Hilbert-Blumenthal and Bianchi quaternionic orbifolds

Geometry, Topology, Group Actions, and Singularities in the Americas

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1. **Outline**

The main aim of this work in progress is to study Möbius transformations in $\mathrm{PSL}(2,\mathcal{O})$ i.e., transformations of the form

$$f(\mathbf{q}) = (a\mathbf{q}+b)(c\mathbf{q}+d)^{-1}$$

where $\mathbf{q}$ belongs to a quaternion K-algebra B of a number field K , $a, b, c, d \in \mathcal{O}$ where $\mathcal{O}$ is an order in the ring of integers of K. number field. We try to generalize in this setting the classical modular groups, Hilbert-Blumenthal and Bianchi groups and the geometric properties of their actions on 4 and 5 dimensional hyperbolic spaces and products of these spaces.

1) Juan Pablo Díaz, Alberto Verjovsky, and Fabio Vlacci. **Quaternionic Kleinian modular groups and arithmetic hyperbolic orbifolds over the quaternions.** *Geom. Dedicata*, 192:127–155, 2018.

2) Joseph Quinn and Alberto Verjovsky. **Cusp shapes of Hilbert-Blumenthal surfaces.** *Geom. Dedicata*, 206:27–42, 2020.

2. **Quaternion algebras**

2.1 - General definitions

Let K be a field of characteristic 0. We will say that a K-algebra $\mathcal{B}$ is a *quaternion algebra* over K, if there exist $\mathbf{i}, \mathbf{j} \in \mathcal{B}$ such that $\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ is a K-basis for $\mathcal{B}$ and

$$\mathbf{i}^2 = a, \ \mathbf{j}^2 = b \text{ and } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad (\text{so that } \mathbf{k}^2 = -ab)$$

for some $a, b \in K^\times$. The quaternion algebra $\mathcal{B}$ is usually denoted by $\left(\frac{a,b}{K}\right)$ or by $(a, b \mid K)$.

In particular, when $K = \mathbb{R}$ and $a = b = -1$ (then $\mathbf{i} = \sqrt{-1} = i$, $\mathbf{j} = \sqrt{-1} = j$ and $\mathbf{ij} = ij = k$), the quaternion algebra $(-1, -1 \mid \mathbb{R})$ is the classical algebra of *Hamilton's quaternions* which is usually denoted by $\mathbb{H}$.

More generally, if $\mathcal{B} = (a, b \mid \mathbb{R})$ is a quaternion algebra over $\mathbb{R}$, then $\mathcal{B} \simeq \mathrm{M}_2(\mathbb{R})$ or $\mathcal{B} \simeq \mathbb{H}$ (this last case occurs if and only if $a, b < 0$). On the other hand, when $K = \mathbb{C}$ we have that $\mathcal{B} = (a, b \mid \mathbb{C}) \simeq \mathrm{M}_2(\mathbb{C})$ for all $a, b \in \mathbb{C}^\times$.

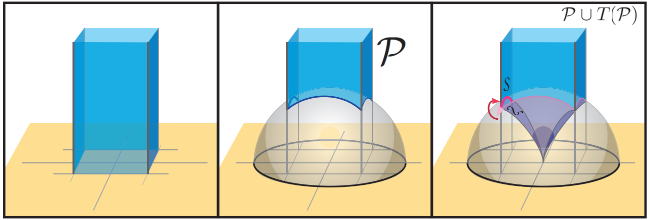

Figure 1. Schematic picture of the chimney which is the fundamental domain of the parabolic group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ (generated by the translations $\tau_{\mathbf{i}}$, $\tau_{\mathbf{j}}$ and $\tau_{\mathbf{k}}$), the polytope $\mathcal{P}$ and the polytope $\mathcal{P}$ and its inversion $T(\mathcal{P})$. The horizontal plane represents the purely imaginary quaternions that forms the ideal boundary $\partial\mathbf{H}^1_{\mathbb{H}} \simeq \mathbb{S}^3$ and above it the open half-space of quaternions with positive real part $\mathbf{H}^1_{\mathbb{H}}$. In the same fashion the ideal boundary of hyperbolic 5-space $\partial\mathbf{H}^5_{\mathbb{R}} \simeq \mathbb{S}^4$

Inspired by the complex conjugation we can define a *standard involution* on $\mathcal{B}$ given by the map

$$^{-} : \mathcal{B} \longrightarrow \mathcal{B}$$

$$\mathbf{q} = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{ij} \longmapsto \overline{\mathbf{q}} = t - x\mathbf{i} - y\mathbf{j} - z\mathbf{ij}.$$

The existence of such involution allows us to define a *reduced trace* $\operatorname{trd} : \mathcal{B} \to K$ by $\operatorname{trd}(\mathbf{q}) := \mathbf{q} + \overline{\mathbf{q}}$ and a *reduced norm* $\operatorname{nrd} : \mathcal{B} \to K$ by $\operatorname{nrd}(\mathbf{q}) := \mathbf{q}\overline{\mathbf{q}}$. Then, we can define $\Re(\mathbf{q}) := \frac{\operatorname{trd}(\mathbf{q})}{2}$ and $|\mathbf{q}| := \operatorname{nrd}(\mathbf{q})$. We remark that the reduced trace is K-linear and the reduced norm is multiplicative on $\mathcal{B}^{\times}$.

Of particular interest to us will be the K-subspace of *pure* elements of $\mathcal{B}$ defined as

$$\mathcal{B}^0 := \{\mathbf{q} \in \mathcal{B} \mid \operatorname{trd}(\mathbf{q}) = 0\}$$

and the normal subgroup

$$\mathcal{B}^1 := \{\mathbf{q} \in \mathcal{B}^\times \mid \operatorname{nrd}(\mathbf{q}) = 1\}$$

of $\mathcal{B}^\times$ of elements of reduced norm 1. When $\mathcal{B} \simeq \mathbb{H}$, we have that $\mathbb{H}^0 \simeq \mathbb{R}^3$ is the subspace of classical *pure Hamiltonians* and $\mathbb{H}^1$ is the classical subgroup of *unit Hamiltonians.*

Remark 1. As a set, the unit Hamiltonians are naturally identified with the 3-sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. The group $\mathbb{H}^1$ acts by rotation on $\mathbb{H}^0 \simeq \mathbb{R}^3$ (on the left) via conjugation $\omega \mapsto \mathbf{q}\omega\mathbf{q}^{-1}$. This action defines a group homomorphism $\mathbb{H}^1 \to \mathrm{SO}(3)$, fitting into the following exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbb{H}^1 \longrightarrow \mathrm{SO}(3) \longrightarrow 1.$$

The interpretation of the quotient group $\mathbb{H}^1/\{\pm 1\}$ as the group of rotations of $\mathbb{R}^3$ is very useful to determine certain algebraic substructures in $\mathbb{H}^1$.

For example, from the classification of finite groups of $\mathrm{SO}(3)$ we have that each finite subgroup of $\mathbb{H}^1$, is isomorphic to one of the following groups:

i. a cyclic group C_n of order n generated by $s_n = \cos(2\pi/n) + i\sin(2\pi/n)$;

ii. a *binary dihedral (dicyclic)* group Q_{4n} of order $4n$ generated by s_{2n} and j;

iii. the *binary tetrahedral* group $2T$ of order 24 with presentation given by

$$\langle r, s, t \mid r^2 = s^3 = t^3 = rst = 1\rangle$$

where $r = i$, $s = \frac{1}{2}(1 + i + j + k)$ and $t = \frac{1}{2}(1 + i + j - k)$;

iv. the *binary octahedral* group $2O$ of order 48 with presentation given by

$$\langle r, s, t \mid r^2 = s^3 = t^4 = rst = 1\rangle$$

where $r = \frac{1}{\sqrt{2}}(i + j)$, $s = \frac{1}{2}(1 + i + j + k)$ and $t = \frac{1}{\sqrt{2}}(1 + i)$; or

v. the *binary icosahedral* group $2I$ of order 120 with presentation given by

$$\langle s, t \mid (st)^2 = s^3 = t^5 = rst = 1 \rangle$$

where $s = \frac{1}{2}(1 + i + j + k)$, $t = \frac{1}{2}(\varphi + \varphi^{-1}i + j)$ and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

[Marie-France Vignéras. *Arithmétique des algèbres de quaternions*, volume 800 of Lect. Notes Math. Springer, Cham, 1980.]

2.2 - Quaternion orders over quadratic fields

Let $K = \mathbb{Q}(\sqrt{n})$ be a quadratic field, for some square-free $n \in \mathbb{N}$, and denote by σ the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$, which acts on $\alpha+\beta\sqrt{n} \in K$ as $\sigma(\alpha+\beta\sqrt{n}) = \alpha-\beta\sqrt{n}$. Let $\mathbb{Z}_K = \mathbb{Z}[\theta] := \mathbb{Z}\oplus\mathbb{Z}\theta$ be the ring of integers of K, where

$$\theta = \begin{cases} \sqrt{n} & \text{if } n \not\equiv 1 \mod 4 \\ \frac{1+\sqrt{n}}{2} & \text{if } n \equiv 1 \mod 4 \end{cases}.$$

Given a place ν of K, we denote by K_ν the completion of K at ν, which is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (at the archimedean places) or to a finite extension of $\mathbb{Q}_p$ (at the non-archimedean places). We say that a quaternion algebra $\mathcal{B}$ over K is *totally definite* if for all real archimedian places ν of K, $\mathcal{B}_\nu := \mathcal{B}\otimes_{\mathbb{Q}} K_\nu \simeq \mathbb{H}$. We remark that, when ν is a complex place, we have that $\mathcal{B}_\nu = \mathcal{B}\otimes_{\mathbb{Q}}\mathbb{C} \simeq \mathrm{M}_2(\mathbb{C})$. Therefore, if $\mathcal{B}$ is a totally definite quaternion algebra over K, then K is necessarily a real quadratic field. In particular, we will denote by $\mathcal{B}_K$ the totally definite quaternion algebra $(-1, -1 \mid K)$ defined over the real quadratic field K.

Definition 2.1. A $\mathbb{Z}_K$-*order* $\mathcal{O}$, in a quaternion algebra $\mathcal{B}$ over K, is a 4-dimensional $\mathbb{Z}_K$-lattice in $\mathcal{B}$ that is also a ring with unity. Moreover, $\mathcal{O}$ is called *maximal* if no other $\mathbb{Z}_K$-order properly contains it.

Remark 2. Maximal $\mathbb{Z}_K$-orders are analogous to rings of integers of number fields but an important difference is that rings of integers are unique, while a quaternion algebra can have many maximal $\mathbb{Z}_K$-orders. For example, if $\mathcal{O} \subseteq \mathcal{B}$ is a maximal $\mathbb{Z}_K$-order and $\mathbf{q} \in \mathcal{B}^\times$, then $\mathbf{q}\mathcal{O}\mathbf{q}^{-1} \subseteq \mathcal{B}$ is a maximal $\mathbb{Z}_K$-order, but as $\mathcal{B}$ is noncommutative, we may have $\mathbf{q}\mathcal{O}\mathbf{q}^{-1} \neq \mathcal{O}$.

