"On the presentability of Artin differentiable stacks"

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Diffrentiable Stacks

Let Euc be the category of open Euclidean sets. One can enlarge our notion of space out of this basic local models in several ways:

1. Euclidean topology $\longrightarrow$ Smooth manifolds, Man
2. Topology of local diff. $\longrightarrow$ Orbifolds, Orbi
3. Topology of local sub. $\longrightarrow$ Artin differentiable stacks, Artin

Each one of this notions generalizes the previous one: Euc $\subset$ Man $\subset$ Orbi $\subset$ Artin

Remark: The two last categories Orbi and Artin are in fact 2-categories.

BASIC EXAMPLES

① FOR ANY LIE GROUP G ONE HAS THE CLASSIFYING STACK $\mathcal{B}G$, SO THAT FOR ANY X THERE IS AN EQUIVALENCE $\mathrm{Prin}_G(X) \simeq \mathrm{Maps}(X, \mathcal{B}G)$

② FOR ANY COMPACT LIE GROUP ACTION $M \circlearrowleft G$, ONE HAS THE STACKY QUOTIENT $M/\!/G$.

LOCAL QUOTIENT STACKS

IT FOLLOWS FROM THE LINEARIZATION THEOREM [ZUNG '06], [CRAINIC, STRUCHINER '13] THAT ANY ARTIN STACK $\mathcal{X}$ IS A LOCAL QUOTIENT STACK, THAT IS, ONE CAN FIND AN OPEN COVER $\mathcal{X} = \bigcup_i \mathcal{U}_i$ SUCH THAT EACH $\mathcal{U}_i \simeq U_i /\!/ G_i$ IS A GLOBAL QUOTIENT STACK FOR A COMPACT LIE GROUP ACTION $U_i \circlearrowleft G_i$.

QUESTION: WHEN IS $\mathcal{X}$ A GLOBAL QUOTIENT STACK $\mathcal{X} \simeq M /\!/ G$?

[Moerdijk '97]

ENOUGH VECTOR BUNDLES [HENRIQUES '05]

BEING A GLOBAL QUOTIEN STACK IS CLOSELY RELATED WITH HAVING ENOUGH VECTOR BUNDLES:

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{E}} \arrow[d] \\ \mathcal{U} \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

IN FACT, WE HAVE THE EQUIVALENCE:

$\mathcal{X}$ HAS ENOUGH VECTOR BUNDLES $\iff$ $\mathcal{X}$ IS A GLOBAL QUOTIENT

SOME RELATED PROPERTIES

① PETER-WEYL PROPERTY: LET $\mathcal{H}^{UNIV}$ BE THE UNIVERSAL HILBERT BUNDLE OVER X. HENCE, X HAS ENOUGH VECTOR BUNDLES IFF THERE IS A SPLITTING $\mathcal{H}^{UNIV} \cong \widehat{\bigoplus_i} \mathcal{E}_i$ INTO FINITE DIMENSIONAL PIECES $\mathcal{E}_i$ [FREED, HOPKINS, TELEMAN '11].

② FREDHOLM COMPLEXES: X HAS ENOUGH VECTOR BUNDLES IFF EVERY FREDHOLM COMPLEX $(\mathcal{H}^\bullet, d^\bullet)$ OVER X CHAIN HOMOTOPY EQUIVALENT TO A FINITE DIMENSIONAL COMPLEX $(\mathcal{H}^\bullet, d^\bullet) \simeq (\mathcal{E}^\bullet, d^\bullet)$ [SEGAL '70].

③ TANNAKIAN RECONSTRUCTION: LET $\mathcal{T}(X)$ BE THE TANNAKIAN DUAL CONSTRUCTED OUT OF THE TENSOR CATEGORY $\mathrm{Vect}(X)$ OF VECTOR BUNDLES OVER X. HENCE, THE CANONICAL MAP $X \to \mathcal{T}(X)$ IS AN EQUIVALENCE IFF X HAS ENOUGH VECTOR BUNDLES [TRENTINAGLIA '10].

EXAMPLES

① **EFFECTIVE ORBIFOLDS:** IT IS A CLASSICAL RESULT [SATAKE '56] THAT EVERY EFFECTIVE ORBIFOLD IS A GLOBAL QUOTIENT. CONSIDER THE TANGENT BUNDLE $T\mathcal{X}$ AND ENDOW $\mathcal{X}$ WITH A RIEMANNIAN METRIC.

$$\begin{tikzcd} Fr(T\mathcal{X}) \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \mathcal{X} \arrow[r] & BO_n \end{tikzcd}$$

FOR EACH $p \in |\mathcal{X}|$, THE MAP $\mathrm{Aut}_{\mathcal{X}}(p) \longrightarrow O_n$ IS INJECTIVE. HENCE, $Fr(T\mathcal{X})$ IS A MANIFOLD AND

$$\mathcal{X} \simeq Fr(T\mathcal{X}) /\!\!/ O_n .$$

② Orbispaces: for non-effective orbifolds there is previous work [Henriques-Metzler '04], [Kališnik '08], [Pronk '10], and culminating in [Pardon '19], which settled the case of orbispaces.

