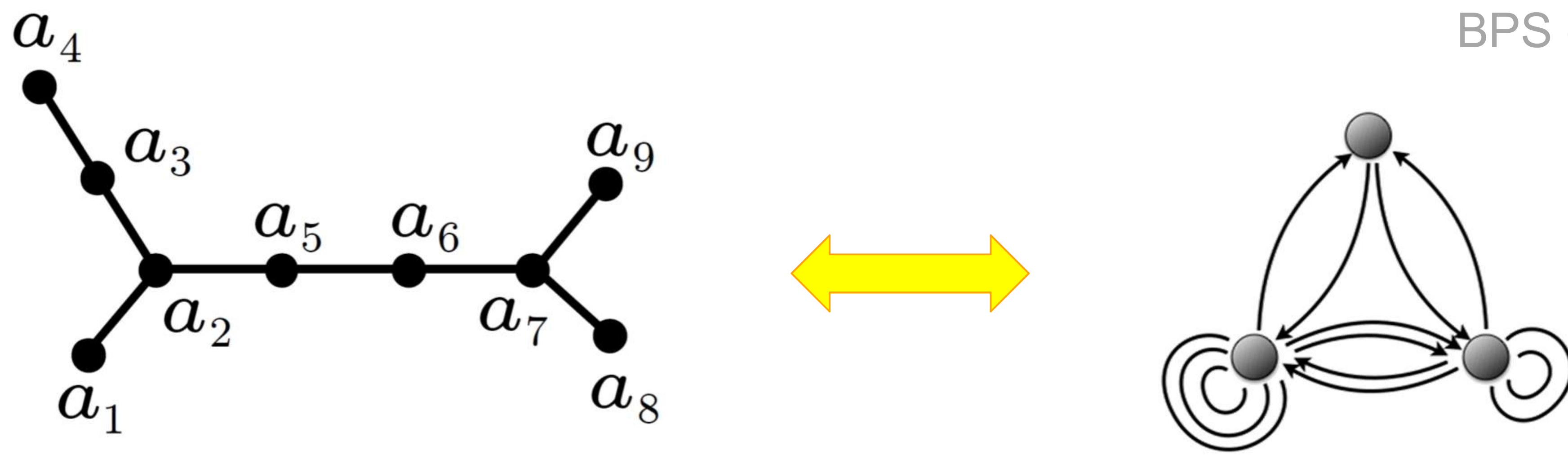
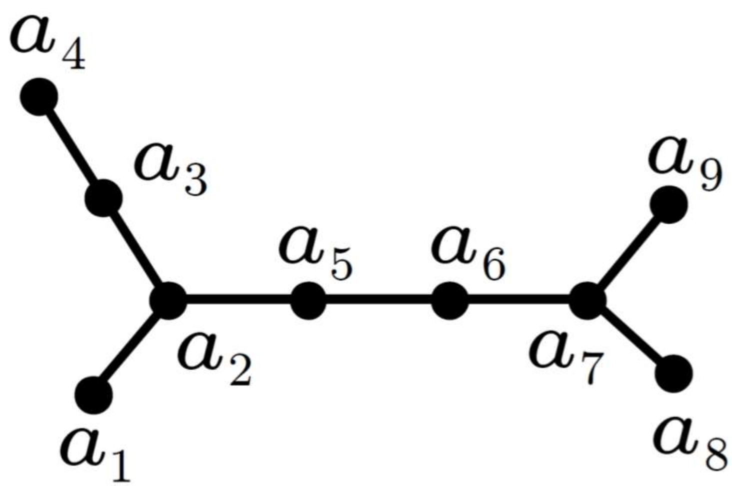
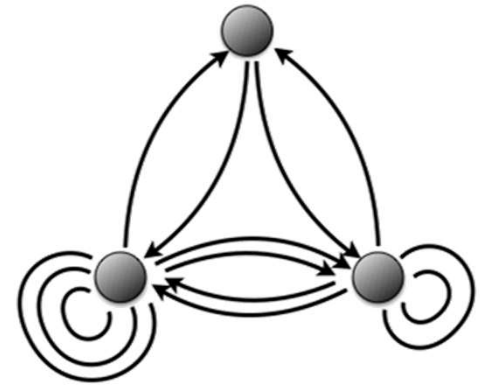
BPS quiver