Example 1. Let $a, b \in \mathbb{Z}_K \backslash \{0\}$ and $\mathcal{B} = (a, b \mid K)$. The most natural example of a $\mathbb{Z}_K$-order is the *standard order* in $\mathcal{B}$

$$\mathcal{O}_{\mathcal{B}} := \mathbb{Z}_K \oplus \mathbb{Z}_K \mathbf{i} \oplus \mathbb{Z}_K \mathbf{j} \oplus \mathbb{Z}_K \mathbf{ij}.$$

In particular, when $\mathcal{B} = \mathcal{B}_K$ (the totally definite quaternion algebra $(-1, -1 \mid K)$ defined over the real quadratic field K) , $\mathbb{H}(\mathbb{Z}_K) := \mathcal{O}_{\mathcal{B}_K} = \mathbb{Z}_K \oplus \mathbb{Z}_K i \oplus \mathbb{Z}_K j \oplus \mathbb{Z}_K k$ is properly contained in the quaternion $\mathbb{Z}_K$-order

$$\mathrm{Hur}(\mathbb{Z}_K) := \mathbb{Z}_K \oplus \mathbb{Z}_K i \oplus \mathbb{Z}_K j \oplus \mathbb{Z}_K \xi,$$

where $\xi = \frac{1+i+j+k}{2}$, showing that $\mathbb{H}(\mathbb{Z}_K)$ is never a maximal $\mathbb{Z}_K$-order in $\mathcal{B}_K$. These orders generalize the rings of Lipschiz and Hurwitz integers $\mathbb{H}(\mathbb{Z})$ and $\mathbb{H}ur(\mathbb{Z})$ studied in

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However, in contrast to the maximality of $\mathbb{H}ur(\mathbb{Z})$ in $(-1,-1 \mid \mathbb{Q})$, $\mathrm{Hur}(\mathbb{Z}_K)$ is not always maximal in $\mathcal{B}_K$.

For example, if $K = \mathbb{Q}(\sqrt{2})$ we can define the *binary octahedral order* of $\mathcal{B}_{\mathbb{Q}(\sqrt{2})}$ as

$$\mathcal{O}_{\mathfrak{O}} := \mathbb{Z}[\sqrt{2}] \oplus \mathbb{Z}[\sqrt{2}]\eta \oplus \mathbb{Z}[\sqrt{2}]\delta \oplus \mathbb{Z}[\sqrt{2}]\eta\delta,$$

where $\eta = \frac{1+i}{\sqrt{2}}$ and $\delta = \frac{1+j}{\sqrt{2}}$, which properly contains $\mathrm{Hur}(\mathbb{Z}[\sqrt{2}])$.

Similarly, when $K = \mathbb{Q}(\sqrt{5})$, we can define the *binary icosahedral order* of $\mathcal{B}_{\mathbb{Q}(\sqrt{5})}$ as

$$\mathcal{O}_{\mathfrak{I}} := \mathbb{Z}[\varphi] \oplus \mathbb{Z}[\varphi]i \oplus \mathbb{Z}[\varphi]\zeta \oplus \mathbb{Z}[\varphi]i\zeta,$$

where φ is the golden ratio and $\zeta = \frac{\varphi+\varphi^{-1}i+j}{2}$, which properly contains $\mathrm{Hur}(\mathbb{Z}[\varphi])$. In fact, $\mathcal{O}_{\mathfrak{O}}$ and $\mathcal{O}_{\mathfrak{I}}$ are maximal $\mathbb{Z}_K$-orders of $\mathcal{B}_{\mathbb{Q}(\sqrt{2})}$ and $\mathcal{B}_{\mathbb{Q}(\sqrt{5})}$ respectively.

2.3 - Unit groups of quaternion orders

In number theory a very important subgroup of $\mathbb{Z}_K$ is the unit group $\mathbb{Z}_K^\times$. When K is a real quadratic field, by Dirichlet's unit theorem, we have that

$$\mathbb{Z}_K^\times = \{\pm\varepsilon^\ell : \ell \in \mathbb{Z}\}, \tag{1}$$

where ε is the fundamental unit of $\mathbb{Z}_K$ normalized so that $\varepsilon > 1$ for the canonical embedding $K \hookrightarrow \mathbb{R}$.

A description of the *unit group* $\mathcal{O}^\times$ of a $\mathbb{Z}_K$-order $\mathcal{O}$ in an arbitrary quaternion algebra is more complicated. However, as we are only interested in $\mathbb{Z}_K$-orders in a totally definite quaternion algebras $\mathcal{B}$ over K, we can describe $\mathcal{O}^\times$ in terms of $\mathbb{Z}_K^\times$ and the *torsion group* $\mathcal{O}^1 := \{\mathbf{u} \in \mathcal{O}^\times : \mathrm{nrd}(\mathbf{u}) = 1\}$ (which is a finite subgroup of $\mathcal{O}^\times$ of $\mathcal{O}$ as follows:

Let $K_+^\times := \{w \in K^\times : w > 0 \text{ and } \sigma(w) > 0\}$ be the group of totally positive elements of $K^\times$ and $\mathbb{Z}_{K+}^\times := \mathbb{Z}_K^\times \cap K_+^\times$ be the group of totally positive units in $\mathbb{Z}_K$. As $\mathrm{nrd}(\mathcal{O}) \subseteq \mathbb{Z}_K$ and $\mathrm{nrd}(\mathcal{B}^\times) \subseteq K_+^\times$, we have that $\mathrm{nrd}(\mathcal{O}^\times) \subseteq \mathbb{Z}_{K+}^\times$. Then, the reduced norm induces the following exact sequence

$$1 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O}^\times \xrightarrow{\mathrm{nrd}} \mathbb{Z}_{K+}^\times$$

which implies that $\mathcal{O}^1$ is a normal subgroup of $\mathcal{O}^\times$. On the other hand, as $Z(\mathcal{B}^\times) = K^\times$, we have that $\mathbb{Z}_K^\times \subseteq Z(\mathcal{O}^\times)$, which implies that $\mathbb{Z}_K^\times$ is a normal subgroup of $\mathcal{O}^\times$. Thus, $\mathbb{Z}_K^\times \mathcal{O}^1$ is a normal subgroup of $\mathcal{O}^\times$ and, as $\mathrm{nrd}(\mathbb{Z}_K^\times \mathcal{O}^1) = \mathbb{Z}_K^{\times 2}$, we have the following embedding

$$\mathcal{O}^\times / \mathbb{Z}_K^\times \mathcal{O}^1 \hookrightarrow \mathbb{Z}_{K+}^\times / \mathbb{Z}_K^{\times 2}.$$

Since K is a real quadratic field, it is well known that $\mathbb{Z}_K^{\times 2} = \langle \varepsilon^2 \rangle$ and

$$\mathbb{Z}_{K+}^\times = \begin{cases} \langle \varepsilon \rangle & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = 1 \\ \langle \varepsilon^2 \rangle & \text{if } N_{K/\mathbb{Q}}(\varepsilon) = -1. \end{cases}$$

Then, if $N_{K/\mathbb{Q}}(\varepsilon) = -1$, we conclude that $\mathcal{O}^\times \simeq \mathbb{Z}_K^\times \mathcal{O}^1$ and if $N_{K/\mathbb{Q}}(\varepsilon) = 1$, we conclude that $\mathcal{O}^\times$ is isomorphic to $\mathbb{Z}_K^\times \mathcal{O}^1$ or to a degree two extension of $\mathbb{Z}_K^\times \mathcal{O}^1$. Finally, as K is a totally real field, the finite subgroup $\mathcal{O}^1$ is embedded in $\mathbb{H}^1 \simeq \mathbb{S}^3$. Then, we can obtain an explicit description of $\mathcal{O}^1$ (and then of $\mathcal{O}^\times$) by using Dirichlet's unit theorem (1) and Remark 1. The binary octahedral group and binary icosahedral group appear only when $K = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(\sqrt{5})$, respectively.

2.4 - Linear groups with coefficients in a quaternion order

Let $\mathrm{M}_2(\mathbb{H})$ be the $\mathbb{H}$-vector space (right or left, according to the setting) of 2×2 matrices with entries in $\mathbb{H}$. It can be proven that, all right-invertible matrices in $\mathrm{M}_2(\mathbb{H})$ are also left-invertible [2, Proposition 2.3]. Then, we can define the *general linear group* $\mathrm{GL}_2(\mathbb{H})$ as the set of all invertible matrices of $\mathrm{M}_2(\mathbb{H})$.

Recall that the *Dieudonné determinant* of a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{H})$ is defined as the non negative real number

$$\det_{\mathbb{H}}(\gamma) := \sqrt{|a|^2|d|^2 + |c|^2|b|^2 - 2\Re(c\overline{a}b\overline{d})}.$$

It can be proven that a matrix $\gamma \in \mathrm{M}_2(\mathbb{H})$ is invertible if and only if $\det_{\mathbb{H}}(\gamma) \neq 0$. Then, $\mathrm{GL}_2(\mathbb{H})$ is precisely the set of all matrices in $\mathrm{M}_2(\mathbb{H})$ having non zero Dieudonné determinant and we can define the *special linear group* $\mathrm{SL}_2(\mathbb{H})$ as the set of all matrices in $\mathrm{GL}_2(\mathbb{H})$ with Dieudonné determinant 1.

Let $K = \mathbb{Q}(\sqrt{n})$ be a real quadratic field, which can be embedded in $\mathbb{R}$. Then, since the quaternion algebra $\mathcal{B}_K$ is totally definite, it can be embedded in $\mathbb{H}$ and, by restriction, we can embed any quaternion $\mathbb{Z}_K$-order $\mathcal{O}$ of $\mathcal{B}_K$ in $\mathbb{H}$. Let $\mathrm{SL}_2(\mathcal{O})$ be the subset of all matrices in $\mathrm{SL}_2(\mathbb{H})$ with coefficients in $\mathcal{O}$ and $\mathrm{PSL}_2(\mathcal{O}) := \mathrm{SL}_2(\mathcal{O})/\{\pm\mathcal{I}\}$, where $\mathcal{I}$ denotes the identity matrix of size 2. It can be shown that $\mathrm{SL}_2(\mathcal{O})$ is a group and therefore $\mathrm{PSL}_2(\mathcal{O})$ is also a group.

Lemma 2.1. $\mathrm{SL}_2(\mathcal{O})$ *is a subgroup of* $\mathrm{SL}_2(\mathbb{H})$.

Proof. It is clear that the right product of matrices in $\mathrm{SL}_2(\mathcal{O})$ is well defined, associative and $\mathcal{I} \in \mathrm{SL}_2(\mathcal{O})$ because $\mathcal{O}$ is a ring with unity. Then, we only need to prove that the right-inverse of each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ lies in $\mathrm{SL}_2(\mathcal{O})$.

