Gromov-Hausdorff convergence of filtered A infinity categories

Kenji Fukaya, SCGP (talk at Miami)

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What should be a homological Mirror symmetry over Λ_0 ?

$$\Lambda_0 = \left\{ \sum a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in (0,\infty), \lim \lambda_i = +\infty \right\}$$

Λ its field of fractions

$$\Lambda_0^{\mathbb{Q}} = \left\{ \sum a_i T^{\lambda_i} \in \Lambda_0 \mid \lambda_i \in \mathbb{Q} \right\}$$

Let us start with a few things in B-side

$$\mathfrak{X} \longrightarrow \mathrm{spec}\mathbb{C}[[T]]$$

a formal deformation

special fiber may be singular   general fiber is assumed to be smooth

$\mathfrak{X}_n \longrightarrow \mathrm{spec}\mathbb{C}[[T^{1/n}]]$ n-fold branched cover

Let $\mathcal{E}_i$ be a chain complex of coherent sheaves over $\mathfrak{X}_n$ for *i* = 1,2.

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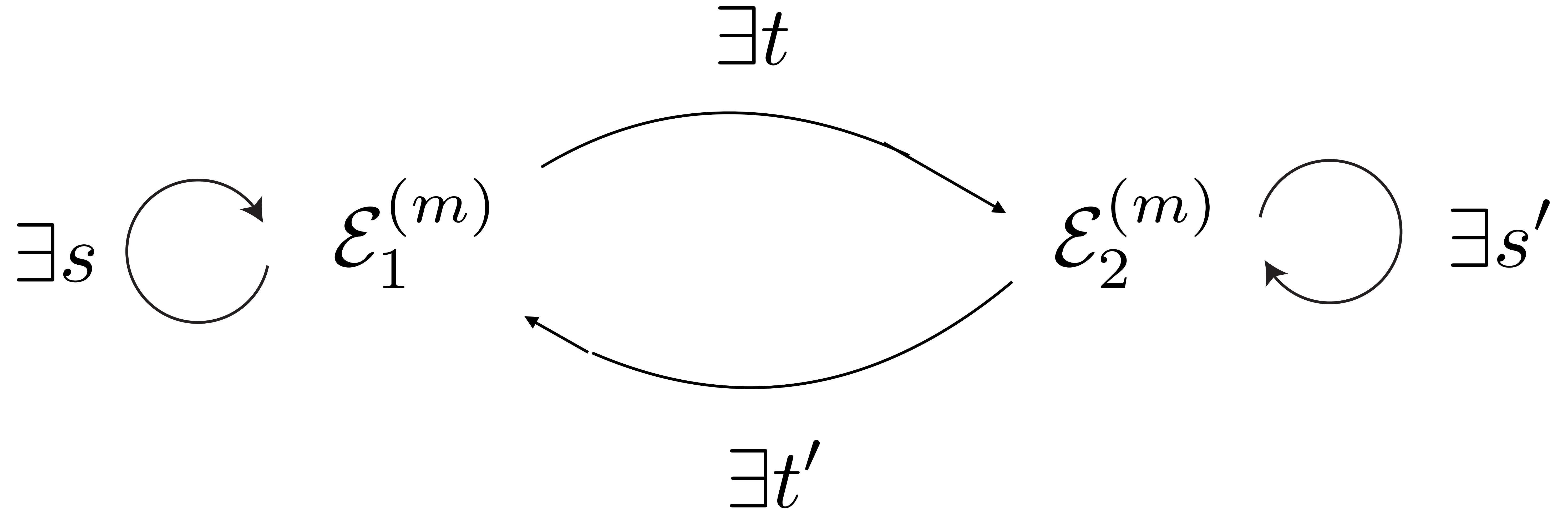
We say $d(\mathcal{E}_1, \mathcal{E}_2) < \epsilon$ if there exists m such that



$$\begin{array}{ccc} \mathcal{E}_i^{(m)} & \longrightarrow & \mathcal{E}_i \\ \downarrow & & \downarrow \\ \mathfrak{X}_{nm} & \longrightarrow & \mathfrak{X}_n \end{array}$$

$$dt = dt' = 0$$

$$t \circ t' + ds = T^{\epsilon}\mathrm{id}$$

$$t' \circ t + ds' = T^{\epsilon}\mathrm{id}$$

$\lim_{n \to \infty} \mathfrak{OB}(\mathbb{D}(\mathfrak{X}_n))$ is a metric space with respect to this metric.

Its completion seems to be related to a version of Berkovich spectra.

