On Mapping Class Groups of Non-Orientable Surfaces

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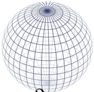
Mapping Class Groups

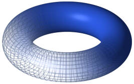
S_g orientable surface, genus g

$$\begin{aligned} Mod(S_g) &:= \pi_0 \ \mathrm{Diff}^+(S_g) \\ &= \ \mathrm{Diff}^+(S_g)/\ \mathrm{Diff}_0(S_g) \end{aligned}$$

- Examples

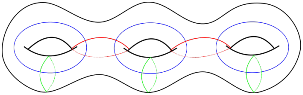


$Mod(S^2) = 0$



$Mod(T) = SL(2, \mathbb{Z})$

- Generated by Dehn twists



- Marked points: $Mod(S_g; k) = \pi_0 \ \mathrm{Diff}^+(S_g; k)$

  where $\quad \mathrm{Diff}^+(S_g; k) =$ diffeo's fixing a set $\{x_1, \ldots, x_k\}$

Nielsen Realization Problem

$$\begin{tikzcd} & & \mathrm{Diff}^+(S_g) \arrow[d, "p"] \\ \text{(finite)} & G \arrow[ur, dashed] \arrow[r, "i"] & \mathit{Mod}(S_g) \end{tikzcd}$$

[Kerckhoff '83] Every finite subgp. of $\mathit{Mod}(S_g)$ can be realized as a gp. of isometries for some hyperbolic structure on S_g.

- $\mathit{Mod}(S_g)$ acts on the Teichmüller space $\mathcal{T}_g = \mathbb{R}^{6g-6}$,
  $\mathcal{T}_g / \mathit{Mod}(S_g)$ = moduli space of Riemann surfaces

[K. '83] Every fin. subgp. of $\mathit{Mod}(S_g) \circlearrowright \mathcal{T}_g$ has a fixed point.

Case of marked points: $\mathcal{T}_k(S_g)$ is defined in a similar way

[Wolpert '87, Masur-Wolf '02] Every finite subgroup of $\mathit{Mod}^{\pm}(S_g; k)$ acting on $\mathcal{T}_k(S_g)$ has a fixed point.

Non-orientable surfaces: $N_g = \underbrace{\mathbb{R}\mathrm{P}^2 \# \dots \# \mathbb{R}\mathrm{P}^2}_{g}$

$$\mathit{Mod}(N_g; k) := \mathrm{Diff}(N_g; k) / \mathrm{Diff}_0(N_g; k)$$

Examples: $\mathit{Mod}(N_2) = \mathbb{Z}/2 \times \mathbb{Z}/2, \qquad \mathit{Mod}(N_3) = GL(2, \mathbb{Z}).$

Theorem (Colin, X.) Every finite subgroup $G \subseteq \mathit{Mod}(N_g; k)$ acting on $\mathcal{T}_k(N_g)$ has a fixed point.

$$\begin{tikzcd}
& \mathrm{Diff}(N_g) \arrow[d, "p"] \\
G \arrow[ur, dashed] \arrow[r, "i"] & \mathit{Mod}(N_g) \arrow[u, bend right, "\nexists s"']
\end{tikzcd}$$

Theorem (Colin, X.) If $g \geq 35$, the projection $p : \mathrm{Diff}(N_g) \to \mathit{Mod}(N_g)$ does not have a section.

Klein Surfaces

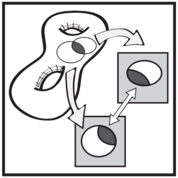
For $f : U \subset \mathbb{C} \to \mathbb{C}$,

$$\partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f)$$

$$\partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$$

- f is analytic if $\partial_{\bar{z}} = 0$
- f is antianalytic if $\partial_z = 0$
- f is dianalytic if $f|_V$ to any connected comp. is either one.

Definition: A *Klein surface* $(\Sigma, \mathfrak{X})$ is a connected surface Σ together with a **dianalytic structure** $\mathfrak{X}$, (i.e. an equivalence class of dianalytical atlases).



