Mirror Structure Constants via Non-archimedean Analytic Disks

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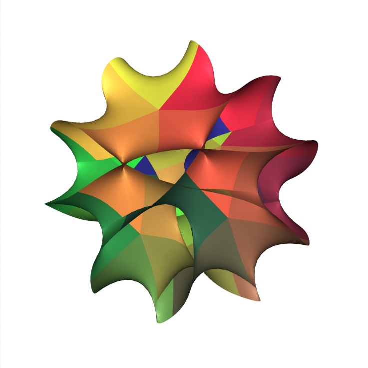
work in progress with S. Keel

Plan:
1. Motivations from SYZ and HMS
2. Heuristics behind the mirror structure constants
3. Non-archimedean SYZ fibration
4. Boundary condition
5. Properness of the moduli space
6. Ingredients in the proof of the main theorem
7. Comparison with punctured log Gromov-Witten invariants by ACGS

1. Motivations from SYZ and HMS



Def: A smooth projective variety $X/\mathbb{C}$ is called Calabi-Yau if its canonical bundle K_X is trivial, i.e. it has a nowhere vanishing holomorphic volume form.

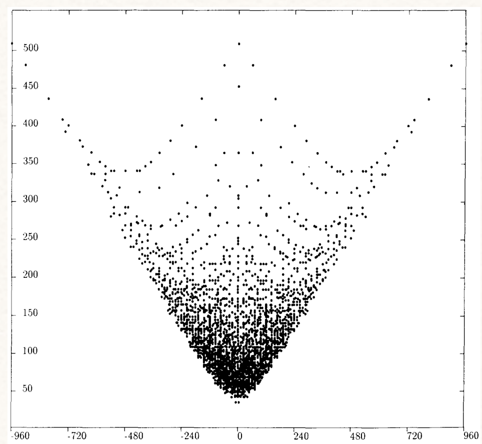
Examples: Elliptic curve, abelian variety, K3 surface, hypersurface of degree $d+1$ in $\mathbb{CP}^d$.

Mirror Symmetry: conjectural duality between Calabi-Yau varieties.

$$\text{Any CY variety } X \longleftrightarrow \exists \text{ mirror variety } \check{X}$$

such that a list of deep geometric relations hold between X and $\check{X}$, involving: Hodge structures, Gromov-Witten invariants, Fukaya categories, derived category of coherent sheaves, SYZ torus fibrations, etc.

Example: Hodge numbers $h^{p,q}(X) = h^{d-p,q}(\check{X})$.



In 3-dim case, $h^{1,1}(X) = h^{2,1}(\check{X}) = h^{1,2}(\check{X})$

$h^{1,2}(X) = h^{2,2}(\check{X}) = h^{1,1}(\check{X})$

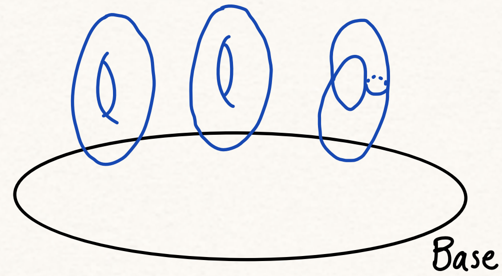
(Candelas et al.)

Two main conjectures in mirror symmetry:

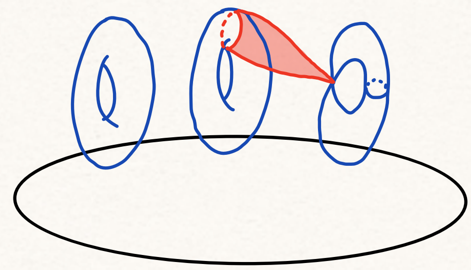
$\begin{cases} \text{SYZ conjecture: Strominger - Yau - Zaslow conjecture} \\ \text{HMS conjecture: Homological mirror symmetry by Kontsevich} \end{cases}$

Rough idea of SYZ conjecture:

(1) In certain asymptotic sense, the CY manifold X should admit a torus fibration called SYZ fibration



(2) The mirror CY manifold $\check{X}$ should be constructed by first taking the dual torus fibration, and then modified using specific counts of holomorphic disks called instanton corrections



Rough idea of HMS conjecture:

$X \rightsquigarrow \mathrm{Fuk}(X)$ Fukaya category

$\begin{cases} \text{objects: Lagrangian submanifolds } L \subset X \\ \text{morphisms: holomorphic disks with boundaries on Lagrangians} \end{cases}$

$\check{X} \rightsquigarrow \mathrm{Coh}(X)$ category of coherent sheaves on $\check{X}$

Then $D^b\mathrm{Fuk}(X) \simeq D^b\mathrm{Coh}(X)$ after passing to the derived categories.