③ Counterexample [Henriques '05]: Let $\alpha \in H^3(S^3, \mathbb{Z})$ be the fundamental class, and let X_α be the total space of the S^1-gerbe over S^3 classified by α. Hence, X_α is an Artin stack which is not a global quotient stack. Note that α is not a torsion class.

As we will see, these turn out to be the only obstruction to global quotient presentability in this sort of situations.

Abelian Gerbes [Giraud '71]

Basically, an S^1-gerbe over X is just a fibration $\tilde{X} \to X$ with fiber BS^1. Such kind of fibrations are classified by $\check{H}^2(X, \underline{S}^1_X)$. Consider the exponential sequence $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$.

Hence, the connecting homomorphism $H^2(X, \underline{S}^1_X) \xrightarrow{\eta} H^3(X, \mathbb{Z})$ turns out to be an isomorphism. If $X_\alpha \to X$ is the S^1-gerbe classified by α, then the class η_α is called its Dixmier-Douady invariant of the gerbe [Dixmier-Douady '63]

$\boxed{\text{Question}}$: When is X_α a global quotient stack?

Projective Bundles

For any positive integer $n > 0$ consider the central extension $1 \to S^1 \to U_n \to PU_n \to 1$. A principal PU_n-bundle P over X is classified by a class in non-abelian cohomology $\check{H}^1(X, \underline{PU_n}_X)$, and the connecting map $\check{H}^1(X, \underline{PU_n}_X) \to \check{H}^2(X, \underline{S^1}_X)$ sends this onto a class $[P] \in \check{H}^2(X, \underline{S^1}_X)$. However, note that $n \cdot [P] = 0$ is torsion, as follows from:

$$\begin{tikzcd}
1 \arrow[r] & S^1 \arrow[r] & U_n \arrow[r] & PU_n \arrow[r] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[u, hook] & SU_n \arrow[r] \arrow[u, hook] & PU_n \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

The Brauer-Grothendieck Group

The torision part $Br(X) := tor(\check{H}^2(X, \underline{S}^1_X))$ is often called the Brauer group of X. Note that, the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces the Bockstein map $H^2(X, \mathbb{Q}/\mathbb{Z}) \to H^3(X, \mathbb{Z})$, whose image is precisely the torsion part $tor(H^3(X, \mathbb{Z})) \cong tor(\check{H}^2(X, \underline{S}^1_X)) = Br(X)$. Thus, every finite dimensional projective bundle produces an element in the Brauer group [Atiyah-Segal '04].

$\boxed{\text{Question}}$: Does every element of the Brauer group arise in this way?

THE INFINITE PROJECTIVE GROUP

LET $PU_\infty = \lim_n PU_n$ BE THE INFINITE PROJECTIVE GROUP. IT IS A CLASSICAL RESULT DUE TO SERRE (WHICH REST ON BOTT'S PERIODICITY) THAT THERE IS A WEAK HOMOTOPY EQUIVALENCE [ATIYAH-SEGAL '04]:

$$BPU_\infty \simeq K(\mathbb{Q}/\mathbb{Z}, 2) \times \prod_{j \geq 1} K(\mathbb{Q}, 2j+2)$$

LET $\mathrm{Proj}^\infty(X)$ BE THE GROUPOID OF PRINCIPAL PU_∞-BUNDLES OVER X. THUS, THERE IS AN EQUIVALENCE OF GROUPOIDS $\mathrm{Proj}^\infty(X) \simeq \mathrm{MAPS}(X, BPU_\infty)$.

On the other hand, every stack map $\mathcal{X} \to BPU_\infty$ induces a continuous map $B\mathcal{X} \to BPU_\infty$ by the geometric realization construction, and we get in this manner a functor $\mathrm{Proj}^\infty(\mathcal{X}) \longrightarrow \pi_1 \mathrm{Maps}(B\mathcal{X}, BPU_\infty)$ into the fundamental groupoid on the mapping space $\mathrm{Maps}(B\mathcal{X}, BPU_\infty)$.