$\mathscr{M}(\Sigma)$ = set of dianalytic structures of Σ that agree with the smooth structure.

Definition:

- A *morphism (or dianalytic map)* $f : (\Sigma, \mathfrak{X}) \to (\Sigma', \mathfrak{Y})$ is a map s.t. $\forall x \in \Sigma$ there are charts $x \in U$, $f(x) \in V$ with

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \qquad \text{dianalytic}$$

- For $f \in$ Diff(Σ), $\mathfrak{X} \in \mathscr{M}(\Sigma)$, *the pullback* $f^*\mathfrak{X}$ is the unique structure such that

$$f : (\Sigma, f^*\mathfrak{X}) \to (\Sigma, \mathfrak{X}) \qquad \text{is a morphism.}$$

Definition: An **orientable double cover** of a non-orientable Klein surface $(\Sigma, \mathfrak{X})$ is a Riemann surface $(S, \mathfrak{X}^0)$ with

- a dianalytic map $\pi : (S, \mathfrak{X}^0) \to (\Sigma, \mathfrak{X})$ unramified double cover
- an antianalytic involution $\sigma : S \to S$ such that $\pi \circ \sigma = \pi$.

Compact case:

$$\begin{tikzcd} S_{g-1} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & S_{g-1} \arrow[dl, "\pi"] \\ & N_g & \end{tikzcd}$$

$$N_g = S_{g-1}/ < \sigma >$$

Remark: Every $f \in \mathrm{Diff}(N_g; k)$ admits exactly two liftings $S_{g-1} \to S_{g-1}$, one of which preserves orientation

$$\widetilde{f} \in \mathrm{Diff}^+(S_{g-1}; 2k)$$

This choice induces

$$\begin{tikzcd} \mathrm{Diff}(N_g; k) \arrow[r, "\rho"] \arrow[d] & \mathrm{Diff}^+(S_{g-1}; 2k) \arrow[d] \\ \mathit{Mod}(N_g; k) \arrow[r, "\phi"] & \mathit{Mod}(S_{g-1}; 2k) \end{tikzcd}$$

Theorem (Hope-Tillmann; Gonçalves-Guaschi-Maldonado)

1. If $g \geq 3$, $\phi : \mathit{Mod}(N_g) \to \mathit{Mod}(S_{g-1})$ is injective.
2. If $k \geq 1$, $\phi : \mathit{Mod}(N_g; k) \to \mathit{Mod}(S_{g-1}; 2k)$ is injective $\forall g$.

Teichmüller Space

Definition: $\mathfrak{X}, \mathfrak{Y} \in \mathscr{M}(\Sigma)$ are *Teichmüller equivalent* if there is $f \in \text{Diff}_0(\Sigma; k)$ such that $f : (\Sigma, \mathfrak{X}) \to (\Sigma; \mathfrak{Y})$ is a morphism.

Teichmüller space:

$$\mathscr{T}_k(\Sigma_g) = \mathscr{M}(\Sigma_g)/\ \text{Diff}_0(\Sigma_g; k) \approx \begin{cases} \mathbb{R}^{6g-6+2k} & \text{orientable} \\ \mathbb{R}^{3g-3+2k} & \text{non-orientable} \end{cases}$$

Lemma: For $\pi : S_{g-1} \to N_g$ orientable double cover of a non-orientable Klein surface N_g,

1. The map is injective

$$\pi^* : \mathscr{T}_k(N_g) \to \mathscr{T}_{2k}(S_{g-1})$$

$$[\mathfrak{X}] \longmapsto [\pi^* \mathfrak{X}]$$

2. The image of π^* is

$$\pi^*(\mathscr{T}_k(N_g)) = \{[\mathfrak{X}] \in \mathscr{T}_{2k}(S_{g-1}) \mid [\sigma^* \mathfrak{X}] = [\mathfrak{X}]\}$$
$$=: \mathscr{T}_{2k}(S_{g-1})_{\sigma^*}$$

Nielsen Realization Theorem

- Have injections

$$\phi : \mathit{Mod}(N_g; k) \to \mathit{Mod}(S_{g-1}; 2k)$$

$$\pi^* : \mathcal{T}_k(N_g) \longrightarrow \mathcal{T}_{2k}(S_{g-1})$$

- $\mathit{Mod}(N_g; k)$ acts on $\mathcal{T}_k(N_g)$ by pullbacks.