I want to study its `Mirror’

(X, ω) Symplectic manifold

$\mathcal{H} : X \times [0,1] \to \mathbb{R}$ $\qquad \mathcal{H}_t(x) = \mathcal{H}(x,t)$

$\mathfrak{X}_{\mathcal{H}_t}$ Hamiltonian vector field of $\mathcal{H}_t$ $\qquad \omega(V, \mathfrak{X}_{\mathcal{H}_t}) = d\mathcal{H}_t(V)$

$\varphi^t_{\mathcal{H}} : X \to X$ is defined by $\qquad \varphi^0_{\mathcal{H}}(x) = x$

$$\frac{d}{dt}\varphi^t_{\mathcal{H}} = \mathcal{X}_{\mathcal{H}_t} \circ \frac{d}{dt}$$

$Ham(X) := \{\varphi^1_{\mathcal{H}} \mid \mathcal{H} : X \times [0,1] \to \mathbb{R}\}$ group of Hamiltonian diffeo.

Hofer metric on $Ham(X)$

$$\mathcal{H} : X \times [0,1] \to \mathbb{R} \qquad \|\mathcal{H}\| := \int_0^1 (\sup \mathcal{H}_t - \inf \mathcal{H}_t) dt$$

$$\varphi \in Ham(X) \qquad \|\varphi\| = \inf\{\|H\| \mid \varphi^1_{\mathcal{H}} = \varphi\}$$

Definition $d_{\mathrm{Hof}}(\varphi, \psi) = \|\psi^{-1}\varphi\|$

This defines a metric on $Ham(X)$ (Hofer)

$d_{\mathrm{Hof}}(\varphi, \psi) = 0$ ⇒ $\varphi = \psi$ is highly nontrivial.

Chekanov metric on $\mathcal{LAG}(X)$

$\mathcal{LAG}(X)$ the space of all Lagrangian submanifolds $L \subset X$

$L, L' \in \mathcal{LAG}(X)$

$$d_{HC}(L, L') := \inf\{\|\varphi\| \mid \varphi \in Ham(X), \varphi(L) = L'\}$$

$$d_{HC}(L, L') \in [0, \infty]$$

d_{HC} is a metric on $\mathcal{LAG}(X)$ (Chekanov)

$\overline{\mathcal{LAG}(X)}$ completion of $\mathcal{LAG}(X)$ with respect to d_{HC}

Homological Mirror symmetry conjecture / Λ_0

If a mirror of (X,ω) is $\mathfrak{X} \longrightarrow \mathrm{spec}\mathbb{C}[[T]]$ then:

(1) $\overline{\mathcal{LAG}(X)} \subseteq$ completion of $\lim_{n\to\infty} \mathfrak{DB}(\mathbb{D}(\mathfrak{X}_n))$

(more precisely we need to include bounding cochain)

(2) (1) is an object part of a homotopy equivalence of filtered A infinity category to a full subcategory.