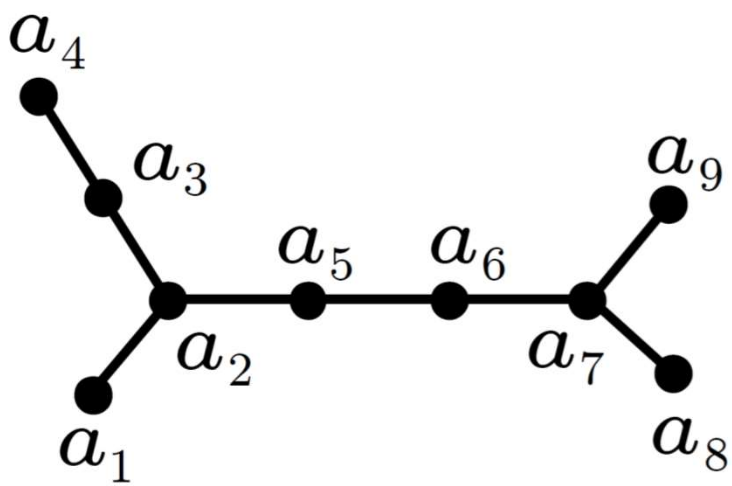

BPS quiver



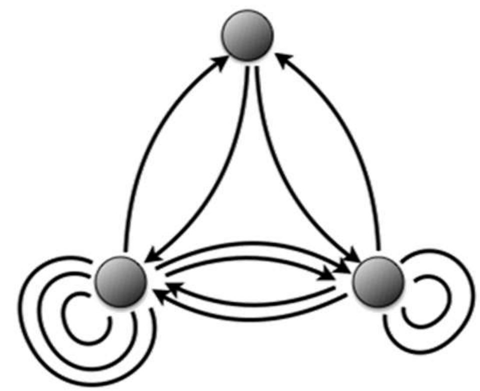



$$Q_{ij} = \begin{cases} a_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$



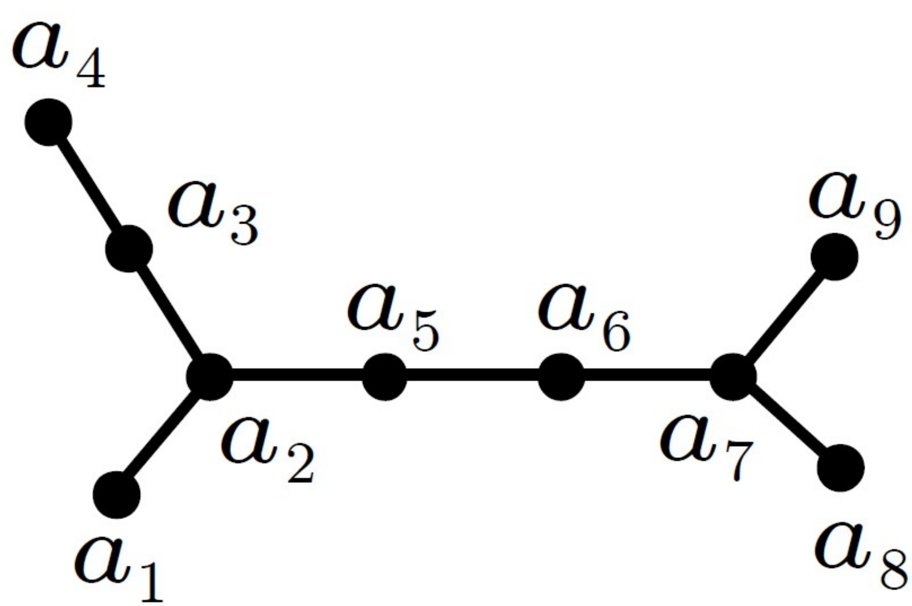

BPS quiver



$$Q_{ij} = \begin{cases} a_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\sum_{d_i \geq 0} \frac{1}{(q)_{\boldsymbol{d}}} q^{\frac{1}{2}\boldsymbol{d}\cdot C\cdot\boldsymbol{d}}$$




$$Q_{ij} = \begin{cases} a_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

# Definition:

vertex 

↔ $$q^{-\frac{a+3}{4}}\left(x-\frac{1}{x}\right)^2$$

edge ↔ $$\frac{1}{\left(x_1-\frac{1}{x_1}\right)\left(x_2-\frac{1}{x_2}\right)}$$

$$\widehat{Z}_b(q) = \text{v.p.}\int_{|x_j|=1} \prod_{j\in\text{Vertices}} \frac{dx_j}{2\pi i x_j} \cdots \prod_{(i,j)\in\text{Edges}} \cdots \Theta_b^Q(\vec{x})$$