Combining SYZ + HMS

$\rightsquigarrow$ heuristic construction of the mirror variety $\check{X}$

The best illustration of this idea is for the case of log Calabi-Yau varieties, because in this case, the mirror $\check{X}$ will be an affine varietry $\check{X} = \operatorname{Spec} A$. We call A the mirror algebra, and it suffices to describe explicitely the underlying vector space of A, and the multiplication rule, i.e. the structure constants.

Remark: The affine mirror variety $\check{X} = \operatorname{Spec} A$ has natural compactifications given by Proj of some graded mirror algebra.

2. Heuristics behind the mirror structure constants

Setup: (Y, D) $\quad$ Y smooth projective variety $/k$ any field of char 0.

D normal crossing divisor

$U := Y \setminus D$

We have log pluricanonical forms $H^0(Y, \omega_Y(D)^{\otimes m}) \subset H^0(U, \omega_U^{\otimes m})$ independent of the compactification.

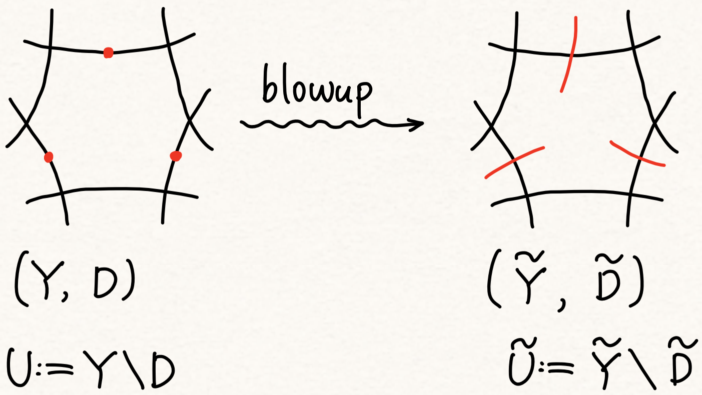
Def: U is log Calabi-Yau if for all m, this subspace is one-dimensional, and generated by a volume form Ω on U.

Example: If $D \in |-K_Y|$, then U is log Calabi-Yau.
In this case, (Y, D) is called a minimal model of U.

Rem: All log Calabi-Yau varieties arise in this way if we allow dlt singularities.

Def: A log Calabi-Yau U has maximal boundary if it has a minimal model (Y, D) with a 0-dimensional log canonical center, i.e. a 0-stratum in the normal crossing case.

Example: (Y, D) toric variety, $U \simeq (\mathbb{C}^*)^n$



$\tilde{D}$ strict transform of D

Goal: Construct the mirror variety $\check{X} = \operatorname{Spec} A$ of any affine log Calabi-Yau.
We will construct the mirror algebra A by generators (as module) and structure constants.

Rem: Without affineness, the mirror will only be formal.

Rem: In fact, we will construct a family of mirror varieties $\begin{array}{c}\operatorname{Spec} A \\ \downarrow \\ \operatorname{Spec} R\end{array}$

Generators of A (as R-module) are indexed by the set

$Sk(U, \mathbb{Z}) :=$ integer points in the essential skeleton of U

$= \{0\} \sqcup \{m\nu \mid m \in \mathbb{N}_{>0},\ \nu$ is an essential divisorial valuation on $k(U)\}$

volume form has 1^{st}- order pole

field of rational functions

Let $R := \mathbb{Z}[NE(Y, \mathbb{Z})] := \bigoplus_{\beta \in NE(Y,\mathbb{Z})} \mathbb{Z} \cdot z^{\beta}$ the monoid ring of $NE(Y, \mathbb{Z})$ over $\mathbb{Z}$.

$$A := R^{(Sk(U,\mathbb{Z}))} := \bigoplus_{p \in Sk(U,\mathbb{Z})} R \cdot \theta_p$$ the free R-module with basis $Sk(U, \mathbb{Z})$.