First assume that $abcd \neq 0$, then one can show that,

$$\gamma^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix}$$

with a^{-1}, b^{-1}, c^{-1}, d^{-1} not necessarily in $\mathcal{O}$. Using the multiplicativity of the norm one proves that

$$|a|^2|d - ca^{-1}b|^2 = |b|^2|c - db^{-1}a|^2 =$$

$$|c|^2|b - ac^{-1}d|^2 = |d|^2|a - bd^{-1}c|^2 = |a|^2|d|^2 + |c|^2|b|^2 - 2\Re(c\overline{a}b\overline{d}) = 1.$$

Then, as $\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}$ and $\overline{\mathbf{q}}^{-1} = \frac{\mathbf{q}}{|\mathbf{q}|^2}$,

$$\gamma^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{(\overline{a} - \overline{bd}^{-1}\overline{c})}{|a - bd^{-1}c|^2} & \frac{(\overline{c} - \overline{db}^{-1}\overline{a})}{|c - db^{-1}a|^2} \\ \frac{(\overline{b} - \overline{ac}^{-1}\overline{d})}{|b - ac^{-1}d|^2} & \frac{(\overline{d} - \overline{ca}^{-1}\overline{b})}{|d - ca^{-1}b|^2} \end{pmatrix} = \begin{pmatrix} \frac{|d|^2}{|d|^2}\frac{(\overline{a} - \overline{bd}^{-1}\overline{c})}{|a - bd^{-1}c|^2} & \frac{|b|^2}{|b|^2}\frac{(\overline{c} - \overline{db}^{-1}\overline{a})}{|c - db^{-1}a|^2} \\ \frac{|c|^2}{|c|^2}\frac{(\overline{b} - \overline{ac}^{-1}\overline{d})}{|b - ac^{-1}d|^2} & \frac{|a|^2}{|a|^2}\frac{(\overline{d} - \overline{ca}^{-1}\overline{b})}{|d - ca^{-1}b|^2} \end{pmatrix}$$

$$= \begin{pmatrix} |d|^2(\overline{a} - \overline{bd}^{-1}\overline{c}) & |b|^2(\overline{c} - \overline{db}^{-1}\overline{a}) \\ |c|^2(\overline{b} - \overline{ac}^{-1}\overline{d}) & |a|^2(\overline{d} - \overline{ca}^{-1}\overline{b}) \end{pmatrix} =$$

$$\begin{pmatrix} |d|^2(\overline{a} - \overline{b}\frac{d}{|d|^2}\overline{c}) & |b|^2(\overline{c} - \overline{d}\frac{b}{|b|^2}\overline{a}) \\ |c|^2(\overline{b} - \overline{a}\frac{c}{|c|^2}\overline{d}) & |a|^2(\overline{d} - \overline{c}\frac{a}{|a|^2}\overline{b}) \end{pmatrix} =$$

$$\begin{pmatrix} |d|^2\overline{a} - \overline{b}d\overline{c} & |b|^2\overline{c} - \overline{d}b\overline{a} \\ |c|^2\overline{b} - \overline{a}c\overline{d} & |a|^2\overline{d} - \overline{c}a\overline{b} \end{pmatrix}$$

with all its coefficients in $\mathcal{O}$.

Now assume that one of the entries of γ is 0. For example, if $a = 0$ and $bc \neq 0$ it follows that

$$\gamma^{-1} = \begin{pmatrix} -c^{-1}db^{-1} & c^{-1} \\ b^{-1} & 0 \end{pmatrix}$$

with b^{-1} and c^{-1} not necessarily in $\mathcal{O}$. Then, as the Dieudonné determinant is equal to 1, we have that $|c|^2|b|^2 = 1$ and

$$\gamma^{-1} = \begin{pmatrix} -c^{-1}db^{-1} & c^{-1} \\ b^{-1} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{\overline{c}}{|c|^2} d \frac{\overline{b}}{|b|^2} & \frac{\overline{c}}{|c|^2} \\ \frac{\overline{b}}{|b|^2} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{1}{|c|^2|b|^2}\overline{c}d\overline{b} & \frac{\overline{c}}{|c|^2}\frac{|b|^2}{|b|^2} \\ \frac{|c|^2}{|c|^2}\frac{\overline{b}}{|b|^2} & 0 \end{pmatrix} = \begin{pmatrix} -\overline{c}d\overline{b} & |b|^2\overline{c} \\ |c|^2\overline{b} & 0 \end{pmatrix}$$

with all its coefficients in $\mathcal{O}$. The other cases are analogous. $\square$

3. **Quaternionic modular groups**

3.1 - Quaternionic Möbius transformations

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{H})$. We define the *quaternionic Möbius transformation* associated to γ as the real analytic function

$$F_\gamma : \mathbb{H} \cup \{\infty\} \longrightarrow \mathbb{H} \cup \{\infty\} = \mathbb{S}^4$$

defined by

$$F_\gamma(\mathbf{q}) := (a\mathbf{q} + b) \cdot (c\mathbf{q} + d)^{-1}, \qquad (2)$$

were we set $F_\gamma(\infty) = \infty$ if $c = 0$, $F_\gamma(\infty) = ac^{-1}$ if $c \neq 0$, and $F_\gamma(-c^{-1}d) = \infty$.

The Möbius transformation associated to γ is an orientation-preserving conformal diffeomorphism of the 4-sphere with its standard metric.

Two elements γ_1 and γ_2 determine the same Möbius transformation if and only it there exists $t > 0$ so that $\gamma_1 = tI\gamma_2$ where $I \in \mathrm{GL}_2(\mathbb{H})$ is the identity matrix. therefore,

$$Conf_+(\mathbb{S}^4) = \mathrm{GL}_2(\mathbb{H})/\{tI,\, t > 0\} \overset{def}{=} \mathrm{PSL}(2, \mathbb{H}).$$

where $Conf_+(\mathbb{S}^4)$ denotes the group of conformal diffeomorphisms of the round 4-sphere. By Poincaré's extension theorem, any conformal diffeomorphism $f : \mathbb{S}^4 \to \mathbb{S}^4$ extends to an isometry $\tilde{f} : \mathbb{B}^5 \to \mathbb{B}^5$ Of the interior of the closed 5-ball $\mathbb{B}^5 \subset \mathbb{R}^5$ where the metric on the open 5-ball is the Poincaré hyperbolic metric.

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_5^2)}{(1 - (x_1^2 + \cdots + x_5^2))^2}$$

Terefore $\mathrm{PSL}(2, \mathbb{H}) = Isom_+(\mathbb{B}^5) = Isom_+(\mathbf{H}^5_{\mathbb{R}})$

Let $b, c \in \mathbb{H}$, $c \neq 0$. We define the *left homothetic transformation* $h_c : \mathbb{H} \to \mathbb{H}$ as the map $\mathbf{q} \mapsto c\mathbf{q}$, the *translation* $T_b : \mathbb{H} \to \mathbb{H}$ as the map $\mathbf{q} \mapsto \mathbf{q} + b$, and the *inversion* I as the map $\mathbf{q} \mapsto \mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}$.

As in the complex case, every quaternionic Möbius transformation is a composition of homotheties, translations and inversions. More precisely $F_\gamma(\mathbf{q})$ can be decomposed as follows:

$$\mathbf{q} \xrightarrow{T_{c^{-1}d}} (\mathbf{q} + c^{-1}d) \xrightarrow{h_c} c\mathbf{q} + d \xrightarrow{I}$$

$$(c\mathbf{q} + d)^{-1} \xrightarrow{h_{b-ac^{-1}d}} (b - ac^{-1}d)(c\mathbf{q} + d)^{-1} \xrightarrow{T_{ac^{-1}}} (b - ac^{-1}d)(c\mathbf{q} + d)^{-1} + ac^{-1}$$
$$= (b - ac^{-1}d)(c\mathbf{q} + d)^{-1} + ac^{-1}(c\mathbf{q} + d)(c\mathbf{q} + d)^{-1} = (a\mathbf{q} + b)(c\mathbf{q} + d)^{-1}.$$

Therefore

$$F_\gamma = T_{ac^{-1}} \circ h_{b-ac^{-1}d} \circ I \circ h_c \circ T_{c^{-1}d}. \tag{3}$$

On the other hand, we define the half-space model of the *one dimensional quaternionic hyperbolic space* as

$$\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \mid \Re(\mathbf{q}) > 0\} \subseteq \mathbb{H}.$$

This space is isometric to the hyperbolic real space $\mathbf{H}^4_{\mathbb{R}} := \{(t, x, y, z) \in \mathbb{R}^4 \mid t > 0\}$ of dimension 4 with the Poincaré metric

$$ds^2 = \frac{dt^2 + dx^2 + dy^2 + dz^2}{t^2}.$$

Let $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \subseteq \mathrm{PSL}_2(\mathbb{H})$ be the subgroup of quaternionic Möbius transformation leave invariant $\mathbf{H}^1_{\mathbb{H}}$. It can be prove that any $F_\gamma \in \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is conformal and preserves orientation, moreover it is an isometry of $\mathbf{H}^1_{\mathbb{H}}$. Then, $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is isomorphic to the groups $Conf_+(\mathbf{H}^1_{\mathbb{H}})$ and $Isom_+(\mathbf{H}^1_{\mathbb{H}})$ of conformal diffeomorphisms and isometries orientation-preserving of the half-space model $\mathbf{H}^1_{\mathbb{H}}$.

Moreover, $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ acts by orientation-preserving conformal transformations on the sphere at infinity of the hyperbolic 4-space defined as $\partial\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \mid \Re(\mathbf{q}) = 0\} \cup \{\infty\}$. Then $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \cong Conf_+(\mathbb{S}^3)$.

The subgroup $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \subseteq \mathrm{PSL}_2(\mathbb{H})$ can be characterized as the group induced by matrices which satisfy one of the following equivalent *BG-conditions* (Bisi-Gentili), introduced by Bisi and Gentili (which, in turn, are a variation of the conditions described by Ahlfors):

$$\left\{\gamma \in \mathrm{PSL}_2(\mathbb{H}) \mid \overline{\gamma}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \tag{4}$$

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{H}) \mid \Re(a\overline{c}) = 0,\ \Re(b\overline{d}) = 0,\ \overline{b}c + \overline{d}a = 1\right\}, \tag{5}$$

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{H}) \mid \Re(c\overline{d}) = 0,\ \Re(a\overline{b}) = 0,\ a\overline{d} + b\overline{c} = 1\right\}. \tag{6}$$

An important subgroup of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is the *affine subgroup* $\mathcal{A}(\mathbb{H})$ consisting of transformations which are induced by matrices of the form $\begin{pmatrix} \lambda a & b \\ 0 & \lambda^{-1} a \end{pmatrix}$ with $|a| = 1$, $\lambda > 0$ and $\Re(\bar{b}a) = 0$ (which clearly satisfy BG-conditions). The group $\mathcal{A}(\mathbb{H})$ is the maximal subgroup of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ which fixes the point at infinity and any $F_\gamma \in \mathcal{A}(\mathbb{H})$ acts as a conformal transformation on $\partial \mathbf{H}^1_{\mathbb{H}}$. Moreover $\mathcal{A}(\mathbb{H})$ is the group of conformal and orientation preserving transformations acting on the space of pure quaternions at infinity which can be identified with $\mathbb{R}^3$ so that $\mathcal{A}(\mathbb{H}) \cong Conf_+(\mathbb{R}^3)$.

