An application of symplectic topology to a problem in algebraic geometry

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Motivation

Non-commutative geometry

Let X be an algebraic variety. Consider the association:

$$X \rightsquigarrow D^b Coh(X) \tag{1}$$

(Here $D^b Coh(X)$ is a pre-triangulated A_∞ category.)

The left hand side is a *classical object*: it is locally modeled on the spectrum of a commutative ring.

The right hand side is a *quantum object*. More precisely, according to the general philosophy of non-commutative geometry, it is reasonable to view pre-triangulated A_∞ categories as *(non-commutative) spaces*. We can therefore regard $D^b Coh(X)$ as a non-commutative partner of the classical object X.

Properties of classical and non-commutative spaces

Given a classical object X as above (or more generally a scheme, a stack, ...), we can talk about properties such as:

- smothness
- properness
- dimension

These classical properties have analogs in the world of non-commutative geometry.

Definition

Let $\mathcal{C}$ be a pre-triangulated A_∞ category over a field.
We say that $\mathcal{C}$ is *proper* if $\dim H^*_{\mathcal{C}}(K, L) < \infty$ for all objects $K, L \in \mathcal{C}$. We say that $\mathcal{C}$ is *smooth* if the diagonal bimodule is perfect.

Dimension

Definition

Let $\mathcal{C}$ be a triangulated category. We say that $G \in \mathcal{C}$ generates $\mathcal{C}$ *in time at most* t if every object of $\mathcal{C}$ can be constructed from G by taking:

- direct sums;
- shifts;
- summands;
- at most t cones.

The generation time of G is the smallest such t.

Definition

The (Rouquier) dimension of $\mathcal{C}$ is the minimal generation time over all $G \in \mathcal{C}$. Finally, if $\mathcal{C}$ is a pre-triangulated A_∞ category, the dimension of $\mathcal{C}$ is the dimension of the associated triangulated category.

Question

Do these non-commutative notions recover their classical counterparts?

Fact

Let X be an algebraic variety. Then X is smooth if and only if $\mathrm{Perf}\, X$ *is smooth. Similarly, X is proper if and only if* $\mathrm{Perf}\, X$ *is proper.*
(Note that if X is smooth, then $\mathrm{Perf}\, X = D^b Coh(X)$*.)*

Concerning dimension: the situation is far from understood.

- there are examples of singular varieties X such that $\dim X < \dim D^b Coh(X)$
- there are also examples of singular varieties X such that $\dim X = \dim D^b Coh(X)$.

Conjecture (Orlov)

Let X be a smooth quasi-projective variety. Then

$$\dim D^b Coh(X) = \dim X. \tag{2}$$

Remark

Rouquier proved that

$$\dim X \leq \dim D^b Coh(X) \leq 2 \dim X, \tag{3}$$

so the content of Orlov's conjecture is to improve the upper bound.

Orlov's conjecture

Orlov's conjecture is known for the following examples:

- affine varieties, projective spaces, projective quadrics (Rouquier)
- curves (Orlov)
- toric surfaces (Ballard–Favero)
- ...

Main Theorem (C–Bai)

Orlov's conjecture is true for any algebraic variety X which admits a Weinstein mirror (M, V), with $\dim M \leq 6$.

New examples:

- toric 3-folds;
- certain log Calabi–Yau surfaces (see Hacking–Keating).

Also gives new proofs for previously known examples. Finally, if X admits a mirror (M^{2n}, V), for $n \geq 3$, then we prove $\dim D^b Coh(X) \leq 2n - 3$.

Some background on Weinstein manifolds

Weinstein manifolds

Definition (Informal)

A *Liouville manifold* is an exact symplectic manifold $M = (M, \lambda)$ which is modeled at infinity on the symplectization of a contact manifold $(\partial_\infty M, \xi_\infty)$. A Liouville manifold is *Weinstein* if it satisfies a tameness condition.

A *Weinstein pair* (M, V) is the data of a Weinstein manifold $M = (M^{2n}, \lambda)$ along with a Weinstein submanifold $V = (V, \lambda_\infty) \subset (\partial_\infty M, \xi_\infty)$.

A *Weinstein sector* is a (type of) Weinstein manifold with boundary.

Example

The cotangent bundle of any closed (boundaryless) manifold is a Weinstein manifold. The cotangent bundle of a compact manifold with boundary is a Weinstein sector.