Multiplication rule:

Given $P_1, \cdots, P_n \in Sk(U, \mathbb{Z})$, we write the product in the mirror algebra A as

$$\theta_{P_1} \cdots \theta_{P_n} = \sum_{Q \in Sk(U,\mathbb{Z})} \sum_{\gamma \in NE(Y,\mathbb{Z})} \chi(P_1, \cdots, P_n, Q, \gamma)\, z^{\gamma} \theta_Q$$

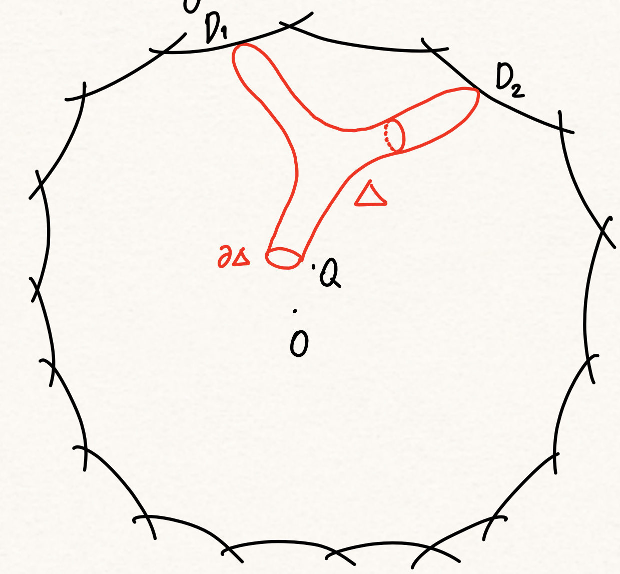
structure constants

SYZ + HMS $\rightsquigarrow$ the structure constants are supposed to be given by the counts of following holomorphic disks in U.

Write $P_j = m_j \nu_j$ for all $P_j \neq 0$.

Assume each ν_j is given by a component $D_j \subset D$ (always possible after a blowup)

Heuristically: $\chi(P_1, \cdots, P_n, Q, \gamma) = \#$ disks Δ in Y



s.t. (i) Δ intersects D_j with order m_j

(ii) $\partial\Delta$ maps to a point near $Q \in Sk(U, \mathbb{Z})$ via the SYZ fibration

(iii) $\partial\Delta$ has homology class $Q \in H_1(\text{fiber})$

(iv) class of $\Delta = \gamma$

In order to make conditions (ii)(iii) above precise, we need to replace the SYZ fibration by the non-archimedean SYZ fibration.

3. Non-archimedean SYZ fibration

We equip our base field k with the trivial absolute value $|\cdot| : k \to \{0, 1\}$

$$|x| = \begin{cases} 1 & \text{for all } x \in k \setminus 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Then k becomes a non-archimedean field !

Berkovich analytification $U \rightsquigarrow U^{an}$ k-analytic space (analogous to complex analytic geometry)

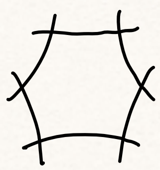
$$U^{an} \underset{=}{\overset{\text{set}}{=}} \left\{ (\xi, \nu) \;\middle|\; \begin{array}{l} \xi \in U \text{ is a scheme-theoretic point} \\ \nu \text{ is an absolute value on the residue field } \kappa(\xi) \\ \text{extending the given one on } k \end{array} \right\}$$

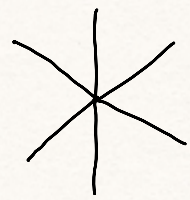
Volume form Ω on U $\rightsquigarrow$ $\|\Omega\| : U^{an} \to \mathbb{R}_{\geq 0}$ upper semicontinuous function

Temkin's Kähler seminorm

Def: The skeleton of U: $Sk(U) :=$ the maximal locus of $\|\Omega\| \subset U^{an}$

