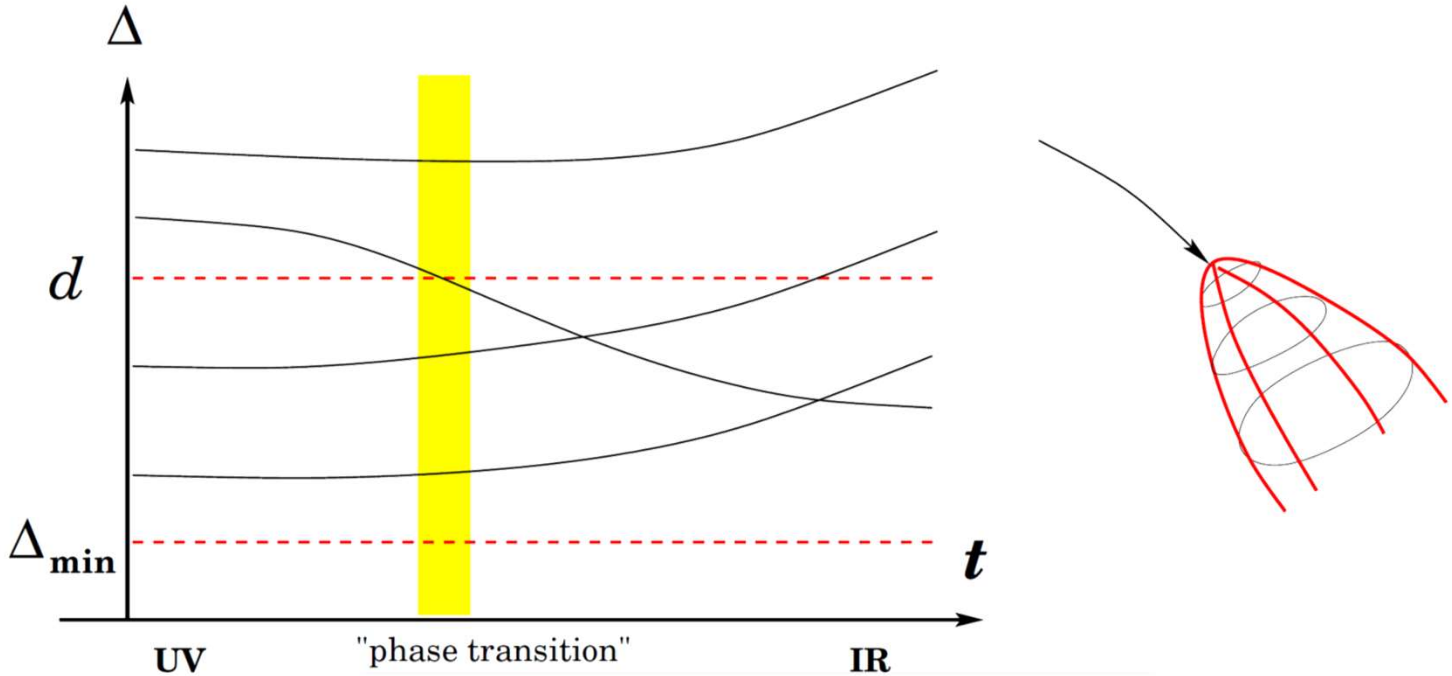
μ-theorem: $\mu(T_{\text{UV}}) > \mu(T_{\text{IR}})$

where $\mu(T) := \#\{\frac{d-2}{2} \leq \Delta_i < d\}$



μ-theorem: $\mu(T_{\mathrm{UV}}) \; > \; \mu(T_{\mathrm{IR}})$

where $\mu(T) := \#\{\frac{d-2}{2} \leq \Delta_i < d\}$



cf. C-theorem $\mathcal{C}(T_{\mathrm{UV}}) > \mathcal{C}(T_{\mathrm{IR}})$

$$\mu(T) := \#\{\tfrac{d-2}{2} \leq \Delta_i < d\}$$

3-state Potts model

μ=3, c=4/5

Tricritical Ising model

μ=2, c=7/10

Ising model

μ=1, c=1/2

$$d = 2 : \quad 0 \leq \Delta_i < 2$$

Spectrum of 2d $\mathcal{N} = 2$ superconformal field theory:

$$\{\Delta_i\} \qquad\qquad 0 \leq \Delta_i < 2$$

$$\Delta_i = R_i$$

"scaling dimensions"
a.k.a. "conformal dimensions"