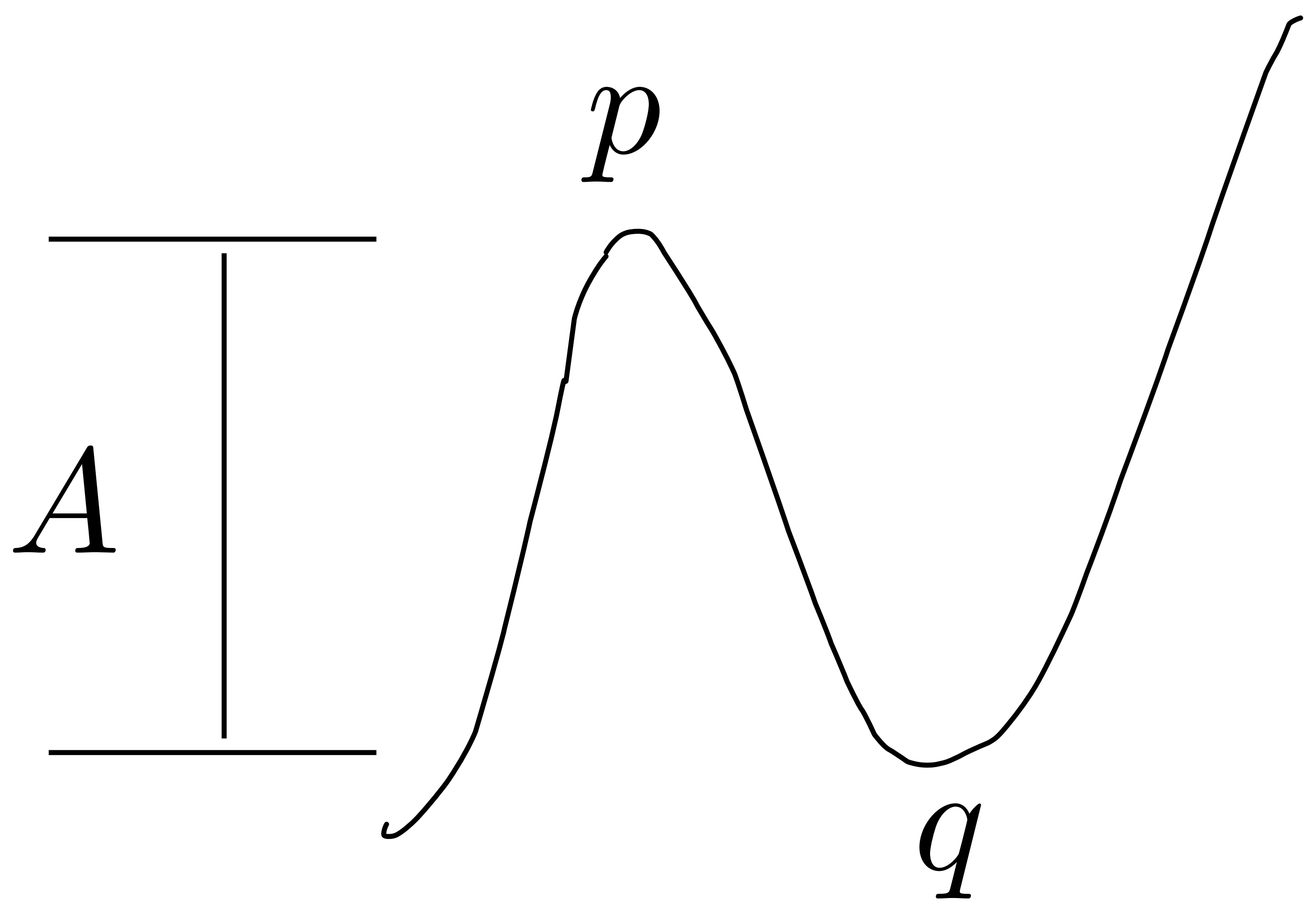
Theorem

$\mathbb{L} \subset \overline{\mathcal{LAG}(X)}$ a separable subset.

There is a filtered A infinity category $\mathfrak{F}(\mathbb{L})$ whose object set is

$$\{(L,b) \mid L \in \mathbb{L}, b \text{ its bounding cochain}\}$$

C^0 Robustness of Morse homology



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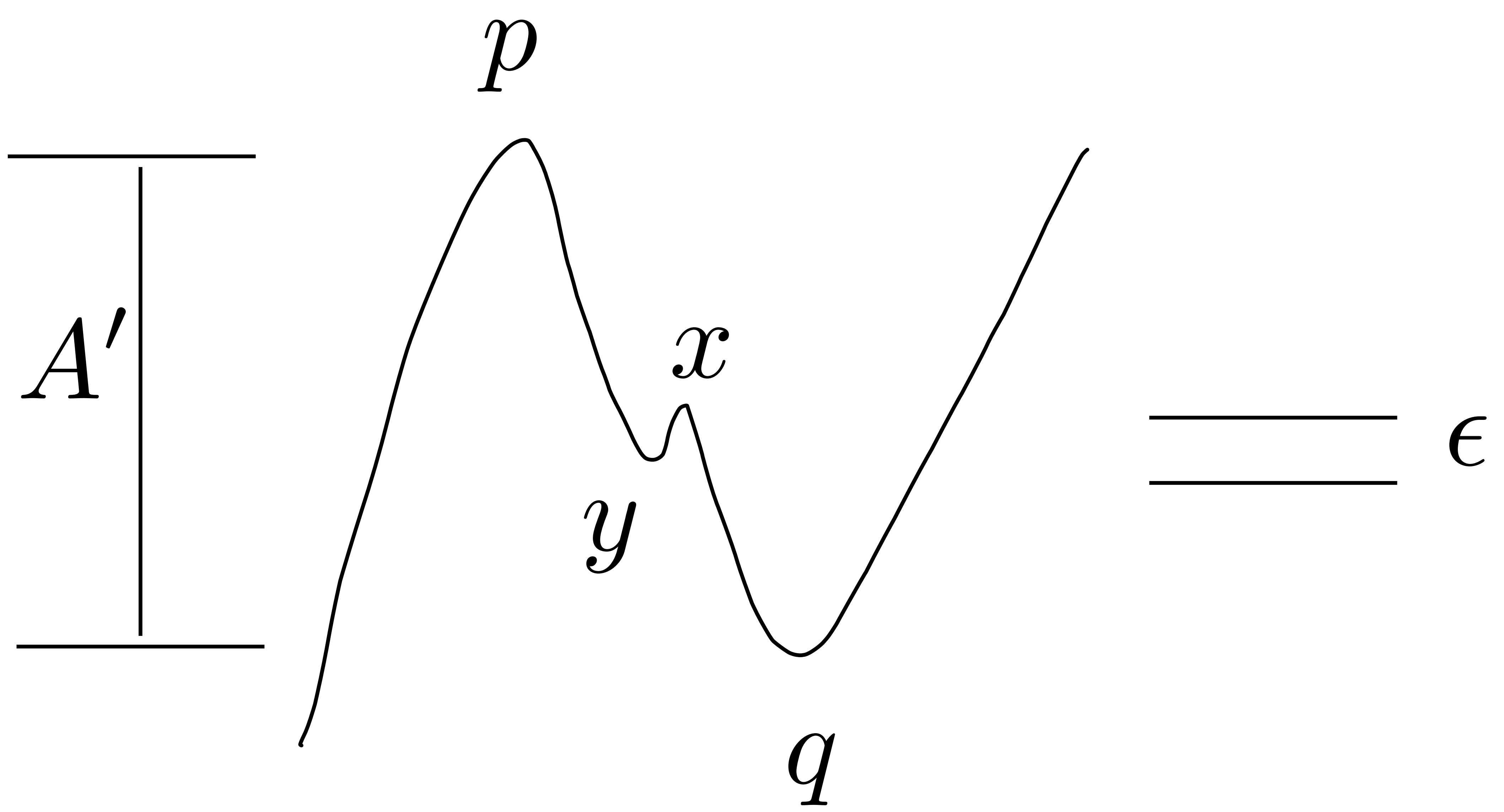