A useful decomposition of the elements of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is the *Iwasawa decomposition* which states that every $\gamma \in \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ can be written in the form

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{7}$$

where $\lambda \in \mathbb{R}^+$, $\omega \in \mathbb{H}^0$, and $\alpha, \beta \in \mathbb{H}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$ and $\Re(\alpha\overline{\beta}) = 0$. The first matrix is a homothety fixing 0 and ∞, the second matrix is a parabolic translation fixing ∞ in the direction of ω and the third matrix is a 4-dimensional rotation. In fact, the set of all matrices of the form $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, is isomorphic to the special orthogonal group SO(4) which is a real compact Lie groups of dimension 6. Then, we can deduce that the set of matrices of $\mathrm{PSL}_2(\mathbb{H})$ satisfying the BG-condition has real dimension 10.

3.2 - The Bianchi quaternionic modular group

If $\mathcal{O}$ an order of $(-1,-1 \mid \mathbb{Q})$ let $\mathrm{PSL}_2(\mathcal{O}) \subset \mathrm{PSL}_2(\mathbb{H})$ denote the subgroup

$$PSL_2(\mathcal{O}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{H}) \mid a, b, c, d \in \mathcal{O} \right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{H})$, then its Poincaré extension is given explicitly in the upper half-space model $\mathbf{H}^5_{\mathbb{R}} := \mathbb{H} \times \mathbb{R}_{>0}$ as follows:

$$\overline{\gamma}(\mathbf{q}, t) = \left(\left(\frac{1}{|c\mathbf{q} + d|^2 + |c|^2 t^2} \right) ((a\mathbf{q} + b)(\overline{\mathbf{q}c} + \overline{d}) + a\overline{c}t^2), \frac{\det_{\mathbb{H}}(\gamma)\mathrm{t}}{|c\mathbf{q} + d|^2 + |c|^2 t^2} \right).$$

Since $PSL_2(\mathcal{O})$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{H})$ it follows from standard facts about Kleinan groups that $PSL_2(\mathcal{O})$ acts properly and discontinuously on hyperbolic 5-space $\mathbf{H}^5_{\mathbb{R}}$.

3.3 - The Hilbert-Blumenthal quaternionic modular group

Let $K = \mathbb{Q}(\sqrt{n})$ be a real quadratic field, $\mathbb{Z}_K$ its ring of integers and $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in the totally definite quaternion algebra $\mathcal{B}_K$. It is well known that the embedding $\mathbb{Z}_K \hookrightarrow \mathbb{R}$ is not discrete, then (contrary to the previous case) we cannot discreetly embed $\mathcal{O}$ into $\mathbb{H}$, and consequently $\mathrm{PSL}_2(\mathcal{O})$ is not a discrete subgroup of $\mathrm{PSL}_2(\mathbb{H})$. However, $\mathbb{Z}_K$ admits a discrete embedding

$$\mathbb{Z}_K \hookrightarrow \mathbb{R} \times \mathbb{R},$$

via the Galois twist $w = \alpha + \beta\sqrt{n} \mapsto (w, \sigma(w)) = (\alpha + \beta\sqrt{n}, \alpha - \beta\sqrt{n})$, which induce a discrete embedding

$$\mathcal{O} \hookrightarrow \mathbb{H} \times \mathbb{H}, \tag{8}$$

given by $\mathbf{q} \mapsto (\mathbf{q}, \sigma(\mathbf{q})) = \big(t + xi + yj + zk,\ \sigma(t) + \sigma(x)i + \sigma(y)j + \sigma(z)k\big)$, and finally (8) extend to a discrete embedding

$$\mathrm{PSL}_2(\mathcal{O}) \hookrightarrow \mathrm{PSL}_2(\mathbb{H}) \times \mathrm{PSL}_2(\mathbb{H}). \tag{9}$$

Then, we can identify the elements of $\mathrm{PSL}_2(\mathcal{O})$ with their image under (9).

Remark 3. We remark that the embedding (8) is more natural from the point of view of Minkowski's geometry of numbers in the sense that as $\mathcal{B}_K$ is a totally definite quaternion algebra over a quadratic real field then $\mathcal{B}_K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \times \mathbb{H}$.

Let $\mathbf{H}^2_{\mathbb{H}} := \mathbf{H}^1_{\mathbb{H}} \times \mathbf{H}^1_{\mathbb{H}} \subseteq \mathbb{H} \times \mathbb{H}$ which is isometric to $\mathbf{H}^4_{\mathbb{R}} \times \mathbf{H}^4_{\mathbb{R}}$ with the Riemannian product metric of Poincaré metrics. As expected, $\mathrm{PSL}_2(\mathbb{H}) \times \mathrm{PSL}_2(\mathbb{H})$ acts on $\mathbb{H} \times \mathbb{H}$ by quaternionic Möbius transformations $(\gamma_1, \gamma_2) \cdot (\mathbf{q}_1, \mathbf{q}_2) = (F_{\gamma_1}(\mathbf{q}_1), F_{\gamma_2}(\mathbf{q}_2))$ but not all $(\gamma_1, \gamma_2) \in \mathrm{PSL}_2(\mathbb{H}) \times \mathrm{PSL}_2(\mathbb{H})$ leaves invariant $\mathbf{H}^2_{\mathbb{H}}$. As in the 1-dimensional case §3.1, the set $\mathcal{M}_{\mathbf{H}^2_{\mathbb{H}}} \subseteq \mathrm{PSL}_2(\mathbb{H}) \times \mathrm{PSL}_2(\mathbb{H})$ of couples (γ_1, γ_2) of matrices that leaves invariant $\mathbf{H}^2_{\mathbb{H}}$ is isomorphic to $Conf_+(\mathbf{H}^2_{\mathbb{H}})$ and $Isom_+(\mathbf{H}^2_{\mathbb{H}})$, and it can be characterized as the set of $(\gamma_1, \gamma_2) \in \mathrm{PSL}_2(\mathbb{H}) \times \mathrm{PSL}_2(\mathbb{H})$ such that both γ_1 and γ_2 satisfy the BG-conditions. Moreover, $\mathcal{M}_{\mathbf{H}^2_{\mathbb{H}}}$ acts on the boundary $\partial \mathbf{H}^2_{\mathbb{H}} \cong \mathbb{S}^3 \times \mathbb{S}^3$ of $\mathbf{H}^2_{\mathbb{H}}$ and $\mathcal{M}_{\mathbf{H}^2_{\mathbb{H}}} \cong Conf_+(\mathbb{S}^3 \times \mathbb{S}^3)$.

Now, we are ready to describe a kind of isometries of $\mathbf{H}^2_{\mathbb{H}}$ lying in

$$\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{O}) := \mathrm{PSL}_2(\mathcal{O}) \cap \mathcal{M}_{\mathbf{H}^2_{\mathbb{H}}}$$

which will be used to define our quaternionic modular group.

Definition 3.1. Let $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in $\mathcal{B}_K$ and $\Im\mathcal{O}$ be the set of pure elements of $\mathcal{B}_K$ lying in $\mathcal{O}$. We define the *subgroup of $\Im\mathcal{O}$-translations* of $\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{O})$ as

$$\mathcal{T}_{\Im\mathcal{O}} := \left\{ T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) \mid \; b \in \Im\mathcal{O} \right\}.$$

A *translation* in $\mathbf{H}^2_{\mathbb{H}}$ is defined as a transformation

$$T_{(b_1,b_2)} := (F_{\gamma_1}, F_{\gamma_2}) : \mathbf{H}^2_{\mathbb{H}} \longrightarrow \mathbf{H}^2_{\mathbb{H}}$$

associated to a couple of matrices of the form

$$(\gamma_1, \gamma_2) = \left(\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \right) \in \mathcal{M}_{\mathbf{H}^2_{\mathbb{H}}},$$

where b_1 and b_2 are such that $\Re(b_1) = 0$ and $\Re(b_2) = 0$. Note that, if $b \in \mathcal{O}$ and $\Re(b) = 0$, then $\Re(\sigma(b)) = 0$ and we can identify the group $\mathcal{T}_{\Im\mathcal{O}}$ with the set of translations in $\mathbf{H}^2_{\mathbb{H}}$ of the form $T_{(b,\sigma(b))}$, with $b \in \Im\mathcal{O}$.

Remark 4. Note that, if $\mathcal{O}$ is the Lipschitz order $\mathbb{H}(\mathbb{Z}_K) := \{\mathbf{q} = t + xi + yj + zk \mid w, x, y, z \in \mathbb{Z}_K\}$ of $\mathcal{B}_K$, we have that

$$\Im\mathbb{H}(\mathbb{Z}_K) = \{\frac{1}{2}(\mathbf{q} - \overline{\mathbf{q}}) \mid \mathbf{q} \in \mathbb{H}(\mathbb{Z}_K)\} = \{xi + yj + zk : x, y, z \in \mathbb{Z}_K\},$$

in analogy with the imaginary part of a complex number. However, $\Im\mathcal{O}$ is not always equal to the set $\{\frac{1}{2}(\mathbf{q} - \overline{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{O}\}$. For example, let $K = \mathbb{Q}(\sqrt{n})$, with $n \not\equiv 1 \mod 4$, then its ring of integers $\mathbb{Z}_K = \mathbb{Z}[\sqrt{n}]$. Let

$$\mathrm{Hur}(\mathbb{Z}[\sqrt{n}]) = \{\mathbf{q} = t + xi + yj + zk \mid t, x, y, z \in \mathbb{Z}[\sqrt{n}] \text{ or } t, x, y, z \in \mathbb{Z}[\sqrt{n}] + \frac{1}{2}\}$$

be the Hurwitz order of $\mathcal{B}_{\mathbb{Q}(\sqrt{n})}$. It is easy to see that $\frac{i+j+k}{2} \in \{\frac{1}{2}(\mathbf{q} - \overline{\mathbf{q}}) \mid \mathbf{q} \in \mathbb{Z}[\sqrt{n}]\} \subseteq \mathcal{B}_K^0$, by taking $\mathbf{q} = \frac{1+i+j+k}{2} \in \mathbb{Z}[\sqrt{n}]$, but $\frac{i+j+k}{2} \notin \mathrm{Hur}(\mathbb{Z}[\sqrt{n}])$ and in particular $\frac{i+j+k}{2} \notin \Im\,\mathrm{Hur}(\mathbb{Z}[\sqrt{n}])$.