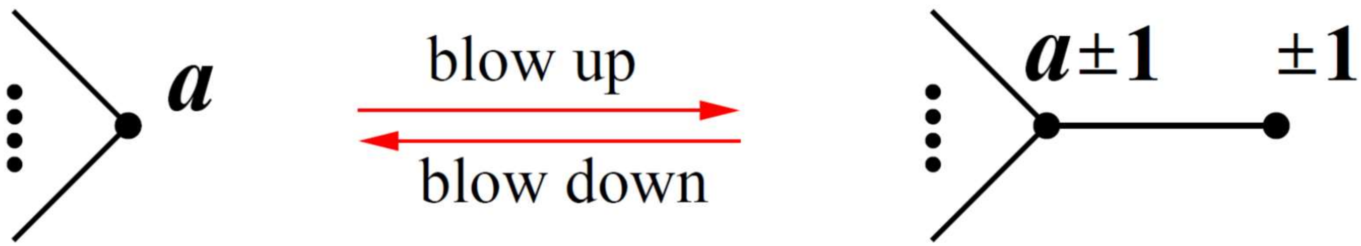
$$\widehat{Z}_b(q) = \text{v.p.} \oint_{|x_j|=1} \prod_{j \in \text{Vertices}} \frac{dx_j}{2\pi i x_j} \left( x_j - \frac{1}{x_j} \right)^{2-\deg(j)} \Theta_b^Q$$

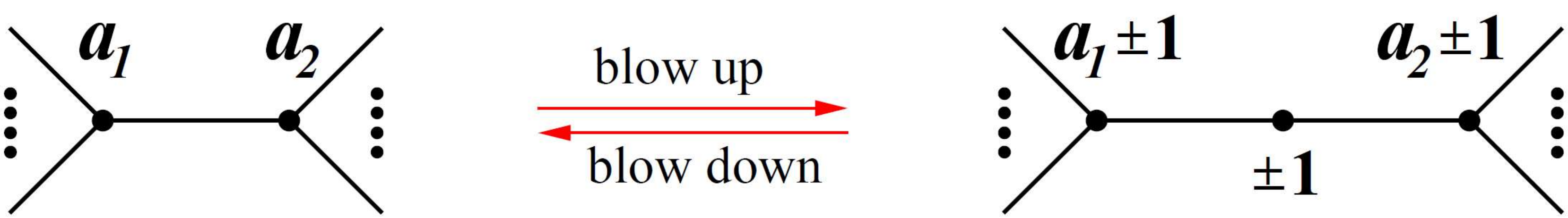
$$\Theta_b^Q = \sum_{\vec{n} \in Q\mathbb{Z}^{|\text{Vert}|}+b} q^{-(n,Q^{-1}n)} \prod_i x_i^{n_i}$$

$$b \in \operatorname{coker} Q$$

Assume $Q$ is negative definite.

# Kirby-Neumann moves




# Theorem [GPPV]:

S.G., D.Pei, P.Putrov, C.Vafa
S.G., C.Manolescu
K.Bringmann, K.Mahlburg, A.Milas

For plumbed 3-manifolds:

- $\widehat{Z}_b(M_3; q)$ converges in $|q| < 1$

  $b \in \mathrm{Spin}^c(M_3)$

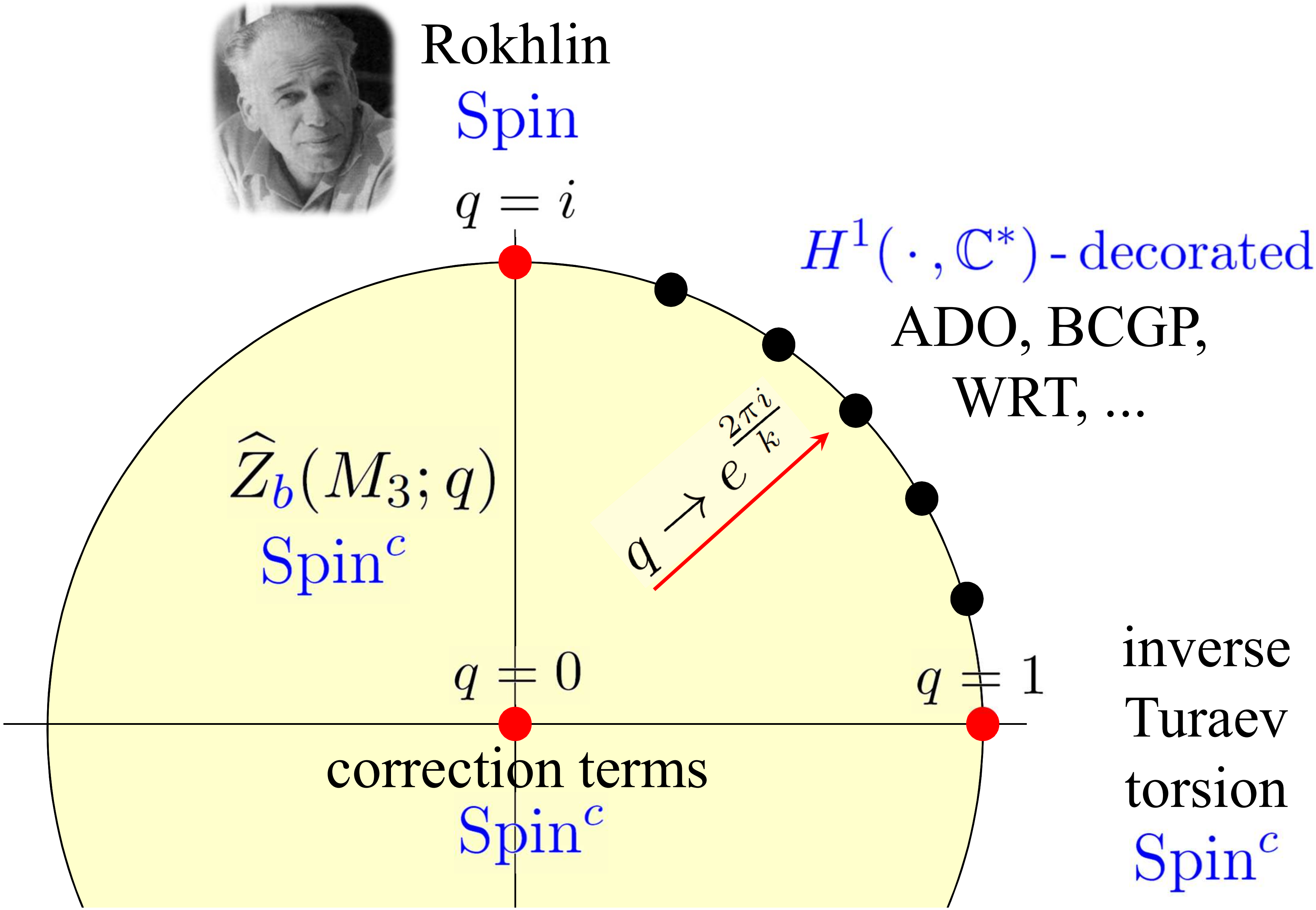
- has integer powers and integer coefficients

$$\widehat{Z}_b = q^{\Delta_b} \left( c_0 + c_1 q + c_2 q^2 + \ldots \right) \in q^{\Delta_b} \mathbb{Z}[[q]]$$

- invariant under Kirby-Neumann moves

- gives familiar quantum group invariants as $q \to e^{2\pi i/k}$

  (WRT, ADO, CGP, Rokhlin, ... )

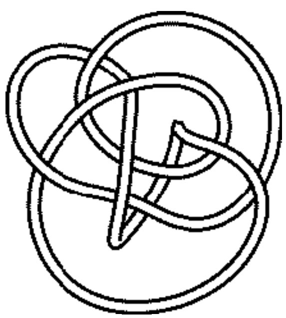


Rokhlin
Spin
$q = i$
$H^1(\cdot, \mathbb{C}^*)$ - decorated
ADO, BCGP,
WRT, ...
$q \to e^{\frac{2\pi i}{k}}$
$\widehat{Z}_b(M_3; q)$
Spin$^c$
$q = 0$
$q = 1$
inverse
Turaev
torsion
Spin$^c$
correction terms
Spin$^c$

$M_3 = S^3_{-1/2}($  $)$ :

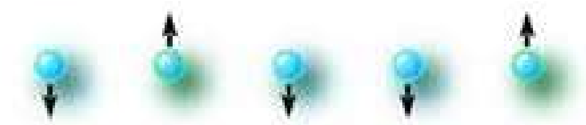
$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

$M_3 = -S^3_{+5}(\mathbf{10_{145}})$:



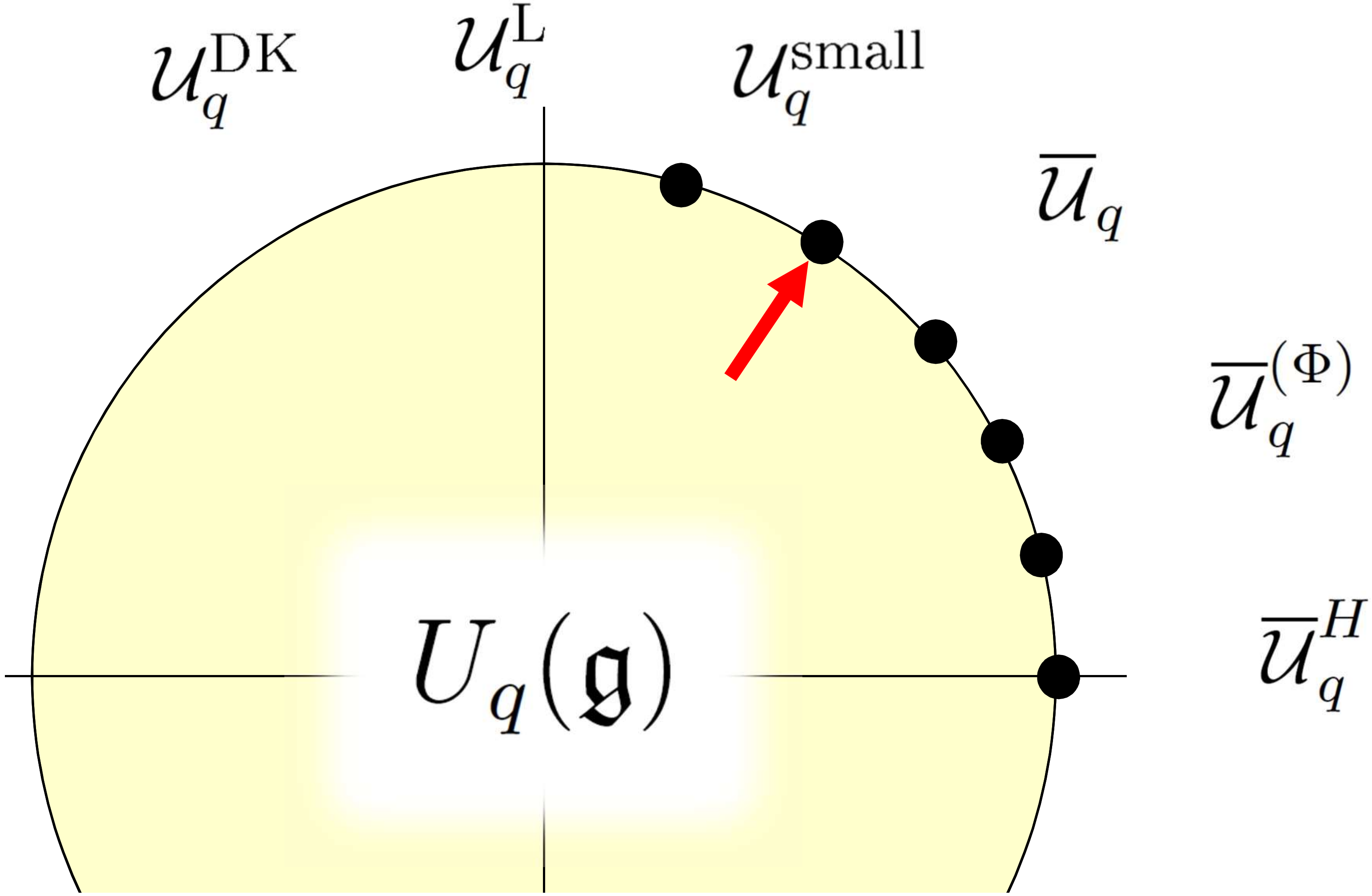
$$
\begin{aligned}
b = 2: &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right) \\
b = 1: &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\
b = 0: &\quad 2q^4 + 2q^5 + 4q^6 + 8q^7 + 14q^8 + \ldots \\
b = -1: &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\
b = -2: &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right)
\end{aligned}
$$