f_ϵ

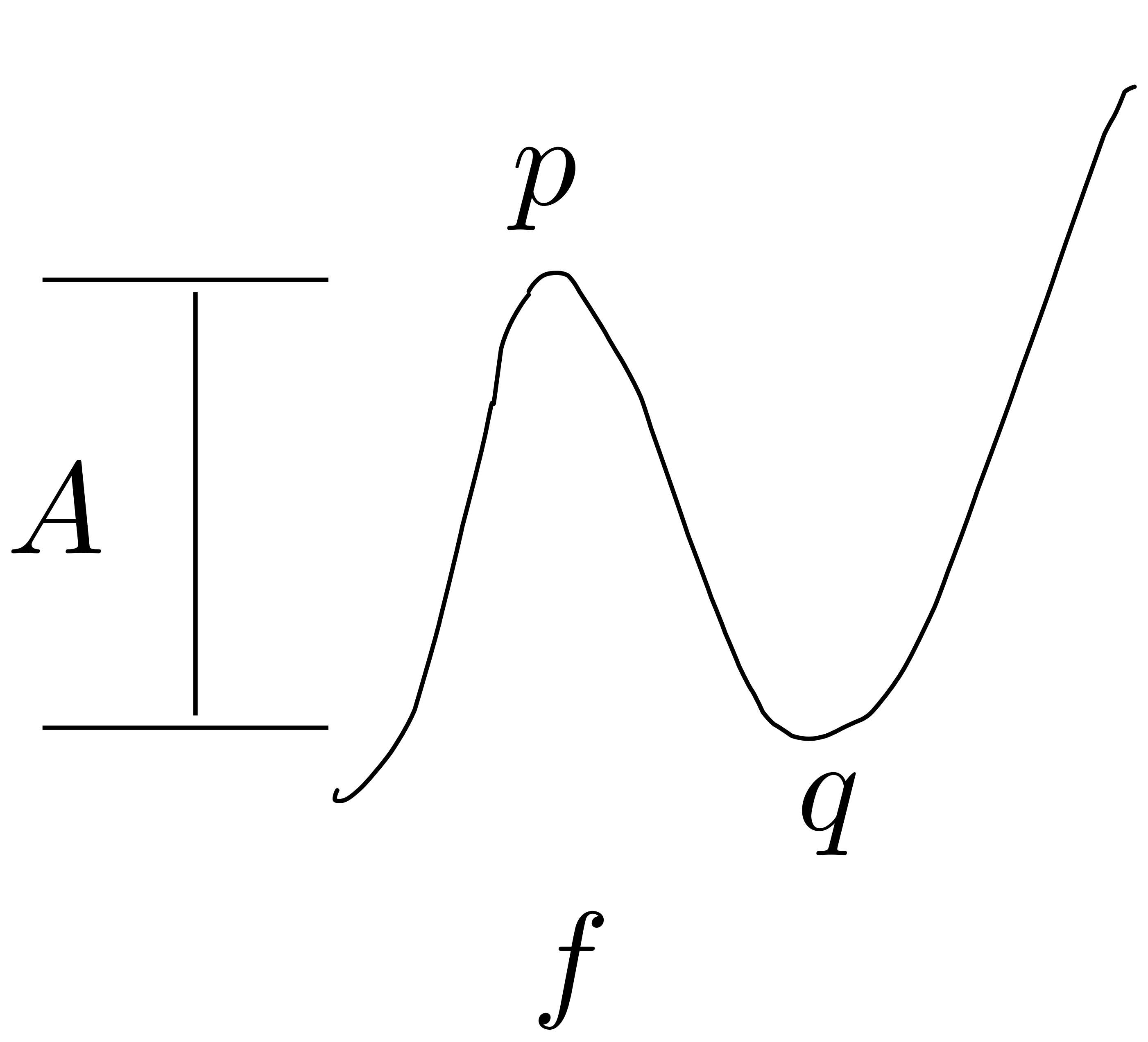
$$|f - f_\epsilon|_{C^0} \leq \epsilon$$