Definition 3.2. Let $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in $\mathcal{B}_K$ and ε be the fundamental unit of $\mathbb{Z}_K$. We define the *scalar unitary subgroup* of $\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{O})$ as the set of matrices

$$\mathcal{U}_\varepsilon(\mathcal{O}) := \left\{ D_\ell := \begin{pmatrix} \varepsilon^\ell & 0 \\ 0 & \varepsilon^{-\ell} \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) \mid \ell \in \mathbb{N} \right\}.$$

A *left bi-homothetic transformation* in $\mathbf{H}^2_{\mathbb{H}}$ is defined as a transformation $h_{(c_1,c_2)} : \mathbf{H}^2_{\mathbb{H}} \to \mathbf{H}^2_{\mathbb{H}}$ given by the map $(\mathbf{q}_1, \mathbf{q}_2) \mapsto (c_1\mathbf{q}_1, c_2\mathbf{q}_2)$, where $c_1, c_2 \in \mathbb{H}$ are such that $\Re(c_1\mathbf{q}_1) > 0$ and $\Re(c_2\mathbf{q}_2) > 0$. Note that $D_\ell \in \mathcal{U}_\varepsilon(\mathcal{O})$ defines the left bi-homothetic transformation $h_{(\varepsilon^{2\ell},\sigma(\varepsilon)^{2\ell})}$ in $\mathbf{H}^2_{\mathbb{H}}$.

Definition 3.3. Let $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in $\mathcal{B}_K$. We define the *torsion unitary subgroup* of $\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{O})$ as the set of matrices

$$\mathcal{U}^1(\mathcal{O}) := \left\{ D_{\mathbf{u}} := \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) \mid \mathbf{u} \in \mathcal{O}^1 \right\}.$$

Recall from Remark 1, that $\mathbb{H}^1$ acts by rotation on $\mathbb{H}^0 \simeq \mathbb{R}^3$ via conjugation. Then, we define a *left bi-rotation* in $\mathbf{H}^2_{\mathbb{H}}$ as a transformation $r_{(\mathbf{u}_1,\mathbf{u}_2)} : \mathbf{H}^2_{\mathbb{H}} \to \mathbf{H}^2_{\mathbb{H}}$ given by the map $(\mathbf{q}_1, \mathbf{q}_2) \mapsto (\mathbf{u}_1\mathbf{q}_1\mathbf{u}_1^{-1}, \mathbf{u}_2\mathbf{q}_2\mathbf{u}_2^{-1})$, where $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{H}^1$. Note that as $\mathcal{O}^1 \subseteq \mathbb{H}^1$ and $\sigma(\mathbf{u}) \in \mathcal{O}^1$ for all $\mathbf{u} \in \mathcal{O}^1$, then $D_{\mathbf{u}} \in \mathcal{U}^1(\mathcal{O})$ defines the left bi-rotation $r_{(\mathbf{u},\sigma(\mathbf{u}))}$ in $\mathbf{H}^2_{\mathbb{H}}$.

Finally the *inversion* in $\mathbf{H}^2_{\mathbb{H}}$ is defined as $(I, I) : \mathbf{H}^2_{\mathbb{H}} \to \mathbf{H}^2_{\mathbb{H}}$, were I is the usual inversion defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 3.4. Let K be a real quadratic field, and $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in the quaternion algebra $\mathcal{B}_K$. We define the *$\mathcal{O}$-quaternionic modular group* $\Gamma(\mathcal{O})$ as the group generated by $\mathcal{U}_\varepsilon(\mathcal{O})$, $\mathcal{U}^1(\mathcal{O})$, $\mathcal{T}_{\Im\mathcal{O}}$ and I. Moreover, we define the *$\mathcal{O}$-affine subgroup* $\mathcal{A}(\mathcal{O})$ of $\Gamma(\mathcal{O})$ as the group generated by $\mathcal{U}_\varepsilon(\mathcal{O})$, $\mathcal{U}^1(\mathcal{O})$ and $\mathcal{T}_{\Im\mathcal{O}}$ (without the involution I).

Remark 5. When $N_{K/\mathbb{Q}}(\varepsilon) = 1$, $\mathcal{O}^\times$ could be a degree two extension of $\mathbb{Z}_K^\times \mathcal{O}^1$ and, in such case, $\mathcal{O}^\times \simeq \mathbb{Z}_K^\times \mathcal{O}^1 \langle 1 + i \rangle$ which follows from [9, Proposition 6] and [Table 4.3, §8][4]. However, we do not include $1 + i$ in Definition 3.2 or in Definition 3.3, then in Definition 3.4, because in bot cases $1 + i$ does not produce a matrix satisfying BG-conditions.

One can show thet $\Gamma(\mathcal{O}) = \mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{O})$.

Definition 3.5. Let K be a real quadratic field, and $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in the quaternion algebra $\mathcal{B}_K$. We define the *Hilbert-Blumenthal quaternionic orbifold* associated to $\mathcal{O}$ as

$$M_{\Gamma(\mathcal{O})} := \Gamma(\mathcal{O}) \backslash \mathbf{H}^2_{\mathbb{H}}.$$

4. Cusps of Hilbert-Blumenthal quaternionic orbifold

In this section we give a description of the cusps of the Hilbert-Blumenthal quaternionic orbifold $M_{\Gamma(\mathcal{O})}$ following [5] and [7].

Recall that an ℓ–torus bundle over an m–torus is the total space of a fiber bundle with base manifold the m-torus T^m and fiber the ℓ-torus T^ℓ. We call such manifolds simply (ℓ, m)–*torus bundles*. We say that M is a *virtual* (ℓ, m)–*torus bundle* if M is finitely covered by an (ℓ, m)–torus bundle.

Let K be a real quadratic field, and $\mathcal{O}$ be a quaternion $\mathbb{Z}_K$-order in the quaternion algebra $\mathcal{B}_K$. We define the *affine Hilbert-Blumenthal quaternionic orbifold* associated to $\mathcal{O}$ as

$$M_{\mathcal{A}(\mathcal{O})} := \mathcal{A}(\mathcal{O})\backslash\mathbf{H}^2_{\mathbb{H}}.$$

Note that in a small neighborhood of (∞, ∞) $M_{\mathcal{A}(\mathcal{O})}$ and $M_{\Gamma(\mathcal{O})}$ coincide. The main goal of this section is to prove the following result:

Theorem 4.1. *A cusp cross-section of* $M_{\mathcal{A}(\mathcal{O})}$ *is a virtual* $(6, 1)$*-torus bundle.*

Proof. First, let's find A in $\mathbb{Z}_K \rtimes \mathbb{Z}_{K,+}^{\times} \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ assuming that $K = \mathbb{Q}(\sqrt{n})$ with $n \not\equiv 1 \mod 4$. Let $\varepsilon = X + Y\sqrt{n}$ be the fundamental unit of $\mathbb{Z}_K^{\times}$. The attaching map comes from the homotheties that translate toward the cusp, which are given by the Möbius action of powers of the matrix $\begin{pmatrix} X + Y\sqrt{n} & 0 \\ 0 & (X + Y\sqrt{n})^{-1} \end{pmatrix}$ on an integer $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{n}]$, where $X + Y\sqrt{n}$ is a fundamental unit. Since this action is

$$a + b\sqrt{n} \mapsto (X + Y\sqrt{n})^2(a + b\sqrt{n}) =$$

$$(X^2 + nY^2)a + 2Ynb + (2Ya + (X^2 + nY^2)b)\sqrt{n},$$

$A = \begin{pmatrix} X^2 + nY^2 & 2Yn \\ 2Y & X^2 + nY^2 \end{pmatrix}$, and we see this as an action on $\mathbb{Z}_K$ via the identification

$$a + b\sqrt{n} \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now, let $\Gamma = \Im\mathcal{O} \rtimes \mathbb{Z}_{K,+}^{\times}$, the cusp cross-section subgroup of the group obtained by omitting the rotations in $\mathcal{O}^1$. Then

$$\mathrm{Stab}_{\Gamma}(\infty) = \left\{ \begin{pmatrix} (X+Y\sqrt{n})^{\ell} & xi+yj+zij \\ 0 & (X+Y\sqrt{n})^{-\ell} \end{pmatrix} \;\middle|\; \ell, x, y, z \in \mathbb{Z} \right\}.$$

This would effect each of xi, yj, zij in the same way that $a + b\sqrt{n}$ was effected in the Hilbert case above. So this gives an action $\mathbb{Z}_K^{\times} \to \mathrm{Aut}(\mathbb{Z}^6)$ implying $\Gamma \cong \mathbb{Z}^6 \rtimes_A \mathbb{Z}$ with

$$A = \begin{pmatrix} (X^2+nY^2) & 2Yn & 0 & 0 & 0 & 0 \\ 2Y & (X^2+nY^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & (X^2+nY^2) & 2Yn & 0 & 0 \\ 0 & 0 & 2Y & (X^2+nY^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & (X^2+nY^2) & 2Yn \\ 0 & 0 & 0 & 0 & 2Y & (X^2+nY^2) \end{pmatrix}. \tag{10}$$

Since this group is torsion-free, it gives a finite covering (which is a manifold) of the cusp cross-sections of our orbifolds. We remark that, extending the analogy from the Hilbert modular varieties, the manifold is a 7-solvmanifold.

When $K = \mathbb{Q}(\sqrt{n})$ with $n \equiv 1 \mod 4$, we can obtain the matrix A by similar calculations but we do not have a general formula through all n. We include the following examples to illustrate this phenomenon. $\square$

Example 2. Let's find A in $\mathbb{Z}_K \rtimes \mathbb{Z}_{K,+}^\times \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for $K = \mathbb{Q}(\sqrt{5})$. The attaching map comes from the homotheties that translate toward the cusp, which are given by the Möbius action of powers of the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{-1+\sqrt{5}}{2} \end{pmatrix}$ on an integer $a + b\theta = a + b\left(\frac{1+\sqrt{5}}{2}\right)$. Since this action is

$$a + b\theta \mapsto \varepsilon^2(a + \theta b) = \varepsilon^2 a + \varepsilon^2 \theta b = \left(\frac{3+\sqrt{5}}{2}\right) a + \left(\frac{3+\sqrt{5}}{2}\right) \theta b$$

$$= a + \left(\frac{1+\sqrt{5}}{2}\right) a + \theta b + \left(\frac{1+\sqrt{5}}{2}\right) \theta b$$

$$= a + \left(\frac{1+\sqrt{5}}{2}\right) a + \left(\frac{1+\sqrt{5}}{2}\right) b + \left(\frac{3+\sqrt{5}}{2}\right) b$$

$$= a + \left(\frac{1+\sqrt{5}}{2}\right) a + b + 2\left(\frac{1+\sqrt{5}}{2}\right) b$$

$$= a + b + (a+2b)\left(\frac{1+\sqrt{5}}{2}\right)$$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and we see this as an action on $\mathbb{Z}_K$ via the identification

$$a + b\theta \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now, let $\Gamma = \Im\mathcal{O} \rtimes \mathbb{Z}_{K,+}^{\times}$, the cusp cross-section subgroup of the group obtained by omitting the rotations in $\mathcal{O}^1$. Then

$$\mathrm{Stab}_\Gamma(\infty) = \left\{ \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{\ell} & xi + yj + zij \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^{-\ell} \end{pmatrix} \,\middle|\, \ell, x, y, z \in \mathbb{Z} \right\}.$$

This would effect each of xi, yj, zij in the same way that $a + b\left(\frac{1+\sqrt{5}}{2}\right)$ was effected in the Hilbert case. So this gives an action $\mathbb{Z}_K^\times \to \mathrm{Aut}(\mathbb{Z}^6)$ implying $\Gamma \cong \mathbb{Z}^6 \rtimes_A \mathbb{Z}$ with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Since this group is torsion-free, it gives a manifold covering of the cusp cross-sections of our orbifolds.