To any Weinstein manifold/pair/sector M we can associate the *wrapped Fukaya category* $\mathcal{W}(M)$, which is an A_∞ category.

Skeleta

Definition

The *skeleton* of a Weinstein manifold M is the subset of points which don't escape to $\partial_\infty M$ under the positive Liouville flow (i.e. the flow of the vector field dual to λ). The skeleton of a Weinstein pair (M, V) is the subset of points which don't escape to $\partial_\infty M - \text{skel } V$ under the positive Liouville flow.

Example

If $M = (T^*N, \lambda_{can})$, for N a closed manifold, then the skeleton is the zero section.

Remark

There is a notion of homotopy for Weinstein manifolds/pairs. These do not change the Fukaya category up to equivalence. However, they do change the skeleton!

The main theorem, revisited

We say that a variety X is *homologically mirror* to a Weinstein pair (M, V) if

$$D^b Coh(X) = \mathcal{W}(M, V). \tag{4}$$

Theorem (The actual main theorem!)

Let (M^{2n}, V) be a polarizable Weinstein pair. If $n \leq 3$, then

$$\dim \mathcal{W}(M, V) \leq n. \tag{5}$$

If $n \geq 3$, we have:

$$\dim \mathcal{W}(M, V) \leq 2n - 3. \tag{6}$$

Remark

(5) is sharp, but we have no reason to expect that (6) is sharp.

(Heuristic summary of) the main ingredients

The cosheaf property for wrapped Fukaya categories

Let M be a Weinstein manifold. (Heuristically), to a subset $\mathcal{U} \subset \mathrm{skel}(M)$, we can associate a Weinstein sector $\mathrm{Thick}(\mathcal{U}) \subset M$ called its *Weinstein thickening*.

Example

If N is a closed manifold and $\mathcal{U} \subset N$ is a submanifold with boundary, then $\mathrm{Thick}(\mathcal{U}) = T^*\mathcal{U} \subset T^*N$.

Fact

The assignment $\mathcal{U} \mapsto \mathcal{W}(\mathrm{Thick}(\mathcal{U})$ forms a cosheaf of A_∞ categories.

This means that if $\{\mathcal{U}_\sigma\}_{\sigma \in \Sigma}$ is a (suitable) cover of $\mathrm{skel}(M)$, then

$$\mathcal{W}(M) = \mathrm{colim}_{\sigma \in \Sigma} \mathcal{W}(\mathrm{Thick}(\mathcal{U}_\sigma)). \quad (7)$$

A precise version of the above fact was proved by Ganatra–Pardon–Shende.

The cosheaf property is relevant to the study of dimension due to the following lemma.

Lemma (Colimit bound)

Let $\{\mathcal{C}\}_{\sigma\in\Sigma}$ be a diagram of A_∞ categories indexed by a finite poset Σ. Then

$$\dim \operatorname{colim}_\Sigma \mathcal{C}_\sigma \leq \operatorname{depth}\Sigma - 1 + \sum_k \max\{\dim(C_\sigma) \mid \operatorname{depth}\sigma = k\}. \tag{8}$$

(Here the depth of an element $\sigma \in \Sigma$ is the length of the longest chain $\sigma_1 < \sigma_2 < \cdots < \sigma_n = \sigma$. The depth of Σ is the maximal depth of all elements of Σ.)

Arborealization

The arborealization program was initiated by Nadler.

To a (signed, rooted) tree T, we can associate a stratified space Arb_T of dimension $|T| - 1$. We say that Arb_T is the *arboreal singularity* determined by T. Arboreal singularities are useful because their wrapped Fukaya category can be computed.

Fact (Informal)

Let T be a (signed, rooted) rooted tree. Then

$$\mathcal{W}(\mathrm{Thick}(\mathrm{Arb}_T)) = \mathrm{Perf}\ k[T]. \tag{9}$$

We say that the skeleton of a Weinstein pair (M, V) is *arboreal* if it has only arboreal singularities.

A major portion of the arborealization program has now been realized.

Theorem (Álvarez-Gavela–Eliashberg–Nadler)

Let (M, V) be a (polarizable) Weinstein pair. Then (after possibly enlarging V) (M, V) can be homotoped so that the resulting skeleton is arboreal.

The key point is that these homotopies change the skeleton, but they don't change the wrapped Fukaya category.

To relate the arborealization program to dimension theory, we have the following fact.