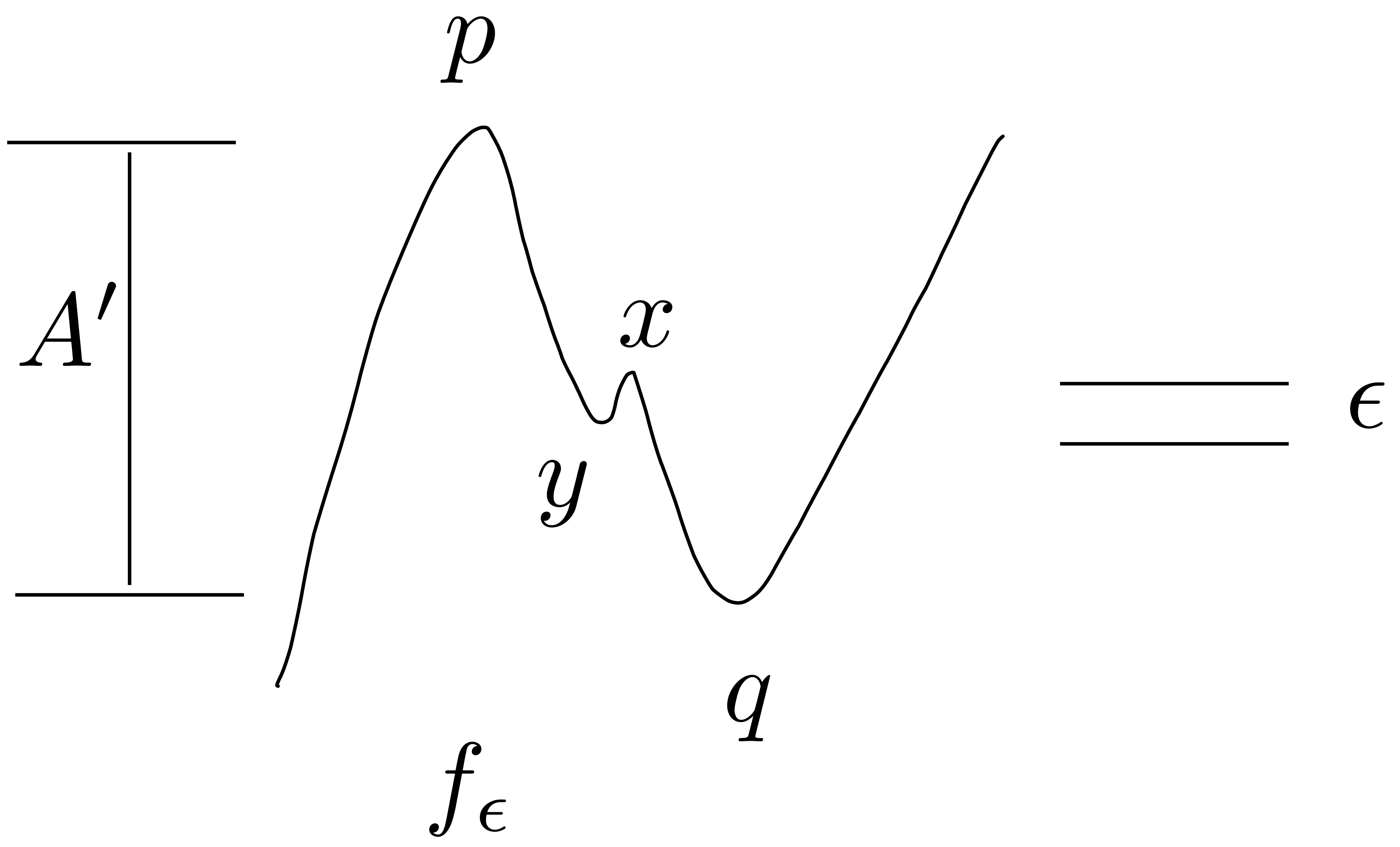
$$|f - f_\epsilon|_{C^0} \leq \epsilon$$ ⇒ Morse homology of f is close to the Morse homology of f_ϵ

$$CF(f;\Lambda_0) = \bigoplus_{p \in Crit(f)} \Lambda_0[p]$$

$$\partial[p] = \sum_q T^{f(p)-f(q)} \# \mathcal{M}(p,q)[q]$$

$\mathcal{M}(p,q)$ gradient lines joining p to q.





$$H(CF(f;\Lambda_0),\partial) = \frac{\Lambda_0}{T^{A}\Lambda_0} \qquad H(CF(f_\epsilon;\Lambda_0),\partial) = \frac{\Lambda_0}{T^{A'}\Lambda_0} \bigoplus \frac{\Lambda_0}{T^{\epsilon}\Lambda_0}$$

Morse homology of f is close to the Morse homology of f_ϵ

A similar story for Lagrangian Floer theory.

