A Categorical Description of Discriminants

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Outline

Classical discriminants

The classical discriminant of a polynomial of degree $\leq n$

$$f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n \in \mathbb{C}[x_1, \ldots x_n]$$

is a polynomial $\Delta(f) \in \mathbb{Z}[a_0, \ldots, a_n]$ such that $\Delta(f) = 0$ if f has a double root.

$$\Delta(a_0 + a_1 x + a_2 x^2) = 4 a_0 a_2 - a_1^2$$

$$\Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3) =$$
$$27 a_0^2 a_3^2 + 4 a_0 a_2^3 + 4 a_1^3 a_3 - a_1^2 a_2^2 - 18 a_0 a_1 a_2 a_3$$

$$\begin{aligned}
&\Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \\
=&256 a_0^3 a_4^3 - 192 a_0^2 a_1 a_3 a_4^2 - 128 a_0^2 a_2^2 a_4^2 + 144 a_0^2 a_2 a_3^2 a_4 - 27 a_0^2 a_4^4 \\
&+ 144 a_0 a_1^2 a_2 a_4^2 - 6 a_0 a_1^2 a_3^2 a_4 - 80 a_0 a_1 a_2^2 a_3 a_4 + 18 a_0 a_1 a_2 a_3^3 \\
&+ 16 a_0 a_2^4 a_4 - 4 a_0 a_2^3 a_3^2 - 27 a_1^4 a_4^2 \\
&+ 18 a_1^3 a_2 a_3 a_4 - 4 a_1^3 a_3^3 - 4 a_1^2 a_2^3 a_4 + a_1^2 a_2^2 a_3^2
\end{aligned}$$

Discriminants à la GKZ

$\mathcal{A}$-sets

- $\mathcal{A} = \{v_1, \dots, v_n\}$ is a collection of elements of the lattice $N \cong \mathbb{Z}^d$, such that $\mathcal{A}$ generates N as a lattice
- There exists a group homomorphism $h : N \to \mathbb{Z}$ such that $h(v_i) = 1$ for any element $v_i \in \mathcal{A}$.

Notation:

$$Q := \mathrm{conv}(A) \subset \mathbb{R}^{d-1}, K := \sum_{1 \leq i \leq n} \mathbb{R}_{\geq 0} v_i = \mathbb{R}_{\geq 0} Q \subset \mathbb{R}^d$$

- ∇_A is the Zariski closure in $\mathbb{C}^n$ of the set of polynomials $f = \sum_{1 \leq i \leq n} a_i x^{v_i}$ in $\mathbb{C}[x_1, \dots, x_d]$ such that there exists some $y \in (\mathbb{C}^\times)^n$ with the property that $f = 0$ is singular at y.
- The discriminant $\Delta_A \in \mathbb{Z}[a_1, \dots, a_n]$ is the irreducible polynomial (defined up to a sign) whose zero set is given by the union of the irreducible codimension 1 components of ∇_A. For the case codim $\nabla_A > 1$, one sets $\nabla_A = 1$.

In the context of toric varieties, Danilov-Khovanskii, Batyrev introduced a more general version of regularity related to the restrictions of the Laurent polynomial f to all the non-empty faces Γ of the polytope Q.

The principal A-determinant E_A is the polynomial in $\mathbb{Z}[a_1, \ldots, a_n]$ defined as

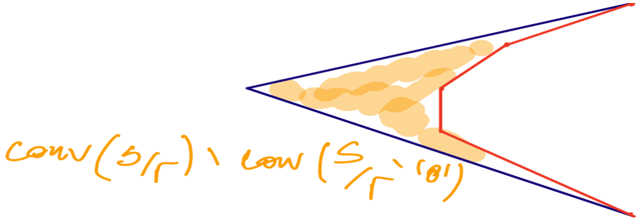
$$E_A := \prod_{\Gamma} (\Delta_{A\cap\Gamma})^{u(\Gamma)\cdot i(\Gamma)},$$

where the product is taken over all the non-empty faces Γ of the polytope $Q = \mathrm{conv}(A)$.

Let Γ be a non-empty face Γ of the polytope $Q = \mathrm{conv}(A)$.