Lemma: For $[\mathfrak{X}] \in \mathcal{T}_k(N_g)$ and $\alpha \in \mathit{Mod}(N_g; k)$

$$\pi^* (\alpha \cdot [\mathfrak{X}]) = \phi(\alpha) \cdot \pi^* [\mathfrak{X}]$$

Theorem (Colin, X) Every finite group $G \subseteq \mathit{Mod}(N_g; k)$ acting on $\mathcal{T}_k(N_g)$ has a fixed point.

Proof:

Let $H \subseteq Mod^{\pm}(S_{g-1}; 2k)$ be the subgp generated by $\phi(G)$ and $[\sigma]$.

$$\begin{tikzcd} S_{g-1} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & S_{g-1} \arrow[dl, "\pi"] \\ & N_g & \end{tikzcd}$$

$\Rightarrow \quad H \cong G \times \mathbb{Z}/2 \subseteq Mod^{\pm}(S_{g-1}; 2k)$

- [Wolpert] $\Rightarrow \quad \exists \quad [\mathfrak{Y}] \in \mathcal{T}_{2k}(S_{g-1})$ fixed by H

  In particular $[\sigma] \cdot [\mathfrak{Y}] = [\sigma^* \mathfrak{Y}] = [\mathfrak{Y}]$

  $$\Rightarrow \quad [\mathfrak{Y}] = \pi^*[\mathfrak{X}] \quad \text{for some} \quad [\mathfrak{X}] \in \mathcal{T}_k(N_g)$$

- Thus, $\forall \alpha \in G$

  $$\begin{aligned} \pi^*(\alpha \cdot [\mathfrak{X}]) &= \phi(\alpha) \cdot \pi^*[\mathfrak{X}] \\ &= \pi^*[\mathfrak{X}] \end{aligned}$$

  π^* monomorphism $\Rightarrow \quad \alpha \cdot [\mathfrak{X}] = [\mathfrak{X}] \qquad \square$

Non-existence of sections

Question: What about infinite subgroups of $Mod(N_g)$?
In particular,

$$p: \ \begin{tikzcd} \mathrm{Diff}(N_g) \arrow[r] & Mod(N_g) \arrow[l, bend right, "\exists s\,?"'] \end{tikzcd}$$

Use characteristic clases:

For $\xi : E \to B$ smooth (orientable) surface bundle,

- T_vE = vertical bundle = $\ker\{d\xi : TE \to TB\}$
- T_vE = 2-dim oriented vector bundle $/E$

$e \in H^2(E;\mathbb{Z})$ **Euler class**

Definition: (Miller-Morita-Mumford classes for ξ)

$$\kappa_n(\xi) := \xi_!\left(e(T_vE)^{n+1}\right) \in H^{2n}(B;\mathbb{Z})$$

where $\xi_! : H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z})$ is the *umkehr map*.

Universal bundle:

$$\begin{tikzcd} S_g \arrow[r, equal] \arrow[d] & S_g \arrow[d] \\ E_g \arrow[d] & E \arrow[l] \arrow[d] \\ B\,\mathrm{Diff}^+(S_g) & X \arrow[l] \end{tikzcd}$$

Have classes:

$$\kappa_n \in H^{2n}(B\,\mathrm{Diff}^+(S_g);\mathbb{Z}) = H^{2n}(\mathit{Mod}(S_g);\mathbb{Z}) \quad (g \geq 2)$$

Theorem (Miller, Morita, Harer)

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(\mathit{Mod}(S_g);\mathbb{Q})$$

which is an iso in the stable range $* \leq \frac{2}{3}(g-1)$.