"R-charges"

Central charge of $\mathcal{N} = 2$ Landau-Ginzburg model:

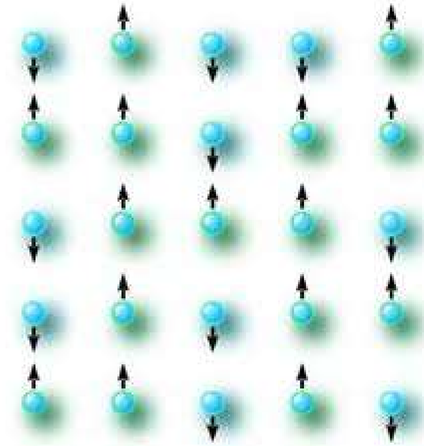
$$c = 3 \sum_i (1 - R_i)$$

$$W(\lambda^{R_i} \Phi_i) = \lambda^2 W(\Phi_i)$$

A_N $\mathcal{N}=2$ minimal model

$$c = 3 - \frac{6}{N+1}$$

$$\Delta_i = \frac{2i}{N+1} \qquad i = 1, \ldots, N$$



can be described as a LG model with the superpotential

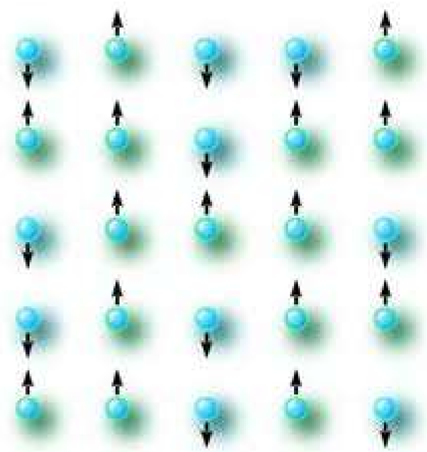
$$W = x^{N+1}$$

cf. $\mathrm{sp}(x^{N+1}) = \{\frac{1}{N+1}, \frac{2}{N+1}, \ldots, \frac{N}{N+1}\}$

chiral ring: $\mathbb{C}[x]/dW = H^*(\mathbb{C}\mathbf{P}^{N-1})$

is the classical cohomology ring of $\mathbb{C}\mathbf{P}^{N-1} = \dfrac{SU(N)}{U(N-1)}$

A_N $\mathcal{N}=2$ minimal model $=$ $\mathcal{N}=2$ Kazama-Suzuki coset model



$$\frac{SU(N)}{SU(N-1)\times U(1)}$$

at level 1

$\mathcal{N}=2$ Kazama-Suzuki model

$$\frac{SU(N)}{SU(N-k)\times SU(k)\times U(1)} \quad \text{at level 1}$$

$$c = \frac{3k(N-k)}{N+1}$$

$\mathcal{N} = 2$ Kazama-Suzuki model

$$\frac{SU(N)}{SU(N-k) \times SU(k) \times U(1)}$$

$$W = x_1^{N+1} + x_2^{N+1} + \ldots + x_k^{N+1}$$

expressed as a polynomial in the elementary symmetric functions

$$z_i = \sigma_i(x_1, \ldots, x_k)$$

e.g.

$$W(x_1, x_2) = x_1^2 + x_2^2 \quad \Rightarrow \quad W(z_1, z_2) = z_1^2 - 2z_2$$

$$z_1 = x_1 + x_2$$
$$z_2 = x_1 x_2$$

chiral ring (Jacobi ring / Milnor algebra):

$$\frac{\mathbb{C}[z_i]}{\{\partial_i W\}} = H^*(Gr(k,N))$$

is the classical cohomology ring of the Grassmannian

cf. Thom-Sebastiani sum:

1	2	3	2	1
$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$

$$sp(x^4+y^4) = \left\{\frac{2}{4},\frac{3}{4},\frac{3}{4},\frac{4}{4},\frac{4}{4},\frac{4}{4},\frac{5}{4},\frac{5}{4},\frac{6}{4}\right\}$$

and similarly for other Brieskorn-Pham singularities.

chiral ring (Jacobi ring / Milnor algebra):

$$\frac{\mathbb{C}[z_i]}{\{\partial_i W\}} = H^*(Gr(k,N))$$

is the classical cohomology ring of the Grassmannian

cf. Thom-Sebastiani sum:

1	2	3	2	1
$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$

$$sp(x^4 + y^4) = \left\{\frac{2}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{4}, \frac{4}{4}, \frac{4}{4}, \frac{5}{4}, \frac{5}{4}, \frac{6}{4}\right\}$$

Note: $sp(W) \subset (0,k)$ while $0 \le \Delta_i < 2$

Definition: An invariant I of a singularity is *semi-continuous* if for each adjacency $f \rightsquigarrow g_1 + g_2 + \ldots + g_N$ one has

$$I(f) \geq \sum_{i=1}^{N} I(g_i)$$

Definition: A subset $S \subset \mathbb{R}$ is called a semi-continuity set if

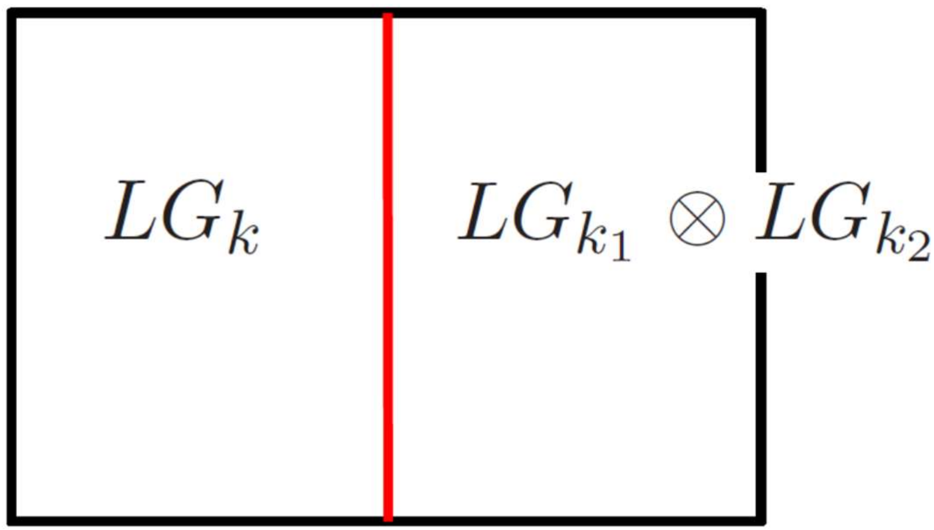
$\#S \cap sp(f)$ is semi-continuous.

Theorem:

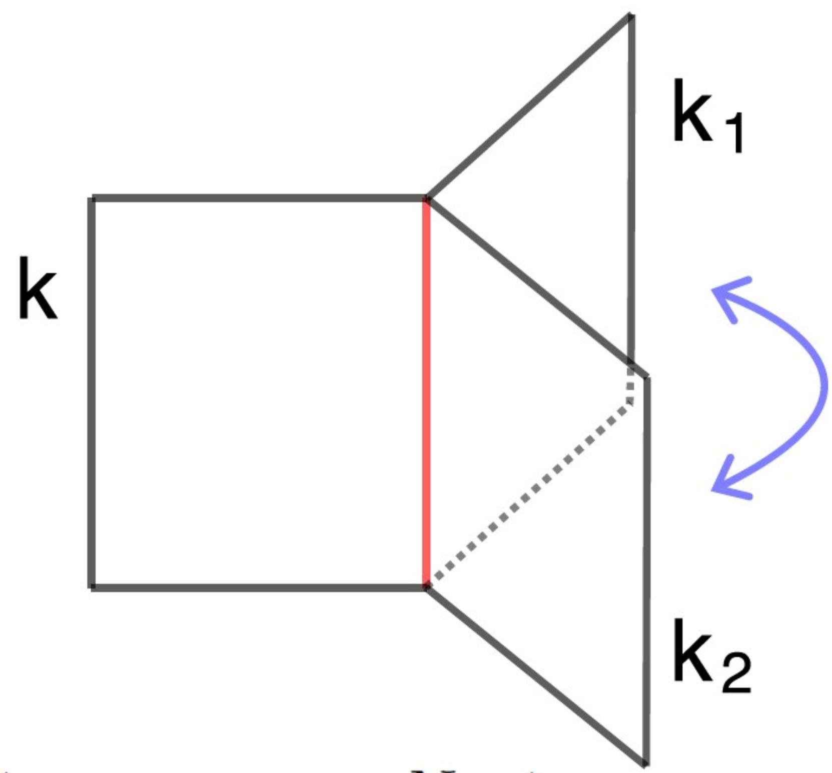
(A. Varchenko, 1983):if f is quasi-homogeneous then each open interval $(a, a+1)$ is a semi-continuity set.