Fact (Gabriel)

Let T be a tree. Then

$$\dim \operatorname{Perf} k[T] = \begin{cases} 0 & \textit{if } T \textit{ is Dynkin} \\ 1 & \textit{otherwise.} \end{cases} \tag{10}$$

Proof idea

To prove our theorem, we roughly implement the following steps.

- arborealize skel(M, V)
- triangulate the skeleton. Cover skel(M, V) by the stars of the vertices.
- then Thick$(Star_\sigma) =$ Perf $k[T_\sigma]$
- now apply cosheaf bound, combined with Gabriel's theorem.

Remark

If $n \leq 3$, then the arboreal singularities in the skeleton are indexed by trees with at most 4 vertices. These are all Dynkin! This explains why the bound is better for $n \leq 3$.

Remark

For technical reasons, we implement the entire proof using microlocal sheaves (Kashiwara–Schapira, Nadler–Shende).

Remarks and further directions

Optimality

Remark

For $n \geq 3$, we don't expect our upper bound to be optimal. In situations where one has explicit control of the skeleton, one could attempt to deform the skeleton so that it has only Dynkin type arboreal singularities. In such cases, the upper bound would be the same as the one for $n \leq 3$ (and would imply Orlov's conjecture in cases where one has homological mirror symmetry).

Remark

It might also be possible to get rid of the polarizability assumption, by allowing deformations of the skeleton which change the underlying Weinstein manifold, but which preserve the category of microlocal sheaves.

Additive invariants

Definition

An *additive invariant* is a functor **F** from the category of pre-triangulated A_∞ categories to abelian groups which:

- sends derived (Morita) equivalences to isomorphisms
- sends semi-orthogonal decompositions to direct sums.

Example

Hochschild homology, cyclic homology, algebraic K-theory, topological Hohcschild homology, etc.

Example

If **F** is an additive invariant and T is a tree with n vertices, then $\mathbf{F}(\mathrm{Perf}\, k[T]) = k^n$.

If (M, V) is a (polarizable) Weinstein pair (of any dimension), then our proof implies

$$\mathcal{W}(M, V) = \mathrm{colim}_{\sigma \in \Sigma} \mathrm{Perf}\, k[T_\sigma], \tag{11}$$

for some trees T_σ. This implies that there is a quotient (localization) map

$$\mathrm{Groth}_{\sigma \in \Sigma} \mathrm{Perf}\, k[T_\sigma] \to \mathcal{W}(M, V), \tag{12}$$

where $\mathrm{Groth}(-)$ denotes the semi-orthogonal gluing of the categories $\mathrm{Perf}\, k[T_\sigma]$ along the diagram Σ.

Note that $\mathrm{Groth}_{\sigma \in \Sigma} \mathrm{Perf}\, k[T_\sigma]$ is smooth and proper. If $\mathcal{W}(M, V)$ is also proper (it is a fortiori smooth), then it follows that there is a semiorthogonal decomposition

$$\mathrm{Groth}_{\sigma \in \Sigma} \mathrm{Perf}\, k[T_\sigma] = \langle \mathcal{W}(M, V), R, \rangle \tag{13}$$

where $R \subset \mathrm{Groth}_{\sigma \in \Sigma} \mathrm{Perf}\, k[T_\sigma]$ is a “semi-orthogonal complement".

If **F** is an additive invariant, then

$$\mathbf{F}(\mathrm{Groth}_{\sigma\in\Sigma}\,\mathrm{Perf}\,k[T_\sigma]) = \oplus_{\sigma\in\Gamma}\oplus_{|T_\sigma|}\mathbf{F}(k) = k^N, \tag{14}$$

for some N. Hence $\mathbf{F}(\mathcal{W}(X, V))$ is a summand of k^N.

Quantitative symplectic topology

Let M be a Weinstein manifold. There are other upper bounds on the dimension of $\mathcal{W}(M)$ coming from geometry.

Example

If $f : M \to \mathbb{C}$ is a Lefschetz fibration, then

$$\dim \mathcal{W}(M) \leq \# \operatorname{crit}(f). \tag{15}$$

Example

Given a (generic) compactly-supported Hamiltonian diffeomorphism $\phi : M \to M$, we have

$$\mathcal{W}(M) \leq \# |\operatorname{skel}(M) \cap \phi(\operatorname{skel}(M))|. \tag{16}$$

If we combine the above geometric upper bounds on $\dim \mathcal{W}(M)$ with lower bounds (coming from other sources), then we obtain applications to quantitative questions in symplectic topology.