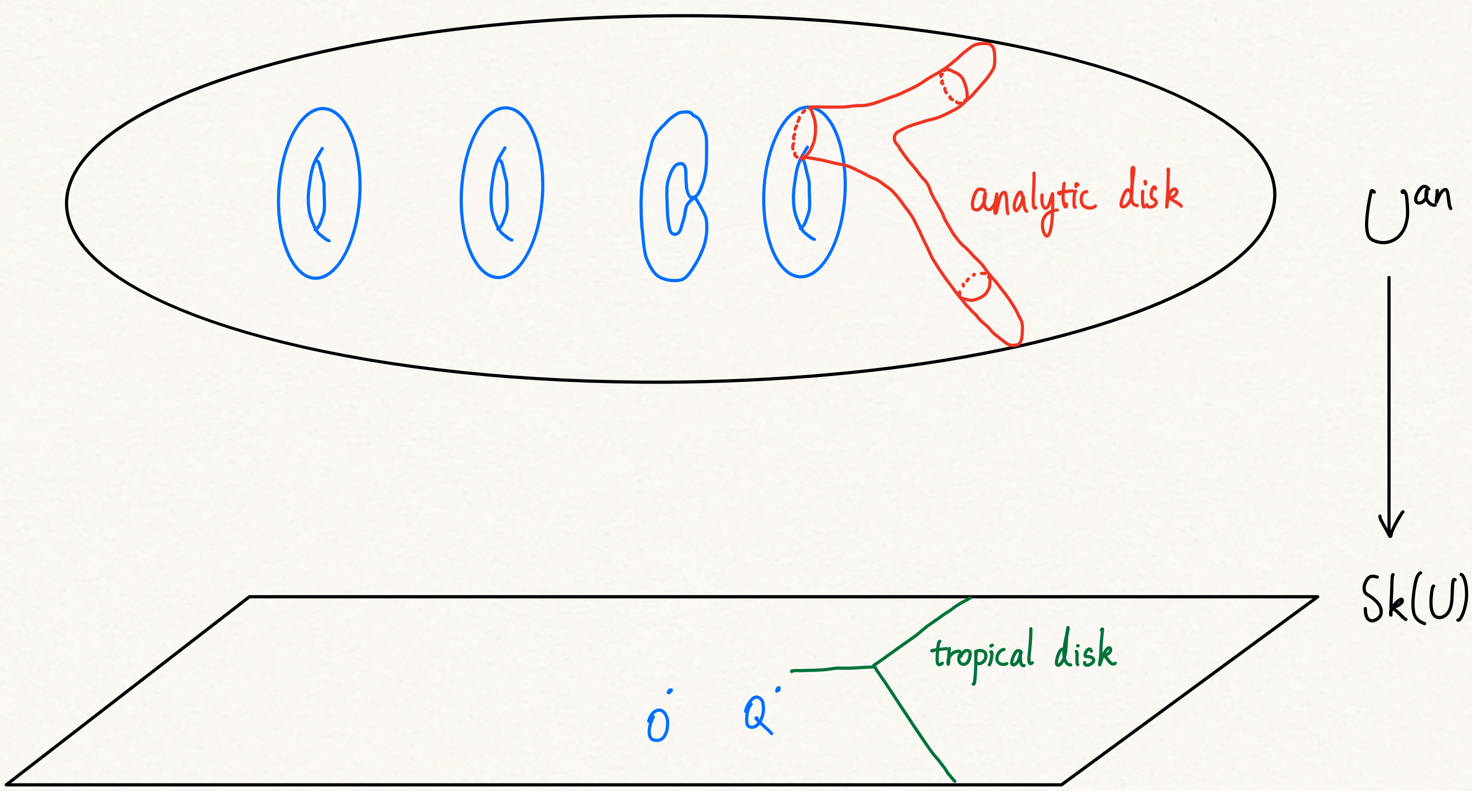
Rem: $Sk(U, \mathbb{Z}) \subset Sk(U)$. valuations on the generic point only

Example: (Y, D)  $Sk(U) \simeq$ dual intersection cone complex of D

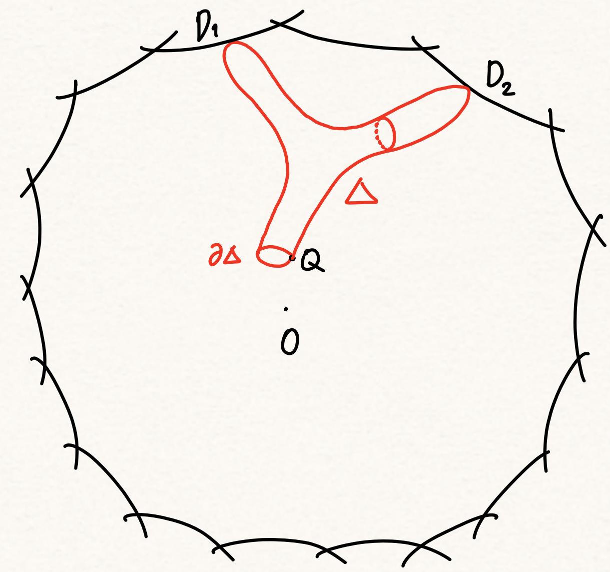


Berkovich, Nicaise, Xu, Yu: strong deformation retraction $\tau : U^{an} \to Sk(U)$, affinoid torus fibration outside codim 2. Non-archimedean SYZ fibration