Symmetries
of integrable
lattice models



Vertex Operator
Algebras

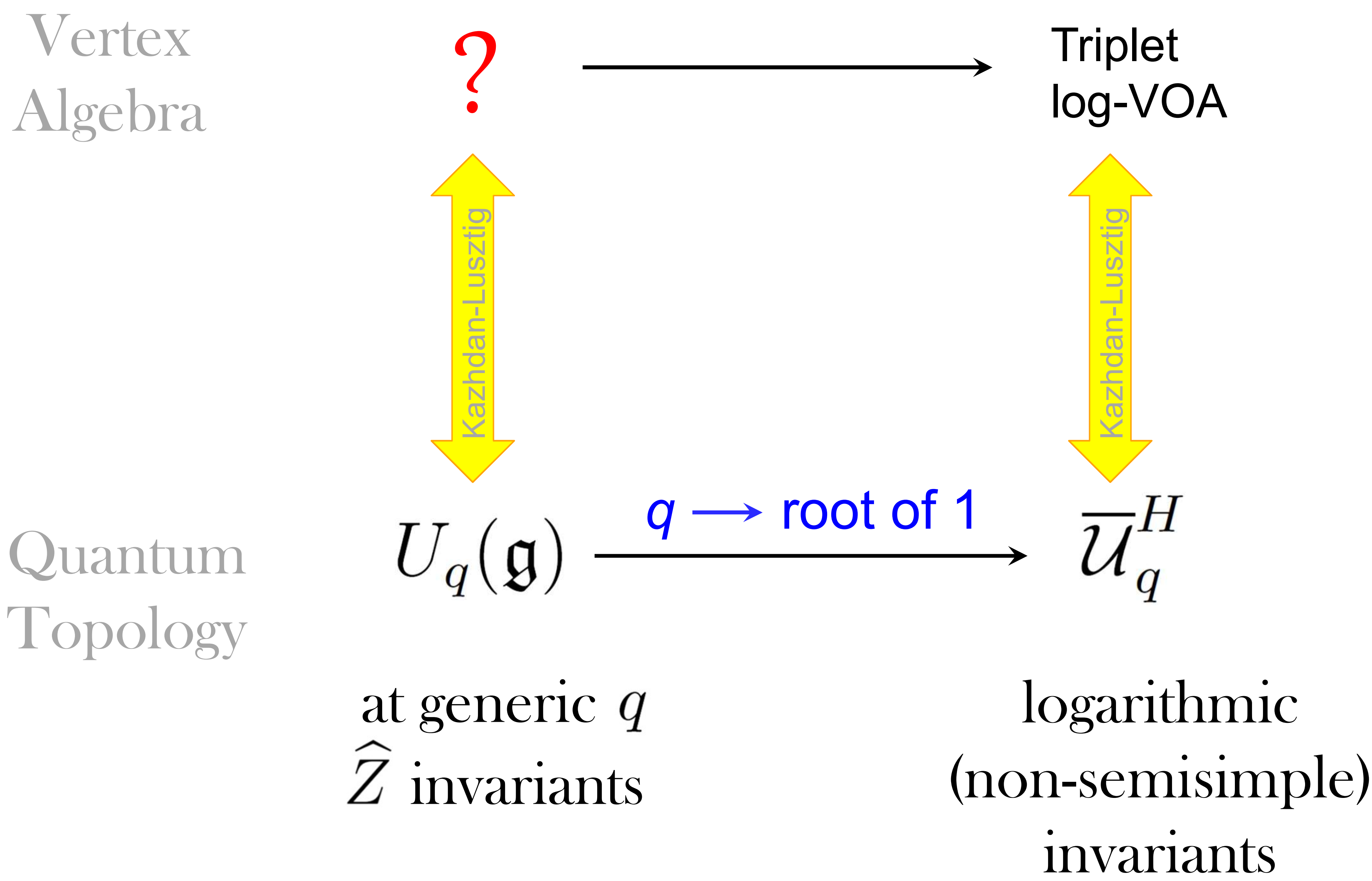
Invariants of knots
and 3-manifolds




$\mathfrak{g} = \mathfrak{sl}_2$ :

$$KEK^{-1} = q^2 E\,,\ KFK^{-1} = q^{-2} F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

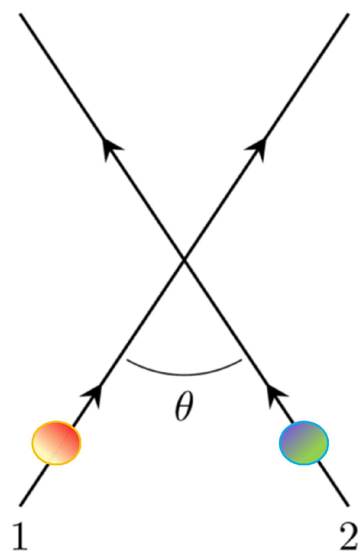

Vertex Algebra
?
Triplet log-VOA
Kazhdan-Lusztig
Kazhdan-Lusztig
Quantum Topology
$U_q(\mathfrak{g})$
$q \longrightarrow$ root of 1
$\overline{\mathcal{U}}_q^H$
at generic $q$
$\widehat{Z}$ invariants
logarithmic (non-semisimple) invariants

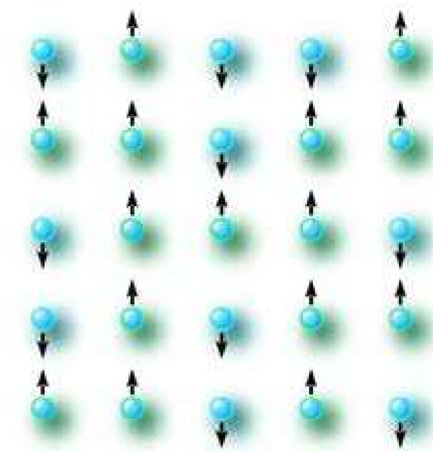
$U_q(\mathfrak{g})$

Quantum groups ⟺ Kazhdan-Lusztig ⟺ Vertex Algebras

2d CFT

Integrable lattice models



H.Bethe (1931)
:
A.Zamolodchikov, Al.Zamolodchikov (1979)
A.Zamolodchikov (1989)
Al.Zamolodchikov (1990)
F.Smirnov (1990)
N.Reshetikhin, F.Smirnov (1990)