(J. Steenbrink, 1985): for general f each interval $(a, a+1]$ is semi-continuity set.

Spectra of walls in KS models



$$\mathcal{I}_k^{k_1,k_2} =$$

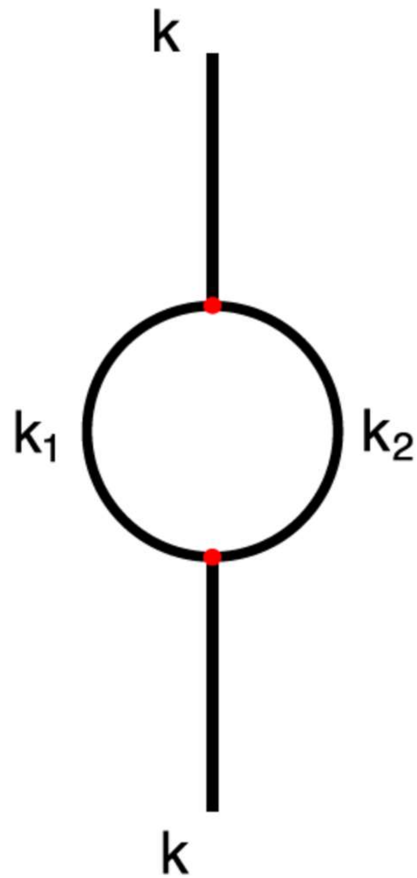


Matrix factorization

$$LG_k : \quad W = x_1^{N+1} + x_2^{N+1} + \ldots + x_k^{N+1}$$

Quantum groups and KS models

$$\mathcal{I}^{k}_{k_1,k_2} * \mathcal{I}^{k_1,k_2}_{k} =$$

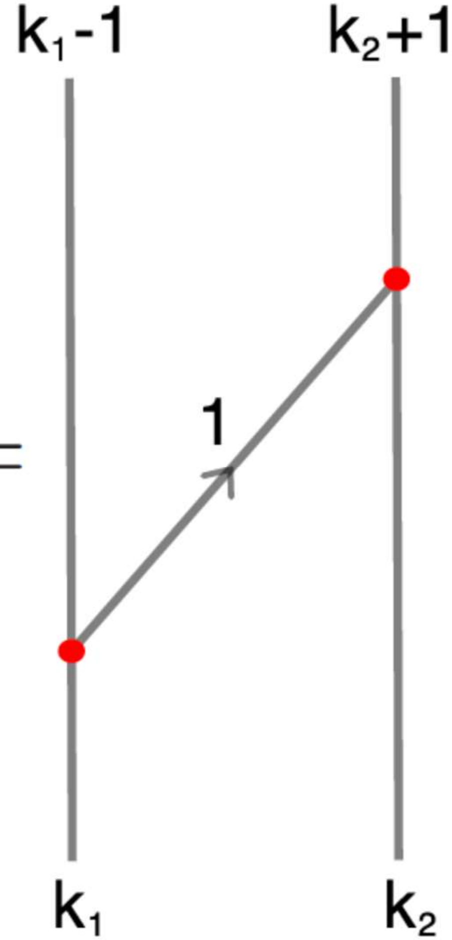


$$=$$

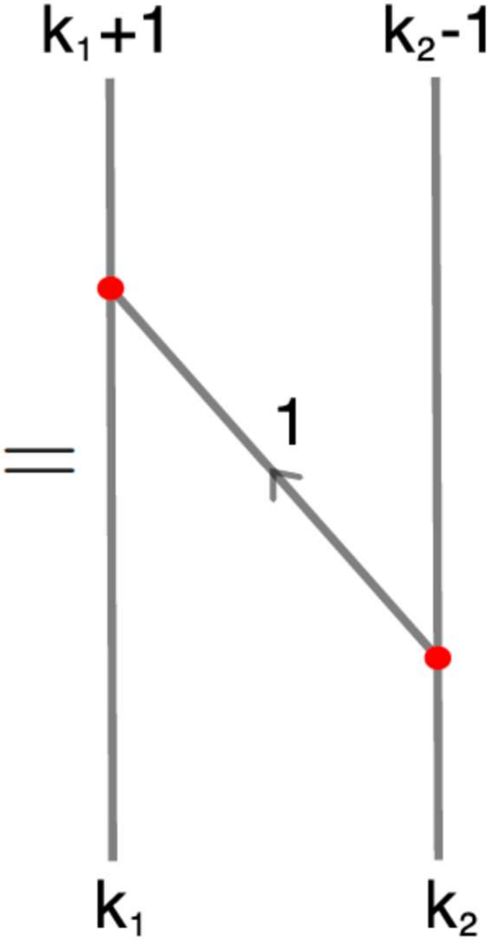
$$\left\{\left[\begin{matrix} k \\ k_1 \end{matrix}\right]\right\}$$