Example 3. Let's find A in $\mathbb{Z}_K \rtimes \mathbb{Z}_{K,+}^\times \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for $K = \mathbb{Q}(\sqrt{13})$. The attaching map comes from the homotheties that translate toward the cusp, which are given by the Möbius action of powers of the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} \frac{3+\sqrt{13}}{2} & 0 \\ 0 & \frac{-3+\sqrt{13}}{2} \end{pmatrix}$ on an integer $a + b\theta = a + b\left(\frac{1+\sqrt{13}}{2}\right)$. Since this action is

$$\begin{aligned} a + b\theta \mapsto \varepsilon^2(a+\theta b) = \varepsilon^2 a + \varepsilon^2 \theta b &= \left(\frac{11+3\sqrt{13}}{2}\right) a + \left(\frac{11+3\sqrt{13}}{2}\right)\theta b \\ &= a(4+3\theta) + b\theta(4+3\theta) \\ &= 4a + 3a\theta + 4b\theta + 3b\theta^2 \\ &= 4a + 3a\theta + 4b\theta + 3b\left(\frac{7+\sqrt{13}}{2}\right) \\ &= 4a + 3a\theta + 4b\theta + 3b(3+\theta) \\ &= 4a + 9b + (3a+7b)\theta \end{aligned}$$

$A = \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}$, and we see this as an action on $\mathbb{Z}_K$ via the identification

$$a + b\theta \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now, let $\Gamma = \Im\mathcal{O} \rtimes \mathbb{Z}_{K,+}^{\times}$, the cusp cross-section subgroup of the group obtained by omitting the rotations in $\mathcal{O}^1$. Then

$$\mathrm{Stab}_{\Gamma}(\infty) = \left\{ \begin{pmatrix} \left(\frac{3+\sqrt{13}}{2}\right)^{\ell} & xi + yj + zij \\ 0 & \left(\frac{3+\sqrt{13}}{2}\right)^{-\ell} \end{pmatrix} \,\middle|\, \ell, x, y, z \in \mathbb{Z} \right\}.$$

This would effect each of xi, yj, zij in the same way that $a + b\left(\frac{1+\sqrt{13}}{2}\right)$ was effected in the Hilbert case. So this gives an action $\mathbb{Z}_K^\times \to \mathrm{Aut}(\mathbb{Z}^6)$ implying $\Gamma \cong \mathbb{Z}^6 \rtimes_A \mathbb{Z}$ with

$$A = \begin{pmatrix} 4 & 9 & 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 9 & 0 & 0 \\ 0 & 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 3 & 7 \end{pmatrix}.$$

Since this group is torsion-free, it gives a manifold covering of the cusp cross-sections of our orbifolds.

Example 4. Let's find A in $\mathbb{Z}_K \rtimes \mathbb{Z}_{K,+}^\times \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for the case where $K = \mathbb{Q}(\sqrt{2})$. The attaching map comes from the homotheties that translate toward the cusp, which are given by the Möbius action of powers of the matrix $\begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{pmatrix}$ on an integer $a + b\sqrt{2}$. Since this action is

$$a + b\sqrt{2} \mapsto (1+\sqrt{2})^2(a + b\sqrt{2}) = 3a + 4b + (2a + 3b)\sqrt{2},$$

$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, and we see this as an action on $\mathbb{Z}_K$ via the identification

$$a + b\sqrt{2} \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}.$$

Notice how only even powers of the fundamental unit correspond to Möbius actions in this way, resembling how $\mathbb{Z}_{K,+}^{\times} = \mathbb{Z}_K^{\times(2)}$.

We can also think of this as the attaching map from the inside to the outside of a thickened torus. Thus the cusp of the Hilbert-Blumenthal surface $(\mathbf{H}_{\mathbb{R}}^2 \times \mathbf{H}_{\mathbb{R}}^2)/\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{2})})$ has fundamental group $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$. Moreover, we have an injection

$$\mathbb{Z}^2 \rtimes_A \mathbb{Z} \hookrightarrow \mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]) : \quad \left(\begin{pmatrix} a \\ b \end{pmatrix}, \ell \right) \mapsto \begin{pmatrix} (1+\sqrt{2})^{\ell} & a + b\sqrt{2} \\ 0 & (-1+\sqrt{2})^{\ell} \end{pmatrix}. \tag{11}$$

Now, let $\Gamma = \Im\mathcal{O} \rtimes \mathbb{Z}_{K,+}^{\times}$, the cusp cross-section subgroup of the group obtained by omitting the rotations in $\mathcal{O}^1$. Then

$$\mathrm{Stab}_\Gamma(\infty) = \left\{ \begin{pmatrix} (1+\sqrt{2})^\ell & xi+yj+zij \\ 0 & (1-\sqrt{2})^\ell \end{pmatrix} \;\middle|\; \ell, x, y, z \in \mathbb{Z} \right\}.$$

This would effect each of xi, yj, zij in the same way that $a + b\sqrt{2}$ was effected in the Hilbert case. More precisel, the attaching map comes from the homotheties that translate toward the cusp, which are given by the Möbius action of powers of the matrix $\begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{pmatrix}$ on a pure quaternion $(a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k$.

$$(a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k \mapsto (1+\sqrt{2})^2((a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k)$$

$$= ((1+\sqrt{2})^2(a+b\sqrt{2}))i + ((1+\sqrt{2})^2(c+d\sqrt{2}))j + ((1+\sqrt{2})^2(e+f\sqrt{2}))k$$

$$= ((3a+4b) + (2a+3b)\sqrt{2})i + ((3c+4d) + (2c+3d)\sqrt{2})j + ((3e+4f) + (2e+3f)\sqrt{2})k$$

So this gives the matrix

$$A = \begin{pmatrix} 3 & 4 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}.$$

Since this group is torsion-free, it gives a manifold covering of the cusp cross-sections of our orbifolds.

Now we will study the Möbius action of powers of the matrix $\begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$ on a pure quaternion $(a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k$, where $\mathbf{u} \in \mathbb{H}(\mathbb{Z}_K)^1 = \{\pm 1, \pm i \pm j, \pm j\}$.

$$(a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k \mapsto \mathbf{u}((a+b\sqrt{2})i + (c+d\sqrt{2})j + (e+f\sqrt{2})k)\mathbf{u}^{-1}$$

$$= (a+b\sqrt{2})\mathbf{u}i\mathbf{u}^{-1} + (c+d\sqrt{2})\mathbf{u}j\mathbf{u}^{-1} + (e+f\sqrt{2})\mathbf{u}k\mathbf{u}^{-1}$$

(if $\mathbf{u} = i$ with $\mathbf{u}^{-1} = -i$)

$$= (a + b\sqrt{2})ii(-i) + (c + d\sqrt{2})ij(-i) + (e + f\sqrt{2})ik(-i)$$

$$= (a + b\sqrt{2})i - (c + d\sqrt{2})j - (e + f\sqrt{2})k$$

which gives the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

similarly if $\mathbf{u} = j$ with $\mathbf{u}^{-1} = -j$,

$$(a + b\sqrt{2})\mathbf{u}i\mathbf{u}^{-1} + (c + d\sqrt{2})\mathbf{u}j\mathbf{u}^{-1} + (e + f\sqrt{2})\mathbf{u}k\mathbf{u}^{-1}$$

$$= (a + b\sqrt{2})ji(-j) + (c + d\sqrt{2})jj(-j) + (e + f\sqrt{2})jk(-j)$$
$$= -(a + b\sqrt{2})i + (c + d\sqrt{2})j - (e + f\sqrt{2})k$$

which gives the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

In general such a matrix are in correspondence with the set of matrices of order 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Now consider $\mathbf{u} \in \mathrm{Hur}(\mathbb{Z}_K)^1 = \mathbb{H}(\mathbb{Z}_K)^1 \cup \left\{\frac{\pm 1 \pm i \pm j \pm k}{2}\right\}$. Then we have for example, if $\mathbf{u} = \frac{1+i+j+k}{2}$ and $\mathbf{u}^{-1} = \frac{1-i-j-k}{2}$

$$(a + b\sqrt{2})\mathbf{u}i\mathbf{u}^{-1} + (c + d\sqrt{2})\mathbf{u}j\mathbf{u}^{-1} + (e + f\sqrt{2})\mathbf{u}k\mathbf{u}^{-1}$$

$$= \frac{1}{4}((a + b\sqrt{2})(1 + i + j + k)i(1 - i - j - k)+$$

$$+(c + d\sqrt{2})(1 + i + j + k)j(1 - i - j - k) + (e + f\sqrt{2})(1 + i + j + k)k(1 - i - j - k))$$

$$= \frac{1}{4}((a + b\sqrt{2})(4j) + (c + d\sqrt{2})(4k) + (e + f\sqrt{2})(4i)$$

$$= (e + f\sqrt{2})i + (a + b\sqrt{2})j + (c + d\sqrt{2})k$$

which gives the matrix of order 3

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

corresponding to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

In general the 16 new matrix corresponds to the non diagonal (but with a diagonal minor) matrices with coefficients ± 1

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \ldots.$$

4.1 - Class number of a quaternion order

Let $K = \mathbb{Q}$ or a real quadratic field, $\mathbb{Z}_K$ its ring of integers, $B_K = (-1, -1 \mid K)$ and $I \subseteq \mathcal{B}_K$ be a $\mathbb{Z}_K$-lattice. Then

$$\mathcal{O}_R(I) := \{\alpha \in \mathcal{B}_K : I\alpha \subseteq I\}$$

is a $\mathbb{Z}_K$-order in $\mathcal{B}_K$ called the *right order of* I. Similarly, we can define the *left order of* I by

$$\mathcal{O}_L(I) := \{\alpha \in \mathcal{B}_K : \alpha I \subseteq I\}.$$

Let $I, J \subseteq \mathcal{B}_K$ be $\mathbb{Z}_K$-lattices. We define the *product* IJ, as the $\mathbb{Z}_K$-submodule of $\mathcal{B}_K$ generated by the set $\{\alpha\beta : \alpha \in I, \beta \in J\}$. In fact, IJ is a $\mathbb{Z}_K$-lattice too. We will say that a $\mathbb{Z}_K$-lattice $I \subseteq \mathcal{B}_K$ is *invertible* if there exists a lattice $I' \subseteq \mathcal{B}_K$ such that

$$II' = \mathcal{O}_L(I) = \mathcal{O}_R(I') \text{ and } I'I = \mathcal{O}_L(I') = \mathcal{O}_R(I).$$

On the other hand, we say that two $\mathbb{Z}_K$-lattices $I, J \subseteq \mathcal{B}_K$ are in the *same right class*, if there exists $\alpha \in \mathcal{B}_K^\times$ such that $\alpha I = J$, and we write $I \sim_R J$. The relation $\sim_R$ defines an equivalence relation on the set of $\mathbb{Z}_K$-lattices in $\mathcal{B}_K$, then we will denote by $[I]_R$ the equivalence class of the $\mathbb{Z}_K$-lattice I. In particular, when I is an invertible lattice, every lattice in the class $[I]_R$ is invertible.