➡ Yangian symmetry,
Bethe ansatz equation, ...

arXiv:hep-th/9301046

# Fermionic Sum Representations for Conformal Field Theory Characters

R. Kedem,[1] T.R. Klassen,[2] B.M. McCoy,[1] and E. Melzer[1]

## 1. Introduction

Recently it was found [1] that characters (or branching functions) of the coset conformal field theories $\frac{(G^{(1)})_1 \times (G^{(1)})_1}{(G^{(1)})_2}$, $G$ a simply-laced Lie algebra, can be represented in the form

$$\sum_{\mathbf{m}}^{Q} \frac{q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t}}{(q)_{m_1}\cdots(q)_{m_r}} \quad , \tag{1.1}$$

Rogers-Ramanujan

S.Kerov, A.Kirillov, N.Reshetikhin (1986)
A.Kirillov, N.Reshetikhin (1988)
:

R.Kedem, B.McCoy (1993)
R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)
S.Dasmahapatra, R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)
R.Kedem, B.McCoy, E.Melzer (1993)
A.Berkovich, B.McCoy, A.Schilling, S.Warnaar (1997)
:

E. Frenkel, A. Szenes (1993)
W.Nahm, A.Recknagel, M.Terhoeven (1993)

Advanced Studies in Pure Mathematics 61, 2011
New Structures and Natural Constructions in Mathematical Physics
pp. 161–184

# Fermionic formulas for characters of $(1, p)$ logarithmic model in $\Phi_{2,1}$ quasiparticle realisation

**Boris Feigin, Evgeny Feigin and Il'ya Tipunin**

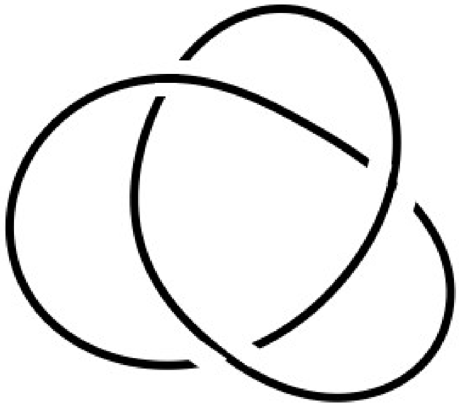
The main result of the paper is formulated as follows.

**Theorem 1.1.** *The characters* (1.7) *can be written in the form*

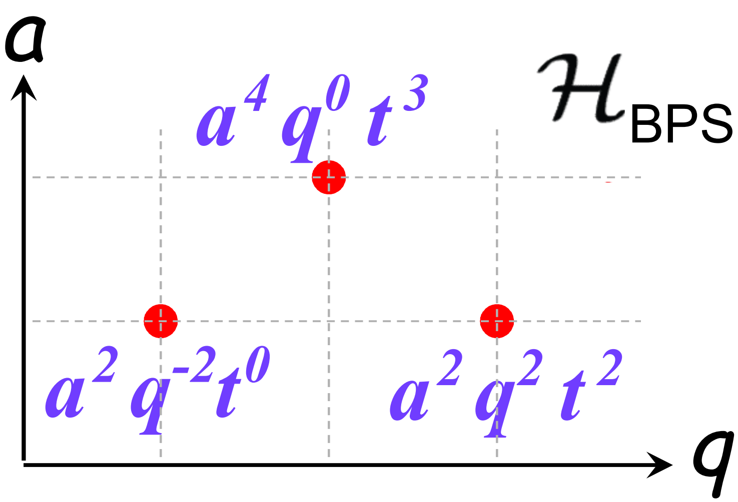
(1.8)

$$\chi_{s,p}(q) = q^{\frac{s^2-1}{4p}+\frac{1-s}{2}-\frac{c}{24}} \sum_{n_+,n_-,n_1,\dots,n_{p-1}\geq 0} \frac{q^{\frac{1}{2}\mathbf{n}\mathcal{A}\cdot\mathbf{n}+\mathbf{v}_s\cdot\mathbf{n}}}{(q)_{n_+}(q)_{n_-}(q)_{n_1}\dots(q)_{n_{p-1}}}$$