- $i(\Gamma) := [N \cap \mathbb{R}\Gamma : \mathbb{Z}(A \cap \Gamma)]$ ($= 0$, if $A \cap \Gamma$ contains a basis of the restriction of N to the face determined by Γ)
- $S = \mathbb{Z}_{\geq 0} A$ is the semigroup generated by A. If S/Γ denotes the image semigroup of S in the quotient free group $N_{\mathbb{R}}/\mathbb{R}\Gamma$, with $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. then

$$u(\Gamma) := \mathrm{vol}\big(\mathrm{conv}\big(S/\Gamma\big) \setminus \mathrm{conv}\big(S/\Gamma \setminus \{0\}\big)\big),$$



To any $\mathcal{A}$-set in the lattice N, we can associate two toric varieties (two sides of mirror summetry).

- $Y := \operatorname{Spec} \mathbb{C}[K^\vee \cap N^\vee]$ is toric affine with Gorenstein singularities (due to the hyperplane condition)
- Any regular triangulation of the polytope Q with vertices among the elements of $\mathcal{A}$ induces a simplicial fan Σ and the associated DM Calabi–Yau stack X_Σ and a natural crepant birational morphism $\pi : X_\Sigma \to Y$.

Different triangulations give rise to different toric birational models X_Σ for the crepant resolution of the toric affine Gorenstein singularity Y.

The secondary polytope

The secondary polytope $S(A)$ is the convex hull in $\mathbb{R}^A = \mathbb{R}^n$ of the characteristics functions ϕ_Σ for all the simplicial fans Σ with

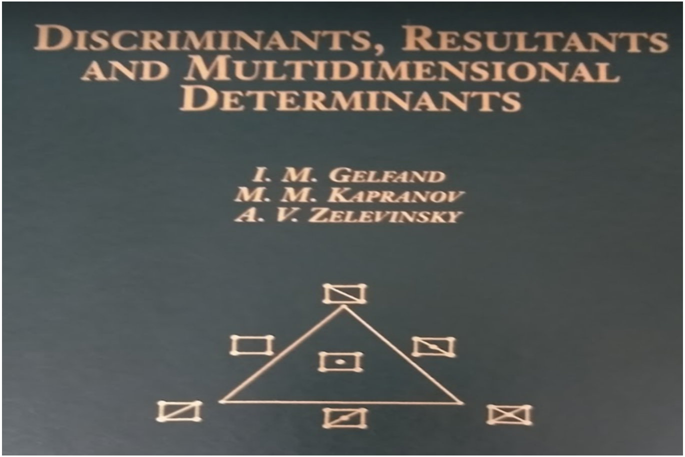
$$\phi_\Sigma(v) := \sum_{v \in \mathrm{Vert}(\sigma)} \mathrm{vol}(\sigma),$$

where the summation is taken over all the maximal cones.
The principal A–determinant E_A and the secondary polytope $S(A)$ are related in a remarkable way as shown by GKZ.

Theorem

For a given set $\mathcal{A}$, the Newton polytope of E_A coincides with the secondary polytope $S(A)$.

$S(A)$ (or its dual secondary fan) provides a toric compactification of the moduli stack of complex structures $\mathcal{M}_{cplex}(f)$.


DISCRIMINANTS, RESULTANTS
AND MULTIDIMENSIONAL
DETERMINANTS
I. M. GELFAND
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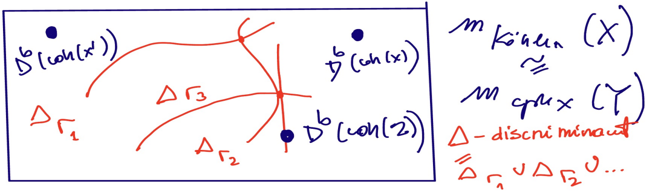
Edges and circuits

- Two simplicial fans such that the corresponding vertices in the secondary polytope are joined by an edge F differ my a modification along a *circuit* in A.
- A *circuit* in A is a minimal dependent subset $\{v_i, i \in I\}$ with $I \subset \{1, \ldots, N\}$. In particular any circuit determines a relation of the form

$$\sum_{i \in I_+} l_i v_i + \sum_{i \in I_-} l_i v_i = 0,$$

with $I = I_+ \cup I_-$, where the two subsets $I_+ := \{i : l_i > 0\}$ and $I_- := \{i : l_i < 0\}$ are uniquely defined by the circuit up to replacing I_+ by I_-.