Becker-Gottlieb transfer: $trf_\xi : \Sigma^\infty B_+ \to \Sigma^\infty E_+$

$$\begin{tikzcd} H^*(B) \arrow[r, "\xi^*"] \arrow[rr, bend right, "\cdot\chi(F)"'] & H^*(E) \arrow[r, "trf_\xi^*"] & H^*(B) \end{tikzcd}$$

Oriented case: $trf_\xi^*(x) = \xi_!\Big(x \cup e(T_vE)\Big)$

$$\Rightarrow \qquad \kappa_n(\xi) = trf_\xi^*(e(T_vE)^n)$$

and in particular

$$\boxed{\kappa_{2n}(\xi) = trf_\xi^*\Big(p_1(T_vE)^n\Big)}$$

p_1 = first Pontryagin class

Non-oriented case: $\eta : E \to B$ non-oriented surface bundle

$$\boxed{\zeta_i(\eta) := trf_\eta^*\Big(p_1(T_vE)^i\Big)} \in H^{4i}(B;\mathbb{Z})$$

Universal bundle:

$$\begin{tikzcd} N_g \arrow[r, equal] \arrow[d] & N_g \arrow[d] \\ E_g & E \arrow[l] \arrow[d] \\ B\,\mathsf{Diff}(N_g) \arrow[from=2-1, to=3-1] & X \arrow[l] \end{tikzcd}$$

Have classes:

$$\zeta_i \in H^{4i}(B\,\mathsf{Diff}(N_g);\mathbb{Z}) = H^{4i}(\mathit{Mod}(N_g);\mathbb{Z}) \quad (g \geq 3)$$

Theorem (Wahl; Galatius-Madsen-Tillmann-Weiss)

$$\mathbb{Q}[\zeta_1, \zeta_2, \dots] \longrightarrow H^*(\mathit{Mod}(N_g);\mathbb{Q})$$

which is iso in the stable range $* \leq \frac{g-3}{4}$.

$$\Rightarrow \quad \zeta_i \neq 0 \quad \text{in} \quad H^{4i}(-;\mathbb{Q}) \quad \text{if} \quad g \geq 16i + 3$$

Theorem (Colin, X.) If $g \geq 35$, the projection $p: \text{Diff}(N_g) \to \mathit{Mod}(N_g)$ does not have a section.

$$\left[\begin{array}{l} \text{Lemma: If } p: \text{Diff}_\delta(N_g) \to \mathit{Mod}(N_g), \text{ then} \\ p^*(\zeta_i) = 0 \text{ in } H^{4i}(\text{Diff}_\delta(N_g); \mathbb{Q}) \quad \text{for } i \geq 2. \end{array}\right]$$

Proof: If there was a section

$$\begin{tikzcd} \text{Diff}_\delta(N_g) \arrow[d, "p"'] \\ \mathit{Mod}(N_g) \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} H^*(\text{Diff}_\delta(N_g); \mathbb{Q}) \arrow[d, shift left, "s^*"] \\ H^*(\mathit{Mod}(N_g); \mathbb{Q}) \arrow[u, shift left, "p^*"] \end{tikzcd} \qquad s^*p^*(\zeta_i) = \zeta_i \neq 0$$

$$g \geq 16i + 3$$

For $i = 2$, $\zeta_2 \neq 0$ if $g \geq 16(2) + 3 = 35$,

but by the Lemma $p^*(\zeta_i) = 0$ for $i \geq 2$. $\square$

Farrell Cohomology

Definition: Let Γ gp with $n = vcd(\Gamma) < \infty$, M any Γ-module

$$\widehat{H}^*(\Gamma; M) := H^*\left(Hom_\Gamma(\widehat{P}; M)\right)$$

- $\widehat{H}^i(\Gamma; M) = H^i(\Gamma; M)$ for $i > n$.
- $\widehat{H}^i(\Gamma; M) = 0$ if Γ is torsion-free.
- $\widehat{H}^i(\Gamma; M)$ are torsion groups

$$\widehat{H}^*(\Gamma; \mathbb{Z}) \cong \prod_p \widehat{H}^*(\Gamma; \mathbb{Z})_{(p)}$$