Now, let $\mathcal{O}$ be a $\mathbb{Z}_K$-order in $\mathcal{B}_K$. A *right fractional $\mathcal{O}$-ideal* is a lattice $I \subseteq \mathcal{B}_K$ such that $\mathcal{O} \subseteq \mathcal{O}_R(I)$. Similarly we can define a left fractional $\mathcal{O}$-ideal. We define the *(right) class set* of $\mathcal{O}$ as the set

$$Cls_R(\mathcal{O}) := \{[I]_R : I \text{ is an invertible right fractional } \mathcal{O}\text{-ideal}\}.$$

The set $Cls_R(\mathcal{O})$ has a distinguished element $[\mathcal{O}]_R \in Cls_R(\mathcal{O})$, so it has the structure of a pointed set. However, in general it does not have the structure of a group under multiplication. For example, for classes $[I]_R$, $[J]_R$, we have that $[\alpha J]_R = [J]_R$ for $\alpha \in \mathcal{B}_K^\times$ but we need not have that $[I\alpha J]_R = [IJ]_R$, because of the lack of commutativity.

We remark that the analogue left relation can be defined and the map $I \mapsto \overline{I}$, induced by the standard involution in $\mathcal{B}_K$, interchanges left and right. Then we will abbreviate

$$Cls(\mathcal{O}) := Cls_R(\mathcal{O}).$$

It can be proven, by using the analogue geometry of numbers [11, Main Theorem 17.7.1], that $Cls(\mathcal{O})$ is finite and we can define the *(right) class number* of $\mathcal{O}$ as

$$h_{\mathcal{O}} := \#Cls(\mathcal{O})$$

Of particular interest to us are when $\mathcal{O}$ is maximal. In this case all lattices are invertible and it can be proven that an invertible right fractional $\mathcal{O}$-ideal can be generated as a right $\mathcal{O}$-ideal by two elements $\alpha, \beta \in \mathcal{B}_K^\times$. The result follows from Exercise 16.6 and Main Theorem 16.7.7 of [11].

4.2 - Motivating example

Let

$$\overline{\mathbf{H}}_{\mathbb{R}}^5 := \{(\mathbf{q}, t) : \mathbf{q} \in \mathbb{H},\ t \geqslant 0\}.$$

As we pointed out above, each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{H})$ acts on $\mathbb{H} \cup \{\infty\}$ by Möbius transformations. This action can be extended to $\overline{\mathbf{H}}^5_{\mathbb{R}}$ as follows (the Poincaré's extension of F_γ):

$$F_{\overline{\gamma}}(\mathbf{q}, t) = \left(\frac{1}{|c\mathbf{q} + d|^2 + |c|^2 t^2} ((a\mathbf{q} + b)(\overline{\mathbf{q}c} + \overline{d}) + a\overline{c}t^2), \det_{\mathbb{H}}(\gamma) \frac{t}{|c\mathbf{q} + d|^2 + |c|^2 t^2} \right).$$

In particular, when $t = 0$ the Poincare extension $F_{\overline{\gamma}}$ of F_γ correspond to the action of F_γ on $\mathbb{H}$.

Let $\mathcal{O}$ be a $\mathbb{Z}$-order in $\mathcal{B}_{\mathbb{Q}} = (-1, -1 \mid \mathbb{Q})$ and consider $\mathrm{PSL}_2(\mathcal{O})$ which is a discrete subgroups of $\mathrm{PSL}_2(\mathbb{H})$. Then we can define the *Bianchi quaternionic orbifold* as

$$M_{\mathcal{O}} := \mathbf{H}^5_{\mathbb{R}} / \mathrm{PSL}_2(\mathcal{O}).$$

which is a real 5-dimensional orbifolds of finite hyperbolic volume. Examples of such orbifolds are described in the extended version of [3] when $\mathcal{O}$ is the ring $\mathbb{H}(\mathbb{Z})$ of Lipschitz integers and the ring $\mathbb{H}ur(\mathbb{Z})$ of Hurwitz integers.

Note that the embedding of $\mathcal{B}_{\mathbb{Q}} \hookrightarrow \mathbb{H}$ induce an embedding

$$\mathbf{P}^1(\mathcal{B}) \hookrightarrow \mathbf{P}^1(\mathbb{H}). \tag{12}$$

The orbits of $\mathbf{P}^1(\mathcal{B})$ under $\mathrm{PSL}_2(\mathcal{O})$ is called the *cusps* of $\mathrm{PSL}_2(\mathcal{O})$.

Proposition 4.2. *$M_{\mathbb{H}ur(\mathbb{Z})}$ has only one cusp.*

Proof. As $\mathbb{H}ur(\mathbb{Z})$ is a lattice of $\mathbb{H}$ we have that $\mathbb{H}ur(\mathbb{Z}) \cdot \mathbb{Q} = \mathbb{H}$ then each element β of $\mathbb{H}$ can be write as $\beta = \alpha c^{-1}$ with $\alpha \in \mathbb{H}ur(\mathbb{Z})$, $0 \neq c \in \mathbb{Z}$ and $gcd(\alpha, c) = 1$. By right Bézout's theorem [11, Corollary 11.3.6] there exists $\mu, \nu \in \mathbb{H}ur(\mathbb{Z})$ such that

$$\alpha\mu - c\nu = 1.$$

This gives $\gamma = \begin{pmatrix} \alpha & \nu \\ c & \mu \end{pmatrix}$ such that $F_\gamma(\infty) = \alpha c^{-1}$. Using Lemma 2.4 of [2] and the commutativity of c, we have that

$$\det_{\mathbb{H}}(\gamma) = \sqrt{|\alpha|^2|\mu|^2 + |c|^2|\nu|^2 - 2\Re(c\overline{\alpha}\nu\overline{\mu})} =$$

$$\sqrt{|c\nu - c\alpha c^{-1}\mu)|^2} = \sqrt{|\alpha\mu - c\nu)|^2} = 1$$

then $\gamma \in \mathrm{PSL}_2(\mathbb{H}ur(\mathbb{Z}))$. Thus the only cusp of $M_{\mathbb{H}ur(\mathbb{Z})}$ is the orbit of ∞. □

From [11, Proposition 11.3.4] we have that $h_{\mathbb{H}ur(\mathbb{Z})} = 1$, then we have the following corollary.

Corollary 4.3. *The number of cusps of $M_{\mathbb{H}ur(\mathbb{Z})}$ is equal to $h_{\mathbb{H}ur(\mathbb{Z})}$.*

The proof of Theorem 4.2 works for any (left) Euclidean order $\mathcal{O}$ in $\mathcal{B}_{\mathbb{Q}}$ (or even in $\mathcal{B}_K$ for a quadratic real field and $M_{\mathcal{O}} := (\mathbf{H}^5_{\mathbb{R}} \times \mathbf{H}^5_{\mathbb{R}})/\mathrm{PSL}_2(\mathcal{O})$. However, there are examples of non-Euclidean orders even in $\mathcal{B}_{\mathbb{Q}}$ as follows:

On the other hand we have the following result:

Proposition 4.4. *$M_{\mathbb{H}(\mathbb{Z})}$ has at least two cusps.*

Proof. (Sketch) $I = 2\mathbb{H}(\mathbb{Z}) + (1 + i + j + k)\mathbb{H}(\mathbb{Z})$
is not a principal ideal of $\mathbb{H}(\mathbb{Z})$. As $\mathbb{H}(\mathbb{Z})$ is not a Bézout ring, it can be proven that $\infty = (1; 0) = \mathbf{P}^1(\mathbb{H})$ and $(1 + i + j + k; 2) \in \mathbf{P}^1(\mathbb{H})$ are in two different orbits of $\mathrm{PSL}_2(\mathbb{H}(\mathbb{Z}))$. □

As $h_{\mathbb{H}(\mathbb{Z})} = 2$ we have the following result.

Corollary 4.5. *The number of cusps of $M_{\mathbb{H}(\mathbb{Z})}$ is $\geqslant h_{\mathbb{H}(\mathbb{Z})}$.*

5. **Hilbert quaternionic varieties (n-dimensional case)**

Let

$$\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \mid \Re(\mathbf{q}) > 0\}$$

be the half-space model of the *one dimensional quaternionic hyperbolic space* which is embedded in the quaternionic projective line $\mathbf{P}^1(\mathbb{H}) := \mathbb{H} \cup \{\infty\}$. This space is isometric to the hyperbolic real space $\mathbf{H}^4_{\mathbb{R}} := \{(t, x, y, z) \in \mathbb{R}^4 \mid t > 0\}$ of dimension 4 with the Poincaré metric $ds^2 = \frac{dt^2+dx^2+dy^2+dz^2}{t^2}$.

Let $\mathrm{GL}_2(\mathbb{H})$ be the set of all invertible matrices of $\mathrm{M}_2(\mathbb{H})$. A quaternionic matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{H})$ acts on $\mathbf{P}^1(\mathbb{H})$ by the *quaternionic Möbius transformation* associated to γ which is defined as the real analytic function

$$F_\gamma : \mathbf{P}^1(\mathbb{H}) \longrightarrow \mathbf{P}^1(\mathbb{H})$$

given by

$$F_\gamma(\mathbf{q}) := (a\mathbf{q} + b) \cdot (c\mathbf{q} + d)^{-1}, \tag{13}$$

were we set $F_\gamma(\infty) = \infty$ if $c = 0$, $F_\gamma(\infty) = ac^{-1}$ if $c \neq 0$, and $F_\gamma(-c^{-1}d) = \infty$.