Intuitive picture (compatible with HMS) of the moduli spaces with a compactification given by the secondary polytope $S(A)$ in the toric case.



$D^b(coh(X)) \cong D^b(coh(X'))$ (Bondal-Orlov, Kawamata,...)
The Aspinwall-Plesser-Wang conjecture (2016) is a proposal about the categorical picture along the components of the discriminant.
It is reminiscent of the (Φ, Ψ) "vanishing–nearby cycle" construction in singularity theory.

Spherical functors

The "wall monodromy" spherical functor.

Theorem

For any edge F of the secondary polytope $S(A)$, there exists a toric DM stack Z_F and an EZ–spherical wall–monodromy functor $D^b(Z_F) \to D^b(X)$ where X is the toric DM stack induced by either one of the simplicial fans corresponding to the edge F.

Let Γ be a non-empty face of the polytope $Q = \mathrm{conv}(A)$ and Σ a stacky fan supported on K.

The stacky fan Σ_Γ is induced by the canonical projection $\pi : N \to N/\mathbb{Z}(A \cap \Gamma)$. The one dimensional cones of this stacky fan Σ_Γ are indpendent of Σ but the cones of the cones of the induced fan Σ_Γ are not.

Theorem

- *For any two choices of stacky fans Σ_Γ and Σ'_Γ as above, the bounded derived categories of coherent categories $D^b(coh(X_{\Sigma_\Gamma}))$ and $D^b(coh(X_{\Sigma'_\Gamma}))$ are equivalent.*
- $\mathrm{rk}\ K_0(X_{\Sigma_\Gamma}) = u(\Gamma) \cdot i(\Gamma)$.

Set $D^b(Z_\Gamma) := D^b(coh(X_{\Sigma_\Gamma}))$

The Conjecture

Conjecture.
1) *(Aspinwall–Plesser–Wang) For each face* Γ*, there exist spherical functors* $D^b(Z_\Gamma) \to D^b(X)$ *for any toric DM stack* X *determined by a triangulation corresponding to a vertex of the secondary polytope.*
2) *(H.–Katzarkov) For any edge* F *of the secondary polytope, the category* $D^b(Z_F)$ *admits a semiorthogonal decomposition consisting of* $n_{\Gamma,F}$ *components* $D^b(Z_\Gamma)$ *for each face* Γ *of the polytope* Q*.*

The first part is a direct consequence of the second: the wall monodromy functors $D^b(Z_F) \to D^b(X)$ are spherical, so a result of Kuznetsov and Halpern-Leistner–Shipman implies that each component of the semiorthogonal decomposition determines a spherical functor.

Theorem (H.–Katzarkov)

For any edge F of the secondary polytope $S(A)$, the following equality holds:

$$\mathrm{rk}(K_0(D^b(Z_F))) = \sum_{\Gamma \subset Q} n_{\Gamma,F} \cdot \mathrm{rk}(D^b(K_0(Z_\Gamma))),$$

for some combinatorially defined non-negative integer multiplicities $n_{\Gamma,F}$.

The proof is based on an analysis of the (asymptotic) properties of the A–determinant E_A. Each edge F determines a circuit I, and discriminant Δ_I. The asymptotic expansions are expressed as powers of Δ_I in two ways corresponding to the two sides of the statement above about K_0 dimensions.

An example

- X is the resolution of the A_3 singularity, with $v_0 = (1,0)$, $v_1 = (1,1)$, $v_2 = (1,2)$, $v_3 = (1,3)$, $v_4 = (1,4)$.
- $S(A)$ is combinatorially equivalent to a cube in $\mathbb{R}^3$.
-

$$\begin{aligned}E_A = a_0 a_4\ \Delta_Q = a_0 a_4 (&256 a_0^3 a_4^3 - 192 a_0^2 a_1 a_3 a_4^2 - 128 a_0^2 a_2^2 a_4^2 \\ &+ 144 a_0^2 a_2 a_3^2 a_4 - 27 a_0^2 a_4^4 + 144 a_0 a_1^2 a_2 a_4^2 \\ &- 6 a_0 a_1^2 a_3^2 a_4 - 80 a_0 a_1 a_2^2 a_3 a_4 + 18 a_0 a_1 a_2 a_3^3 \\ &+ 16 a_0 a_2^4 a_4 - 4 a_0 a_2^3 a_3^2 - 27 a_1^4 a_4^2 \\ &+ 18 a_1^3 a_2 a_3 a_4 - 4 a_1^3 a_3^3 - 4 a_1^2 a_2^3 a_4 + a_1^2 a_2^2 a_3^2).\end{aligned}$$