$\boxed{\text{Question}}$: When is the functor $\mathrm{Proj}^\infty(\mathcal{X}) \longrightarrow \pi_1 \mathrm{Maps}(B\mathcal{X}, BPU_\infty)$ an equivalence?

EXAMPLES

① IF $X = M$ IS A SMOOTH MANIFOLD, THEN $\mathrm{Proj}^{\infty}(M) \to \pi_1 \mathrm{Maps}(M, BPU_\infty)$ IS AN EQUIVALENCE.

② IF $X = \mathcal{B}G$ IS THE CLASSIFING STACK OF A FINITE GROUP G, THEN $\mathrm{Proj}^{\infty}(\mathcal{B}G) \to \pi_1 \mathrm{Maps}(BG, BPU_\infty)$ IS ALSO AN EQUIVALENCE. THIS IS BECAUSE $\mathrm{Proj}^{\infty}(\mathcal{B}G)$ CLASSIFIES PROJECTIVE REPRESENTATIONS $\rho: G \to PU_\infty$ UP TO CONJUGATION, WHILE $\pi_1 \mathrm{Maps}(BG, BPU_\infty) \simeq \pi_1(\mathrm{Maps}(BG, K(\mathbb{Q}/\mathbb{Z}, 2)))$ CLASSIFIES CENTRAL EXTENSIONS $0 \to \mathbb{Q}/\mathbb{Z} \to \tilde{G} \to G \to 1$ UP TO ISOMORPHISM. IN PARTICULAR, ONE HAS A BIJECTION $\pi_0 \mathrm{Proj}^{\infty}(\mathcal{B}G) \cong H^2(G, \mathbb{Q}/\mathbb{Z})$.

NOTE THAT, THIS IS A LOCAL QUESTION. IN FACT, IF $X = \bigcup_i \mathcal{U}_i$ IS AN OPEN COVER AND $\mathcal{U}_{i_1,\dots,i_k} := \bigcap_{l=1}^{k} \mathcal{U}_{i_l}$ ARE THE k-FOLD INTERSECTIONS, THEN IT IS ENOUGH TO SHOW THAT EACH $\mathrm{Proj}^\infty(\mathcal{U}_{i_1,\dots,i_k}) \longrightarrow \pi_1 \mathrm{Maps}(B\mathcal{U}_{i_1,\dots,i_k}, BPU_\infty)$ IS AN EQUIVALENCE.

THIS IS DUE TO THE EQUIVALENCES:

1. $\mathrm{Proj}^\infty(X) \simeq \mathrm{hcol}_i\, \mathrm{Proj}^\infty(\mathcal{U}_i)$; (DESCENT)

2. $\pi_1 \mathrm{Maps}(BX, BPU_\infty) \simeq \mathrm{hcol}_i\, \pi_1 \mathrm{Maps}(B\mathcal{U}_i, BPU_\infty)$; (VAN KAMPEN)

Note that whenever $\mathcal{X}$ is an orbifold, one can choose a cover $\mathcal{X} = \bigcup_i \mathcal{U}_i$ such that each $\mathcal{U}_{i_1,\dots,i_k} \simeq U_{i_1,\dots,i_k} /\!/ G_{i_1,\dots,i_k}$ is an equivariant contractible orbifold chart. Hence, we have:

Theorem (B-Lupercio) Whenever $\mathcal{X}$ is an orbifold, the functor $\mathrm{Proj}^{\infty}(\mathcal{X}) \longrightarrow \pi_1 \mathrm{Maps}(B\mathcal{X}, BPU_\infty)$ is an equivalence.

As a direct consequence, we get:

Theorem (B-Lupercio) Every class in the Brauer group of an orbifold can be realized by a finite dimensional projective bundle.

Turning back to Presentability

Suppose that $\mathcal{X} \simeq M/\!\!/G$ is presentable. Hence, any finite dimensional projective bundle $\mathcal{P}$ over $\mathcal{X}$ can be realized by a G-equivariant bundle P over M. Note that we get a presentation $\mathcal{P} \simeq P/\!\!/G$ in this manner. Moreover, if $\alpha := [\mathcal{P}]$, we get a presentation $\mathcal{X}_\alpha \simeq U_n \backslash\!\!\backslash P/\!\!/G$ by lifting to a unitary action on P via the projection $U_n \longrightarrow PU_n$. Therefore, we get:

Theorem (B-Lupercio) Suppose that $\mathcal{X}$ is a global quotient stack. Then, the total space $\mathcal{X}_\alpha$ of an S^1-gerbe is presentable as a global quotient iff its defining class α is torsion.

THANK YOU