$L, L' \in \mathcal{LAG}(X)$ transversal

$$HF((L,b),(L',b');\Lambda_0) = \bigoplus_i \frac{\Lambda_0}{T^{a_i}\Lambda_0}$$

$$a_i \in [0,\infty] \qquad a_1 \geq a_2 \geq a_3 \ldots \qquad \exists k\ a_k = 0$$

We write the right hand side $\Lambda_0(\vec{a})$

A similar story for Lagrangian Floer theory.

Theorem (FOOO 2009)

$$HF((L,b),(L',b');\Lambda_0) = \Lambda_0(\vec{a})$$

$$HF((\varphi(L),\varphi_* b),(L',b');\Lambda_0) = \Lambda_0(\vec{b})$$

$\Rightarrow$ $|a_i - b_i| \le \|\varphi\|$

Lagrangian Floer homology depends continuously on Hofer-Chekanov metric.

A similar results by Albers, Usher, Polterovich etc. and Biran-Cornea

Lagrangian Floer homology depends continuously on Hofer-Chekanov metric.

Filtered A infinity category depends continuously on Hofer-Chekanov metric.

$\mathfrak{C}$ filtered A infinity category

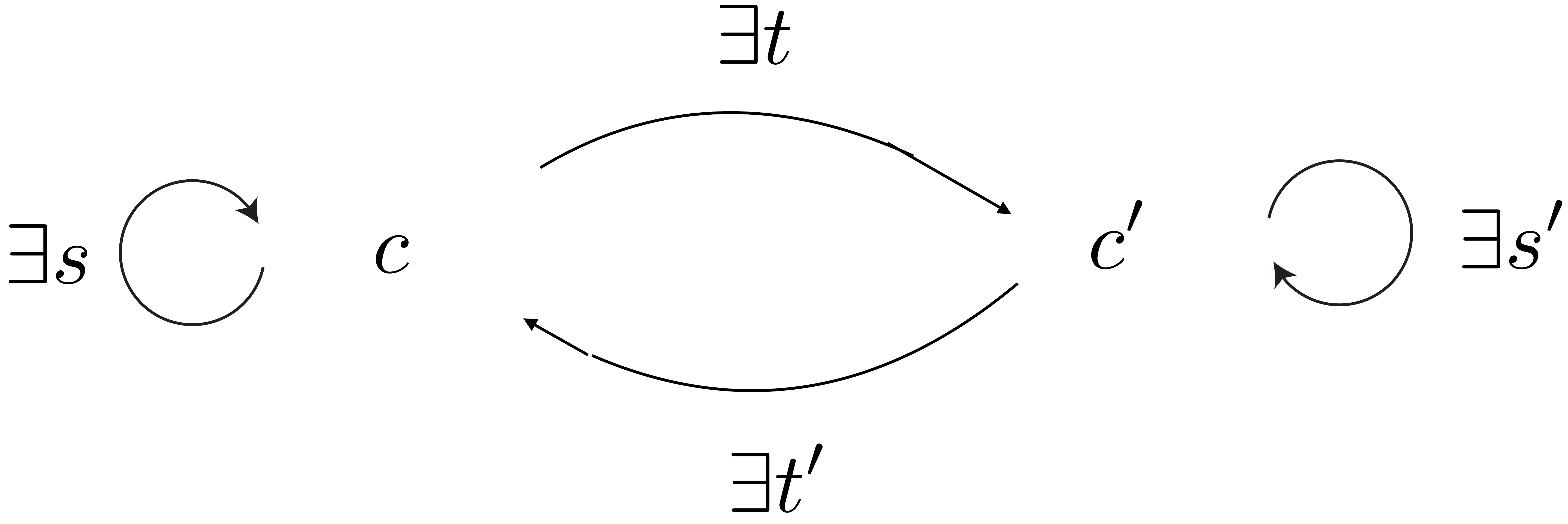
$\mathfrak{OB}(\mathfrak{C})$ Set of objects

$c, c' \in \mathfrak{OB}(\mathfrak{C})$ ➡ $\mathfrak{C}(c, c')$ morphism spaces, free Λ_0 module