- Z_Q is the point $\mathrm{Spec}\,\mathbb{C}$. $X = [\mathbb{C}^2/\mathbb{Z}_4]$ is the stacky resolution determined by the cone (v_0, v_4).
- F_1 edge corresponding to the birational transformation $X \leftrightarrow X_1$, where X_1 is the toric DM stack with cones determined by the pairs v_0, v_1 and v_1, v_4.
- The associated polyhedral subdivision is $(\mathrm{conv}\{v_0, v_4\}, \{0, 1, 4\})$, and the circuit relation I is $3v_0 - 4v_1 + v_4 = 0$.
- Δ_I is $256a_0^3a_4 - 27a_1^4$
- The leading term with respect to the edge F_1 in Δ_Q is

$$256a_0^3a_4^3 - 27a_1^4a_4^2 = a_4^2 \cdot \Delta_I.$$

- The spherical functor is

$$D^b(\mathrm{Spec}\,\mathbb{C}) \to D^b([\mathbb{C}^2/\mathbb{Z}_4]).$$

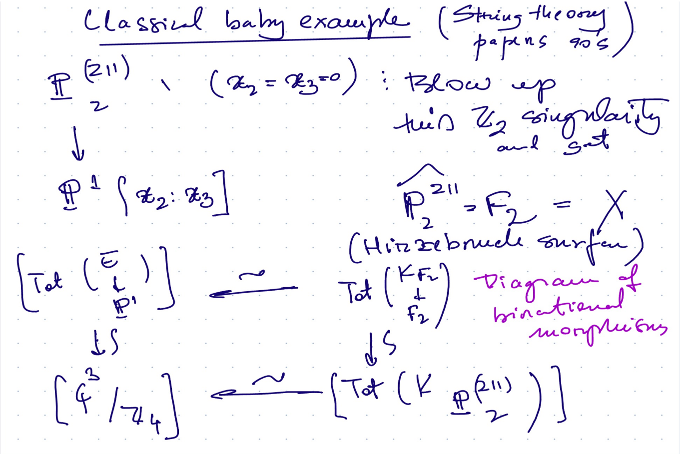
- F_2 denote the edge corresponding to the birational transformation $X \leftrightarrow X_2$, where X_2 is the toric DM stack with cones determined by the pairs v_0, v_2 and v_2, v_4.
- The circuit relation I is $v_0 - 2v_2 + v_4 = 0$. The discriminant Δ_I is $4a_0a_4 - a_2^2$.
- The the leading term with respect to the edge F_2 in the quartic discriminant Δ_Q is

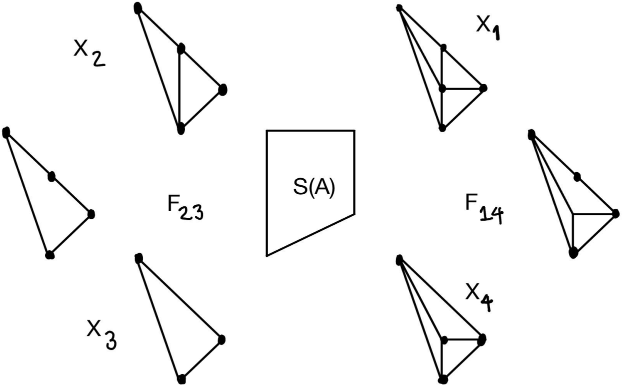
$$256a_0^3a_4^3 - 128a_0^2a_2^2a_4^2 + 16a_0a_2^4a_4 = 16a_0a_4 \cdot \Delta_I^2.$$