locally given by $(\mathbb{G}_m^n)^{an} \longrightarrow \mathbb{R}^n$, taking coordinatewise valuations.



Recall: $\mathcal{X}(P_1, \cdots, P_n, Q, \gamma) = \#$ disks Δ in Y



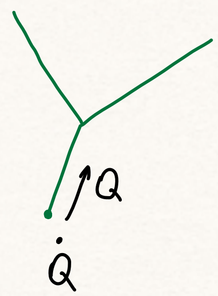
s.t. (i) Δ intersects D_j with order m_j

(ii) $\partial\Delta$ maps to a point near $Q \in Sk(U, \mathbb{Z})$ via the SYZ fibration

(iii) $\partial\Delta$ has homology class $Q \in H_1(\text{fiber})$

(iv) class of $\Delta = \gamma$

Now we reformulate conditions (ii) (iii) via the non-archimedean SYZ fibration:

tropical disk  $\subset Sk(U)$

Having formulated the precise conditions, we are ready to count analytic disks satisfying these conditions.

Trouble: The moduli space of such disks is ∞-dimensional.

We must impose further conditions to cut the dimension down to 0.

In particular, we must impose a regularity condition on the boundary $\partial\Delta$ to discard most of the non-archimedean analytic disks.

Rough idea: The boundary $\partial\Delta$ lies in the place of U^{an} with affinoid torus fibration, i.e. locally isomorphic to $(\mathbb{G}_m^n)^{an} \longrightarrow \mathbb{R}^n$.

We want the map $\partial\Delta \longrightarrow (\mathbb{G}_m^n)^{an}$ to be as simple as possible.

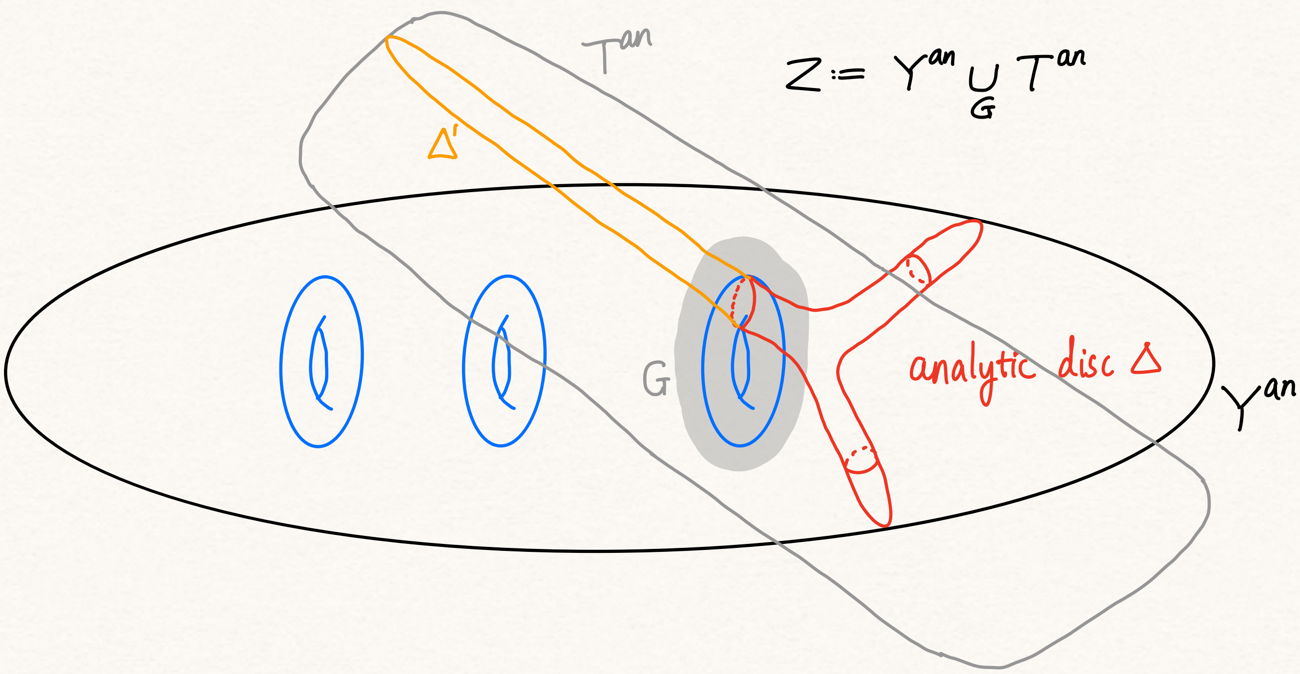
After analyzing low-dimensional examples, we proposes the following boundary condition:

$$\begin{tikzcd} \partial\Delta \arrow[r] \arrow[d, "\text{opposite orientation}"'] & (\mathbb{G}_m^n)^{an} \arrow[d, hook] \\ \Delta' \arrow[r, "g"] & T^{an} \end{tikzcd}$$

toric compactification

s.t. the tropical curve associated to g is a straight line.

Geometrically, it means that we are gluing the toric variety T^{an} to Y^{an} along a small domain G of trivial affinoid torus fibration



Now we count closed rational curves $C = \Delta \underset{A := \Delta\cap\Delta'}{\cup} \Delta'$ in Z s.t.

$$\begin{cases} \Delta \longrightarrow Y^{an} & \text{satisfies (i)-(iv) as before} \\ \Delta' \longrightarrow T^{an} & \text{has straight tropical curve} \end{cases}$$

5. Properness of the moduli space

Worry: The new target space $Z := Y^{an} \underset{G}{\cup} T^{an}$ is not projective, not even proper, not even separated.

How is it ever possible to count curves in such a space?

Idea: As long as we can keep the circle $A = \Delta \cap \Delta'$ away from the boundary of the domain G, the rest of the curve C will not feel the non-separated locus of Z.

↑ we glued $Z = Y^{an} \underset{G}{\cup} T^{an}$ along G

More precisely, we fix a marked point q on the circle A, as well as marked points p_i where C touches the boundary of Y^{an} and T^{an}.

Let $M(U, \beta)$ denote the moduli space of such curves in $Z = Y^{an} \underset{G}{\cup} T^{an}$

Consider $\Phi : M(U, \beta) \xrightarrow{(\mathrm{dom},\, ev_q)} V_M \times G$

$$V_M \subset M_{0,n}^{an}$$

Main theorem: Φ is finite étale over an open neighborhood of $Sk(V_M \times G)$.

Its degree $\rightsquigarrow$ the desired count of non-archimedean analytic disks

$\rightsquigarrow$ structure constant $\chi(P_1, \cdots, P_n, Q, \gamma)$

$\rightsquigarrow$ commutative associative mirror algebra A.

Further result: $\operatorname{Spec} A \to \operatorname{Spec} R$ is a flat family of Gorenstein semi log canonical log Calabi-Yau varieties, with normal and log canonical generic fibers.

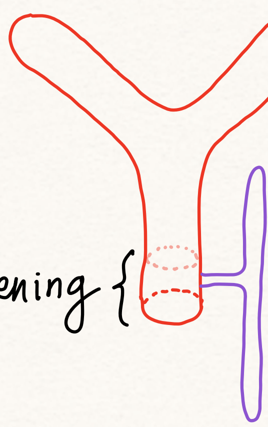
6. Ingredients in the proof of the main theorem

Finite étale =
- smooth : non-archimedean deformation theory based on Porta–Y
- relative dimension 0 : gluing of volume form
- proper =
  - topologically proper : use formal model
  - boundaryless : use formal model

In fact, for boundaryless, we need to introduce an auxiliary moduli space $M'(U,\beta)$ asking not only the circle $A = \Delta \cap \Delta'$ to map to G, but also an open thickening of the circle to map to G.

Finally we use the theory of skeletal curves to identify $M(U,\beta)$ with $M'(U,\beta)$ over $Sk(V_M \times G)$.

Geometric idea:



thickening

need to prevent bad branches growing out of the annulus when we pass by walls in the skeleton $Sk(U)$

7. Comparison with punctured log Gromov–Witten invariants by ACGS