$\mathfrak{m}_k, k = 1, 2, 3, \ldots$ A infinity operations

$\mathfrak{OB}(\mathfrak{C})$ has a pseudo metric *d*

$$d(c, c') < \epsilon$$



$$dt = dt' = 0$$

$$t \circ t' + ds = T^{\epsilon}\mathrm{id}$$

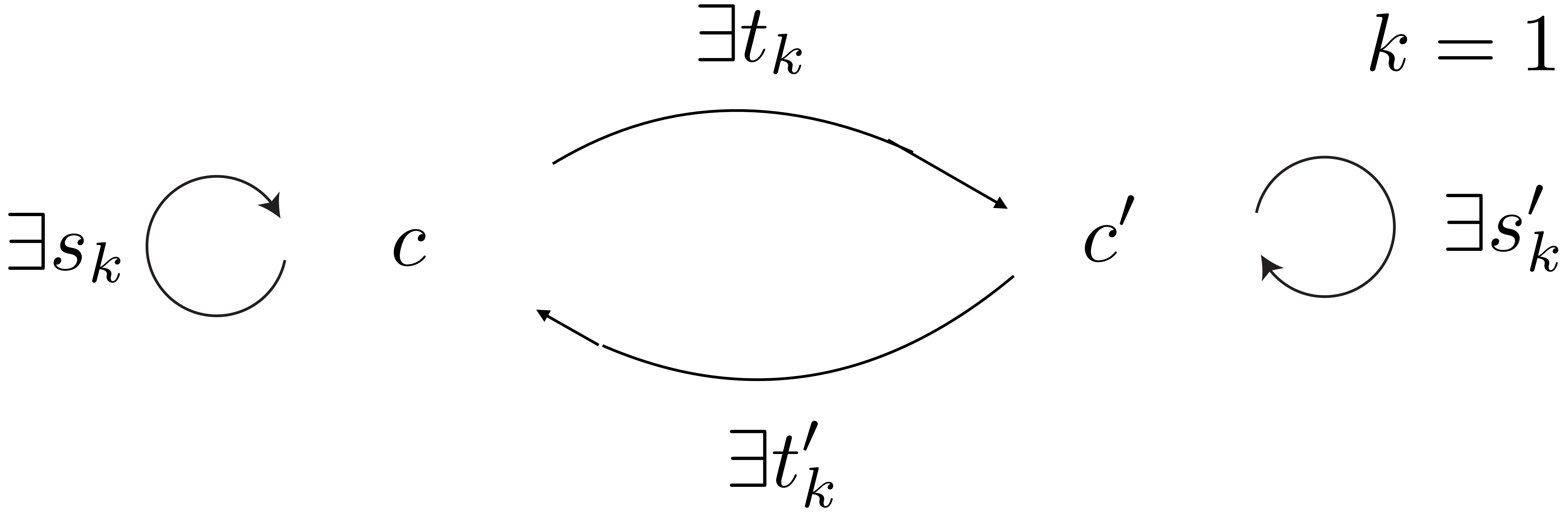
$$t' \circ t + ds' = T^{\epsilon}\mathrm{id}$$

$$d = \pm\mathfrak{m}_1 \qquad \circ = \pm\mathfrak{m}_2$$

$\mathfrak{O}\mathfrak{B}(\mathfrak{C})$ need to strengthen to d_∞

$d_\infty(c, c') < \epsilon$

$k = 1, 2, 3, \ldots$



$$dt_1 = dt'_1 = 0 \qquad t_1 \circ t'_1 + ds_1 = T^\epsilon \qquad t'_1 \circ t_1 + ds'_1 = T^\epsilon$$

$$s_1 \circ t_1 + t_1 \circ s'_1 + ds_2 = 0$$

$$s'_1 \circ t'_1 + t'_1 \circ s_1 + ds'_2 = 0$$

a similar relation for higher *s*'s and *t*'s

Theorem d_∞ satisfies the triangle inequality.

$\mathbb{L}$ a finite set of Lagrangian submanifolds.
Two elements are transversal.

$\mathfrak{F}(\mathbb{L})$ a filtered A infinity category whose object is (L, b)

$L \in \mathbb{L}$ b a bounding cochain of *L.*

Theorem $\varphi \in Ham(X)$ $L, \varphi(L) \in \mathbb{L}$

⇒ $$d_\infty((L,b),(\varphi(L),\varphi_* b)) \le \|\varphi\|$$

$\mathfrak{C}_1 \quad \mathfrak{C}_2$ filtered A infinity category

Definition

$$d_{GH}(\mathfrak{C}_1, \mathfrak{C}_2) < \epsilon$$

iff there exists a filtered A infinity category $\mathfrak{C}$ and $\mathfrak{F}_1, \mathfrak{F}_2$

$$\mathfrak{C}_1 \xrightarrow{\mathfrak{F}_1} \mathfrak{C} \xleftarrow{\mathfrak{F}_2} \mathfrak{C}_2$$

(1) $\mathfrak{F}_1, \mathfrak{F}_2$ are homotopy equivalences to a full subcategory.