- Γ has p-periodic cohomology if $\widehat{H}^i(\Gamma; \mathbb{Z})_{(p)} \cong \widehat{H}^{i+d}(\Gamma; \mathbb{Z})_{(p)}$
- Brown's Formula:

$$\widehat{H}^*(\Gamma; \mathbb{Z})_{(p)} \cong \prod_{\mathbb{Z}_p \in S} \widehat{H}^*(N(\mathbb{Z}_p); \mathbb{Z})_{(p)}$$

Let Γ gp. of finite vdc and $\pi \leq \Gamma$ of odd prime order p.

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] & \Gamma \arrow[r] & \Gamma/\Delta \arrow[r] & 1 \\ & & \pi \arrow[u] \arrow[r, "\cong"] & \pi \arrow[u] & \end{tikzcd}$$

$$H^*(\pi;\mathbb{F}_p) = E[x_1] \otimes \mathbb{F}_p[u_2]$$

$$H^*(\Gamma;\mathbb{Z}) \longrightarrow H^*(\pi;\mathbb{Z}) \xrightarrow{\text{mod } p} \mathbb{F}_p[u] \subseteq H^*(\pi;\mathbb{F}_p)$$

- $\exists$ a max. $m = m(\pi,\Gamma)$ such that

$$im\Big(H^k(\Gamma;\mathbb{Z}) \to H^k(\pi;\mathbb{Z})\Big) \subseteq \mathbb{F}_p[u^m] \subseteq H^*(\pi;\mathbb{F}_p)$$

- If Γ p-periodic gp, then p-period is given by

$$p(\Gamma) = l.c.m.\{\, 2 \cdot m(\pi,\Gamma) \mid \pi \leq \Gamma \text{ order } p \,\}$$

Theorem (Colin, X.) Let $g > 2$, p odd prime. If $Mod(N_g;k)$ contains p-torsion, then the p-period is 4.

Idea of Proof:

$$
\begin{tikzcd}
GL_2^+(\mathbb{R}) \arrow[r] & GL_2(\mathbb{R}) \\
\widetilde{\pi} \arrow[u, "\text{Faithful rep.}"] \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{Diff}(N_g;1) \arrow[u] \arrow[d] \\
\pi \arrow[r, hook] & \mathit{Mod}(N_g;1)
\end{tikzcd}
\qquad
\begin{tikzcd}
df_{x_0} : T_{x_0}N_g \to T_{x_0}N_g \\
f \arrow[u, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {\color{red}c^2} & & \\
c \arrow[d, mapsto, "\text{First Chern class}"'] & H^*BGL_2^+(\mathbb{R}) \arrow[d] & H^*BGL_2(\mathbb{R}) \arrow[l] \arrow[d] & {\color{red}p_4} \arrow[ull, bend right] \arrow[dd, bend left] \\
0 \neq v & H^*(B\widetilde{\pi}) & H^*B\,\mathrm{Diff}(N_g;1) \arrow[l] & \\
 & H^*(B\pi) \arrow[u, "\cong"] & H^*B\,\mathit{Mod}(N_g;1) \arrow[l] \arrow[u, "\cong"'] & {\color{red}\bullet} \arrow[dll, bend left] \\
 & {\color{red}0 \neq \bullet} & &
\end{tikzcd}
$$

Fixed point data

Fixed point data for diffeo's: Let $\phi \in$ Diff$^+(S_g)$ of order p,

$$< \phi > = \mathbb{Z}/p \circlearrowright S_g$$

- $Sing(<\phi>) = \{x_i\}$ = (finite) set of fixed points
- ϕ acts by rotation on $T_{x_i}(S_g)$ wrt a fixed RS structure
- Let $0 < \beta_i < p$ s.t. ϕ^{β_i} acts by mult. by $e^{2\pi i/p}$

$$\delta(\phi) := (\beta_1, \ldots, \beta_t)$$

- [Nielsen] ϕ_1, ϕ_2 of order p, conjugated $\Leftrightarrow$ $\delta(\phi_1) = \delta(\phi_2)$.
- [Symonds] $\delta(\phi)$ depends only on the isotopy class of ϕ.