arXiv: 1507.06318

$$\mathcal{E}_{k_1,k_2} :=$$



$$\mathcal{F}_{k_1,k_2} :=$$



$$\mathcal{E}_{k_1,k_2} := \left(\mathrm{Id}_{k_1-1} \otimes \mathcal{I}^{k_2+1}_{1,k_2}\right) * \left(\mathcal{I}^{k_1-1,1}_{k_1} \otimes \mathrm{Id}_{k_2}\right)$$

$$\mathcal{F}_{k_1,k_2} := \left(\mathcal{I}^{k_1+1}_{k_1,1} \otimes \mathrm{Id}_{k_2-1}\right) * \left(\mathrm{Id}_{k_1} \otimes \mathcal{I}^{1,k_2-1}_{k_2}\right)$$

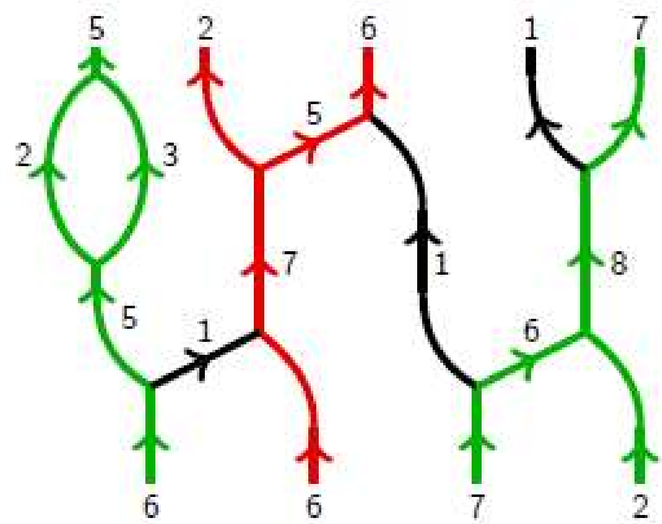
arXiv: 1507.06318

Categorification of Quantum Groups

$$\mathcal{E}_{k_1+1,k_2-1} * \mathcal{F}_{k_1,k_2} \cong \mathcal{F}_{k_1-1,k_2+1} * \mathcal{E}_{k_1,k_2} \oplus \mathrm{Id}_{k_1,k_2}\{[k_2-k_1]\}$$

and, similarly, for $k_1 \geq k_2$:

$$\mathcal{F}_{k_1-1,k_2+1} * \mathcal{E}_{k_1,k_2} \cong \mathcal{E}_{k_1+1,k_2-1} * \mathcal{F}_{k_1,k_2} \oplus \mathrm{Id}_{k_1,k_2}\{[k_1-k_2]\}$$



arXiv: 1507.06318

$$f$$

singularity, its orbifold, ...