(2) $d_H(\mathfrak{OB}(\mathfrak{C}_1), \mathfrak{OB}(\mathfrak{C}_2)) < \epsilon$

$$d_H(\mathfrak{OB}(\mathfrak{C}_1), \mathfrak{OB}(\mathfrak{C}_2)) < \epsilon$$

Hausdorff distance as subspaces of $(\mathfrak{OB}(\mathfrak{C}), d_\infty)$ is $< \epsilon$

namely:

$$\forall p \in \mathfrak{OB}(\mathfrak{C}_1) \quad \exists q \in \mathfrak{OB}(\mathfrak{C}_2) \qquad d_\infty(p, q) < \epsilon$$

$$\forall q \in \mathfrak{OB}(\mathfrak{C}_2) \quad \exists p \in \mathfrak{OB}(\mathfrak{C}_1) \qquad d_\infty(p, q) < \epsilon$$

Theorem $d_{GH}(\mathfrak{C}_1, \mathfrak{C}_2) + d_{GH}(\mathfrak{C}_2, \mathfrak{C}_3) \geq d_{GH}(\mathfrak{C}_1, \mathfrak{C}_3)$

Theorem

$$\sum_{i=1,2,\ldots} d_{GH}(\mathfrak{C}_i, \mathfrak{C}_{i+1}) < \infty \quad \Longrightarrow \quad \mathfrak{C}_i \text{ has a limit.}$$

The limit is unique in the following sense.

$\mathfrak{C}$ $\mathfrak{C}'$ are limits then $d_{GH}(\mathfrak{C}, \mathfrak{C}') = 0$

$d_{GH}(\mathfrak{C}, \mathfrak{C}') = 0$ implies the following

(1) $(\mathfrak{OB}(\mathfrak{C}), d_\infty)$ is isometric to $(\mathfrak{OB}(\mathfrak{C}'), d_\infty)$

(2) If $c_i \in \mathfrak{OB}(\mathfrak{C})$ corresponds to $c'_i \in \mathfrak{OB}(\mathfrak{C}')$ then

$H(\mathfrak{C}(c_1, c_2), \mathfrak{m}_1)$ is almost isomorphic to $H(\mathfrak{C}'(c'_1, c'_2), \mathfrak{m}_1)$

(3) (2) preserves multiplicative structures $\mathfrak{m}_k$

$H(\mathfrak{C}(c_1, c_2), \mathfrak{m}_1)$ is almost isomorphic to $H(\mathfrak{C}'(c_1', c_2'), \mathfrak{m}_1)$

For any $\epsilon > 0$

$$\exists \varphi_\epsilon \qquad H(\mathfrak{C}(c_1, c_2), \mathfrak{m}_1) \longrightarrow H(\mathfrak{C}'(c_1', c_2'), \mathfrak{m}_1)$$

st. $\quad T^\epsilon \mathrm{Ker} \varphi_\epsilon = 0$ almost injective

$$\mathrm{Im} \varphi_\epsilon \subseteq T^\epsilon H(\mathfrak{C}'(c_1', c_2'), \mathfrak{m}_1)$$ almost surjective

Generating Criteria over Λ_0

$\mathfrak{C}$ filtered A infinity category

A infinity Yoneda functor

$$\mathfrak{YON} \qquad \mathfrak{C} \longrightarrow \text{Right Mod}\,\mathfrak{C}$$

$\mathbb{L}$ a finite set of Lagrangian submanifolds.
Two elements are transversal.

$\mathbb{L}$ a finite set of Lagrangian submanifolds of (X, ω)
Two elements are transversal.

$\mathbb{L}'$ another such finite set consider $\mathfrak{YON}_{\mathbb{L}}$

$$\mathfrak{F}(\mathbb{L}') \longrightarrow \mathfrak{F}(\mathbb{L} \cup \mathbb{L}') \longrightarrow \text{Right module}\,\mathfrak{F}(\mathbb{L} \cup \mathbb{L}') \longrightarrow \text{Right module}\,\mathfrak{F}(\mathbb{L})$$

Definition $\mathbb{L}$ is an ϵ weak generator iff

$$d_{GH}(\mathfrak{F}(\mathbb{L}'), \mathfrak{YON}_{\mathbb{L}}(\mathfrak{F}(\mathbb{L}'))) < \epsilon \quad \text{for all } \mathbb{L}'$$

weak generator $\epsilon = 0$

Conjecture $\mathfrak{p} : HH(\mathfrak{F}(\mathbb{L})) \to H(X)$ open closed map

$T^{\epsilon}1 \in \mathrm{Im}\mathfrak{p}$ ⇒ $\mathbb{L}$ is an ϵ weak generator.

Example

$X = S^2$

$\mathbb{L}$ the set of all great circles contains north pole and south pole.

The image of $\mathfrak{p} : HH(\mathfrak{F}(\mathbb{L})) \to H(X)$ contains

$T^\epsilon 1$ for any $\epsilon > 0$

can be generalized to direct products of S^2 and direct products of $\mathbb{L}$