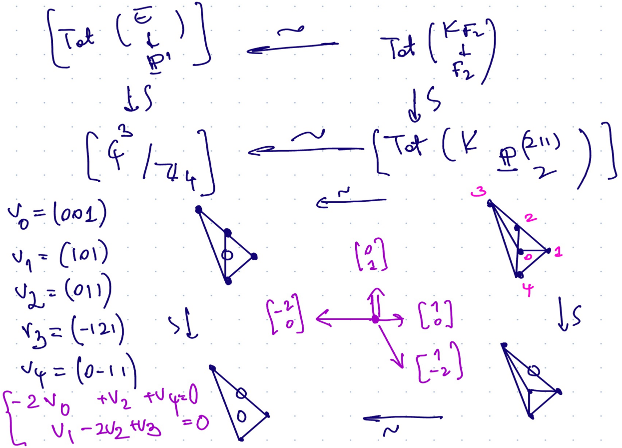
- $n_{Q,F_2} = 2$ and the spherical functor is

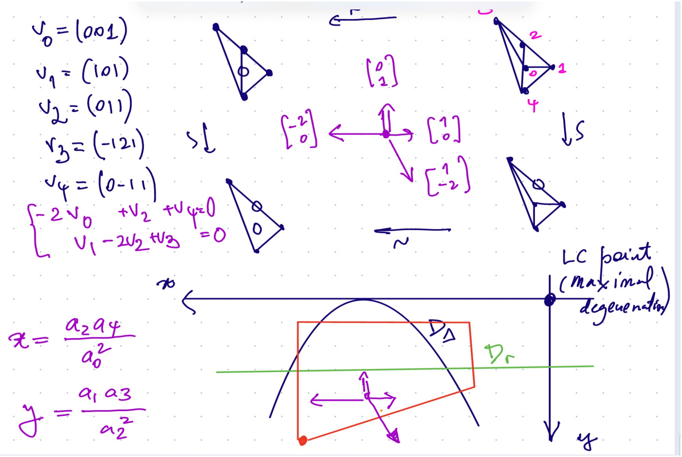
$$D^b([\mathrm{Spec}\,\mathbb{C}/\mathbb{Z}_2]) \to D^b([\mathbb{C}^2/\mathbb{Z}_4]).$$

Another example

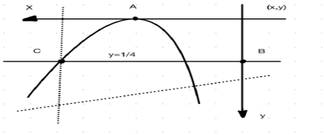



X2
X1
F23
S(A)
F14
X3
X4


Tot
Tot
K_{F_2}
F_2
$v_0 = (001)$
$v_1 = (101)$
$v_2 = (011)$
$v_3 = (-121)$
$v_4 = (0-11)$
$-2v_0 + v_2 + v_4 = 0$
$v_1 - 2v_2 + v_3 = 0$


$v_0 = (0\,0\,1)$
$v_1 = (1\,0\,1)$
$v_2 = (0\,1\,1)$
$v_3 = (-1\,2\,1)$
$v_4 = (0\,-1\,1)$
$-2v_0 + v_2 + v_4 = 0$
$v_1 - 2v_2 + v_3 = 0$
LC point (maximal degeneration)
$x = \frac{a_2 a_4}{a_0^2}$
$y = \frac{a_1 a_3}{a_2^2}$
D_Δ
D_r

Wall monodromies around the component $y = 1/4$



$x = k - -\text{constant}$, k very small.

$$\begin{array}{ccc} E = \mathbb{A}^1 \times \mathbb{P}^1 & \hookrightarrow & X_1 \\ \downarrow q & & \\ Z = \mathbb{A}^1 & & \end{array}$$

$x = k - -\text{constant}$, k very large.

$$\begin{array}{ccc} E = [\mathbb{A}^1/\mathbb{Z}_2] \times \mathbb{P}^1 & \hookrightarrow & X_3 = [\mathbb{C}^3/\mathbb{Z}_4] \\ \downarrow q & & \\ Z = [\mathbb{A}^1/\mathbb{Z}_2] & & \end{array}$$

Questions and Remarks

- Is there an analogous story for higher dimensional faces of the secondary polytope $S(A)$?
- The conjectured semi-orthogonal decompositions encode the braid monodromy associated to the complement of the discriminant locus (Aspinwall-Horja-Karp). Braid monodromy categorification.
- Wall–crossing phenomena along the components of the discriminant locus; Riemann–Hilbert correspondence
- What is the associated schober?
- Is there a Landau–Ginzburg version?
- Relation to Bridgeland's stability conditions and the theory of limiting stability conditions (Katzarkov)

Thank you for your attention!