LG model

$$W = f$$

Algebra

sigma-model

$$f = 0 \qquad \{\Delta_i\}$$

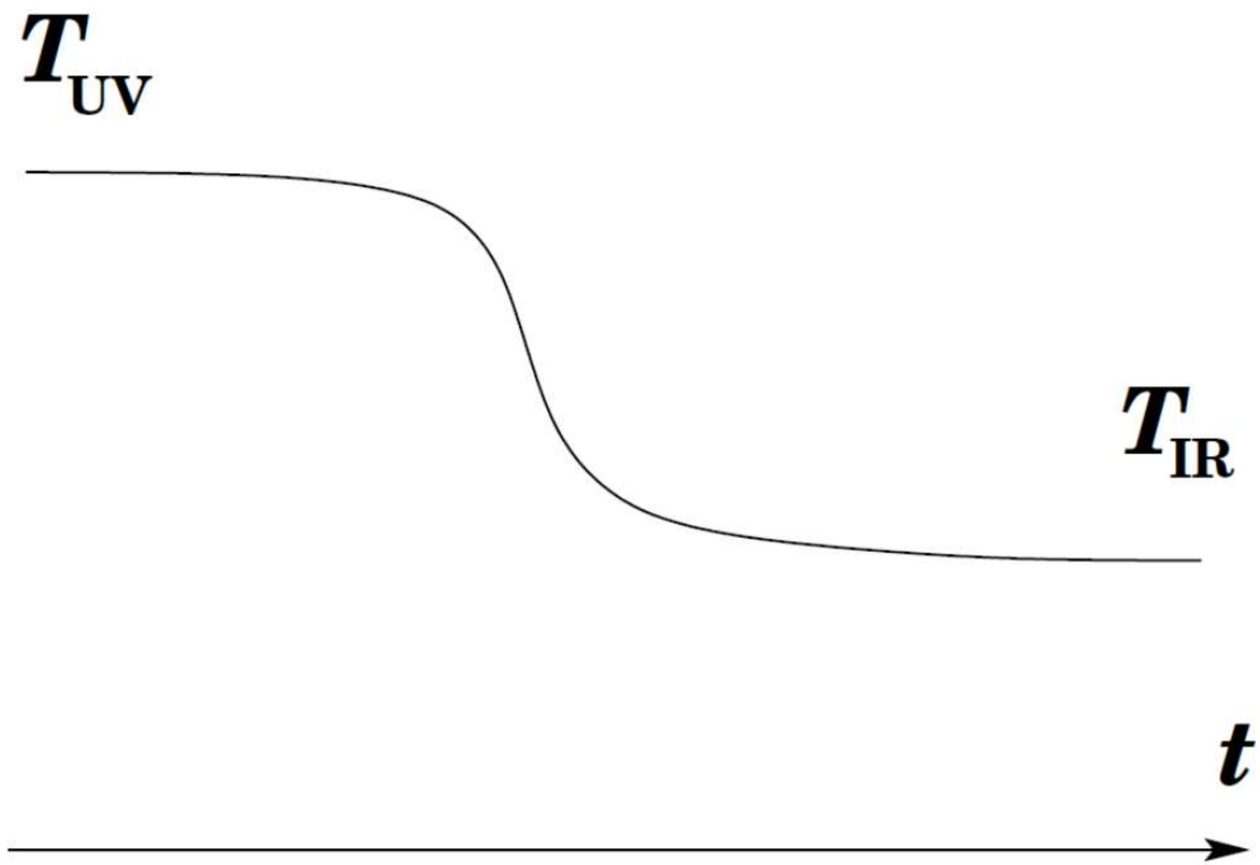
Geometry

Also a product of $\mathcal{N} = 2$ minimal models and Kazama-Suzuki models!

$$\frac{SL(2)}{U(1)} \bigotimes \mathrm{LG}(W = f)$$

$$\mathcal{H}^{BPS}_{\mathrm{UV}} \xrightarrow[\text{spectral sequence}]{?} \mathcal{H}^{BPS}_{\mathrm{IR}}$$

Categorification of semicontinuity?



Thanks for listening.

Questions?

Coset theory	c	Degrees of generators
$\dfrac{\mathrm{SU}(n+m)}{\mathrm{SU}(n)\otimes \mathrm{SU}(m)\otimes \mathrm{U}(1)}$	$\dfrac{3nm}{m+n+1}$	$1,2\ldots,\min(n,m)$
$\dfrac{\mathrm{SO}(n+2)}{\mathrm{SO}(n)\otimes \mathrm{U}(1)}$, ($n$ even)	$\dfrac{3n}{n+1}$	$1,\ n/2$
$\dfrac{\mathrm{SO}(2n)}{\mathrm{SU}(n)\otimes \mathrm{U}(1)}$	$\dfrac{3n(n-1)}{2(2n-2+1)}$	$4i-2,\ \begin{cases} i=1,\ldots,n/2\ (n\text{ even}) \\ i=1,\ldots,(n-1)/2\ (n\text{ odd}) \end{cases}$
$\dfrac{E_6}{\mathrm{SO}(10)\otimes U(1)}$	$\dfrac{48}{13}$	$1,4$
$\dfrac{\mathrm{E}_7}{\mathrm{E}_6\otimes \mathrm{U}(1)}$	$\dfrac{81}{19}$	$1,5,9$

e.g. $\mathrm{SO}(n+2)/\mathrm{SO}(n)\times \mathrm{SO}(2)$:

W.Lerche, C.Vafa, N.P.Warner

$$W = x_1^{n+1} + x_1 x_2^2$$

D_{n+2} minimal model