So, for $[\phi] \in Mod(S_g)$ $\quad \delta([\phi]) := (\beta_1, \ldots, \beta_t)$

- $[\phi_1], [\phi_2] \in Mod(S_g)$ conjugated $\Leftrightarrow$ $\delta([\phi_1]) = \delta([\phi_2])$.

Non-orientable case: For $\phi \in$ Diff(N_g) of order p

$$\delta(\phi) := (\beta_1, \ldots, \beta_t)$$

- Well defined up to sign.
- $\delta(\phi) \cong \delta(\phi') \;\Leftrightarrow\; (\beta_1, \ldots, \beta_t) = (\varepsilon_1 \beta_1', \ldots, \varepsilon_q \beta_t'), \quad \varepsilon_i = \pm 1$

Non-orientable case, marked points: For $\phi \in$ Diff$(N_g; k)$ of order p

$$\delta_k(\phi) := (\beta_1, \ldots, \beta_k \mid \beta_{k+1}, \ldots, \beta_t)$$

where

- $(\beta_1, \ldots, \beta_k)$ ordered k-tuple, fixed point data of marked points.
- $(\beta_{k+1}, \ldots, \beta_t)$ unordered $(t-k)$-tuple.
- Similar $\cong$ notion.
- Well defined on *Mod*(N_g) and *Mod*$(N_g; k)$.
- $[\phi_1], [\phi_2] \in$ *Mod*$(S_g; k)$ conjugated $\Leftrightarrow$ $\delta_k([\phi_1]) = \delta_k([\phi_2])$.

Theorem (Colin, X.)

1. *Mod*$(N_g; k)$ contains a subgroup of order p if and only if the Riemann-Hurwitz equation

$$g - 2 = p(h - 2) + t(p - 1)$$

has an integer solution with $t \geq k$, $h \geq 1$.

2. For all $g > 2$ and odd prime p, if *Mod*$(N_g; k)$ has p-torsion then it has p-periodic cohomology.

Theorem Let $g > 2$, $k \geq 1$ and $t > 1$ an integer satisfying the equation

$$g - 2 = p(h - 2) + t(p - 1),$$

then,

$$\left\{ \begin{array}{c} \text{Congruence classes of } t\text{-tuples} \\ (1, \beta_2, \ldots, \beta_k \mid \beta_{k+1}, \ldots, \beta_t) \\ \text{with } 0 < \beta_j < p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes of order } p \\ \text{subgps of } \mathit{Mod}(N_g, k) \text{ acting} \\ \text{on } N_g \text{ w/ } t \text{ fixed points} \end{array} \right\}$$

Example: Case $g = p$ Only solution: $(h,t) = (1,2)$

Theorem: Let $\mathbb{Z}_p \leq \mathit{Mod}(N_p;k)$, with $k = 1,2$. Then $N(\mathbb{Z}_p) \cong D_{2p}$ and thus

$$\widehat{H}^i(N(\mathbb{Z}_p);\mathbb{Z})_{(p)} = \begin{cases} \mathbb{Z}_p & i \equiv 0 \mod 4 \\ 0 & i \equiv 1,2,3 \mod 4. \end{cases}$$

Theorem: Let p be an odd prime. Then, for $k = 1,2$

$$\widehat{H}^i(\mathit{Mod}(N_p,k);\mathbb{Z})_{(p)} = \begin{cases} (\mathbb{Z}_p)^{\frac{p-1}{2}} & i \equiv 0 \mod 4 \\ 0 & i \equiv 1,2,3 \mod 4. \end{cases}$$