Let $\mathrm{SL}_2(\mathbb{H})$ as the set of all matrices in $\mathrm{GL}_2(\mathbb{H})$ with Dieudonné determinant 1 and $\mathrm{PSL}_2(\mathbb{H}) := \mathrm{SL}_2(\mathbb{H})/\pm\mathcal{I}$. As we saw before the subgroup $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \subseteq \mathrm{PSL}_2(\mathbb{H})$ of quaternionic Möbius transformation leave invariant $\mathbf{H}^1_{\mathbb{H}}$ can be characterized as the group induced by matrices which satisfy one of the following equivalent BG-conditions

$$\left\{\gamma \in \mathrm{PSL}_2(\mathbb{H}) \mid \overline{\gamma}^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \tag{14}$$

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{H}) \mid \Re(a\overline{c}) = 0,\ \Re(b\overline{d}) = 0,\ \overline{b}c + \overline{d}a = 1\right\}, \tag{15}$$

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{H}) \mid \Re(c\overline{d}) = 0,\ \Re(a\overline{b}) = 0,\ a\overline{d} + b\overline{c} = 1\right\}. \tag{16}$$

Now consider the n-fold cartesian product

$$\mathbf{H}^n_{\mathbb{H}} = \mathbf{H}^1_{\mathbb{H}} \times \cdots \times \mathbf{H}^1_{\mathbb{H}}$$

and let $\mathfrak{U}$ the group of "biholomorphic maps = isometries" $\mathbf{H}^n_{\mathbb{H}} \to \mathbf{H}^n_{\mathbb{H}}$. The connected component of the identity of $\mathfrak{U}$ is equal to the n-fold direct product $\mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}} = \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \times \cdots \times \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ and we have the following exact sequence

$$1 \longrightarrow \mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}} \longrightarrow \mathfrak{U} \longrightarrow S_n \longrightarrow 1.$$

By using coordinates $\mathbf{q}_1, \ldots, \mathbf{q}_n$ in $\mathbf{H}^n_{\mathbb{H}}$ with $q_\ell = t_\ell + x_\ell i + y_\ell j + z_\ell k$, we have a metric invariant under $\mathfrak{U}$ given by:

$$\sum_{\ell=1}^{n} ds_\ell^2 = \sum_{\ell=1}^{n} \frac{dt_\ell^2 + dx_\ell^2 + dy_\ell^2 + dz_\ell^2}{t_\ell^2}.$$

From now on, we will study discrete subgroups of $\mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}}$. Let K be a totally real field of degree n and $\sigma_\ell : K \hookrightarrow \mathbb{R}$, $\ell = 1, \ldots, n$, denotes the n different embeddings of K in $\mathbb{R}$. Let

$$\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{B}_K) := \mathrm{PSL}_2(\mathcal{B}_K) \cup \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}},$$

which acts on $\mathbf{H}_{\mathbb{H}}^n$ by

$$\begin{aligned} F_\gamma(\mathbf{q}_1, \ldots, \mathbf{q}_n) &:= (\mathbb{F}_{\sigma_1(\gamma)}, \ldots, \mathbb{F}_{\sigma_n(\gamma)}) \\ &= ((\sigma_1(a)\mathbf{q}_1 + \sigma_1(b)) \cdot (\sigma_1(c)\mathbf{q}_1 + \sigma_1(d))^{-1}, \ldots, (\sigma_1(a)\mathbf{q}_n + \sigma_n(b)) \cdot (\sigma_n(c)\mathbf{q}_n + \sigma_n(d))^{-1}), \end{aligned}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{B}_K)$. Then we can consider $\mathrm{PSL}_2^{\mathrm{BG}}(\mathcal{B}_K) \subseteq \mathcal{M}_{\mathbf{H}_{\mathbb{H}}^n}$. Let $\mathcal{O}$ be a $\mathbb{Z}_K$-order of $\mathcal{B}_K$. Interesting discrete subgroups of $\mathcal{M}_{\mathbf{H}_{\mathbb{H}}^n}$ are the following:

- The $\mathcal{O}$-quaternionic modular group $\Gamma(\mathcal{O}) = \langle \mathcal{U}_\varepsilon(\mathcal{O}), \mathcal{U}^1(\mathcal{O}), \mathcal{T}_{\Im\mathcal{O}}, I \rangle$,
- the $\mathcal{O}$-affine subgroup $\mathcal{A}(\mathcal{O}) = \langle \mathcal{U}_\varepsilon(\mathcal{O}), \mathcal{U}^1(\mathcal{O}), \mathcal{T}_{\Im\mathcal{O}} \rangle$ and
- the subgroup $\Delta(\mathcal{O}) = \langle \mathcal{U}_\varepsilon(\mathcal{O}), \mathcal{T}_{\Im\mathcal{O}} \rangle$.

Conjecture 5.1. *The (finite) volume of the orbifold $M_{\Gamma(\mathcal{O})} = \mathbf{H}_{\mathbb{H}}^n/\Gamma(\mathcal{O})$ should be related with* *certain value of the zeta function*

$$\zeta_{\mathcal{O}}(s) = \sum_{I \subseteq \mathcal{O}} \frac{1}{N(I)^s}$$

of $\mathcal{O} \in B_K$*, similarly to the classical case (See [11, §26.5] for special values of* $\zeta_{\mathcal{O}}$ *and [11, §26.8] for its functional equation).*

CLASSICAL CASE:

$$\int_{M_{\Gamma(\mathcal{O})}} \omega = 2\zeta_K(-1)$$

where ω is the Gauss-Bonnet form

$$\omega = (-1)^n \frac{1}{(2\pi)^n} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \cdots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$$

Remark 6. Note that in the classical case we write ζ_K because $\mathbb{Z}_K$ is the unique maximal order. However in the quaternionic case $\zeta_{\mathcal{O}}$ depends on the choice of the (maximal) order $\mathcal{O}$ in $\mathcal{B}_K$.

Hypothesis 5.2 (Hirzebruch)**.** *We assume that* Γ *is a discrete subgroup of* $\mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}}$ *and that* $M_\Gamma := \mathbf{H}^n_{\mathbb{H}}/\Gamma$ *has finite volume.*

We will say that Γ is irreducible if it contains no elements $\gamma = (\gamma_1, \cdots, \gamma_n)$ such that $\gamma_\ell = 1$ for some ℓ and $\gamma_{\ell'} \neq 1$ for some ℓ'.

An element of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is *parabolic* if has exactly one fixed point in $\mathbf{P}^1\mathbb{H}$. This belongs to $\mathbf{P}^1\mathbb{H}^0 := \mathbb{H}^0 \cup \infty$ ¿why? Then an element $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}}$ is called *parabolic* if all γ_ℓ are parabolic, and this element has exactly one fixed point in $\mathbf{P}^1\mathbb{H}$ which belongs to $\mathbf{P}^1\mathbb{H}^0$. So, the *parabolic points* of Γ are the fixed points of the parabolic elements of Γ. Finally, the orbits of parabolic points under the action of Γ on $\mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}}$ are called *cusps*. When Γ is irreducible there are only finitely many cusps.

Proposition 5.3. $\Gamma(\mathcal{O})$ *is irreducible, or at least* $\Gamma = \langle \mathcal{U}_\varepsilon(\mathcal{O}), \mathcal{T}_{\Im\mathcal{O}}, I \rangle$ *(or* $\Gamma = \Gamma(\mathcal{O}) \backslash \mathcal{U}^1(\mathcal{O})$*).*

From now on we will assume that Γ is irreducible. If $p \in \mathbf{P}^1\mathbb{H}^0$ is a parabolic point of Γ, we transform it to ∞ by an element δ of $\mathcal{M}_{\mathbf{H}^n_{\mathbb{H}}}$ not necessarily belonging to Γ . Then $\delta p = \infty$.

Let $\Gamma_p := \{\gamma \in \Gamma : \gamma p = p\}$ be the isotropy group of p. Then any element of $\delta\Gamma_p\delta^{-1}$ is contained in $\mathcal{A}(\mathcal{O})$. If

$$\delta\Gamma_p\delta^{-1} \subseteq \Delta(\mathcal{O})$$

we have a natural homomorphism

$$\varphi : \delta\Gamma_p\delta^{-1} \longrightarrow \Lambda := \{(t_1, \cdots, t_n) \in \mathbb{R}^n_{>0} : \prod_{\ell=1}^{n} t_\ell = 1\} \cong \mathbb{R}^{n-1}.$$

whose image is a discrete subgroup Λ_p of rank $n-1$ (in our case $\Lambda_p \cong \mathbb{Z}^\times_{K,+} \cong \mathbb{Z}^{n-1}$). The kernel of ϕ consist of all translations of the form

$$\ker(\phi) = \left\{ \left(\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, \cdots \begin{pmatrix} 1 & b_n \\ 0 & 1 \end{pmatrix} \right) \in \mathcal{T}_{\Im\mathcal{O}} \right\}$$

Then it is isomorphic to a certain discrete subgroup T_p of $\mathbb{R}^{3n}$ of rank $3n$. In our case $T_p \cong \Im\mathcal{O} \cong \mathbb{Z}^{3n}$. So, we have the exact sequence

$$0 \longrightarrow T_p \longrightarrow \delta\Gamma_p\delta^{-1} \longrightarrow \Lambda_p \longrightarrow 0$$

For any positive number d, the group $\delta\Gamma_p\delta^{-1}$ acts freely on

$$W := \left\{ \mathbf{q} \in \mathbf{H}^n_{\mathbb{H}} : \prod_{\ell=1}^{n} \Re(\mathbf{q}_\ell) \geqslant d \right\}$$

The orbit space $W/\delta\Gamma_p\delta^{-1}$ is a (non-compact) manifold with compact boundary $N = \partial W/\delta\Gamma_p\delta^{-1}$. Since ∂W is a principal homogeneous space for the semi-direct product $E = \mathbb{R}^{3n} \rtimes \Lambda$ of all transformations $\mathbf{q}_\ell \mapsto t_\ell \mathbf{q}_\ell + b_\ell$ with $t \in \Lambda$ and $b \in \mathbb{R}^{3n} \cong (\mathbb{H}^0)^n$ we can consider N as the quotient space of the group E (homeomorphic to $\mathbb{R}^{4n-1}$) by the discrete subgroup $\delta\Gamma_p\delta^{-1}$.

The $(4n-1)$-dimensional manifold is a torus bundle over the $(n-1)$-dimensional torus Λ/Λ_p.

The fibre is the torus $\mathbb{R}^{3n}/T_p$, and N is obtained by the action of Λ_p on $\mathbb{R}^{3n}/T_p$ which is induced by the action of $\delta\Gamma_p\delta^{-1}$ on $\mathbb{R}^{3n}$. The we have the following result

Proposition 5.4. *The cusp cross-section of* $\mathbf{H}^n_{\mathbb{H}}/\Delta(\mathcal{O})$ *is an* $(3n, n-1)$*-torus bundle. Consequently, the cusp cross-section of* $M_{\Gamma(\mathcal{O})}$ *is a virtual* $(3n, n-1)$*-torus bundle.*

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