# Knot

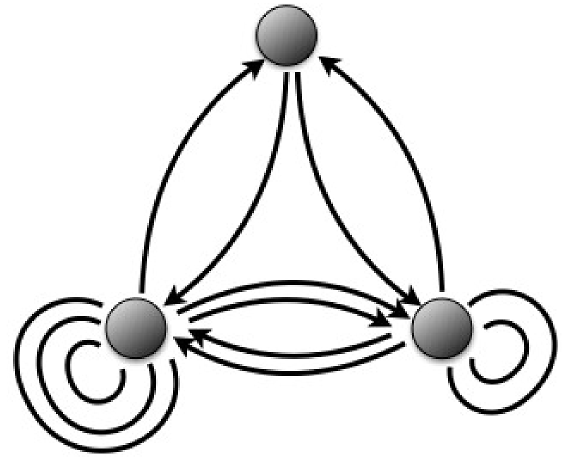


HOMFLY-PT homology




N.Dunfield, S.G., J.Rasmussen

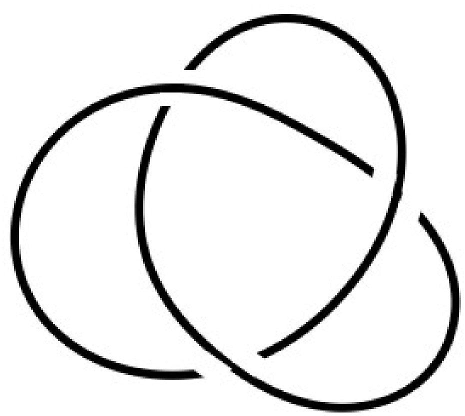
# Quiver

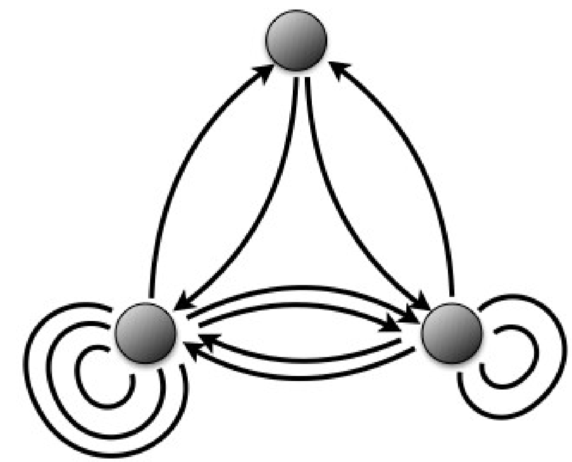


$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

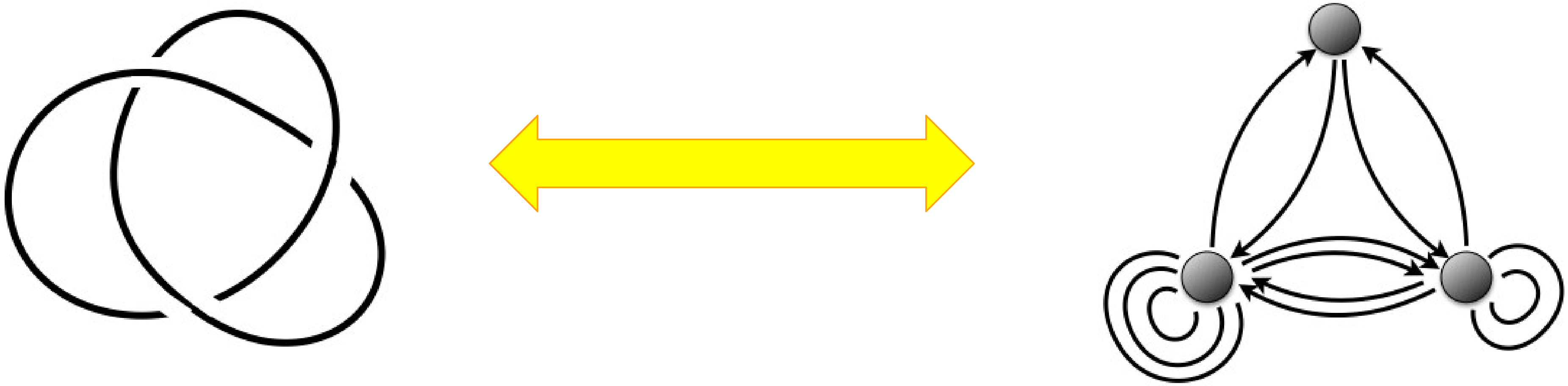
Knot

Quiver





| **Knots** | **Quivers** |
|---|---|
| Homological degrees, framing | Number of loops |
| Colored HOMFLY-PT | Motivic generating series |
| LMOV invariants | Motivic DT-invariants |
| Classical LMOV invariants | Numerical DT-invariants |
| Algebra of BPS states | Cohom. Hall Algebra |



# Surprise #1:

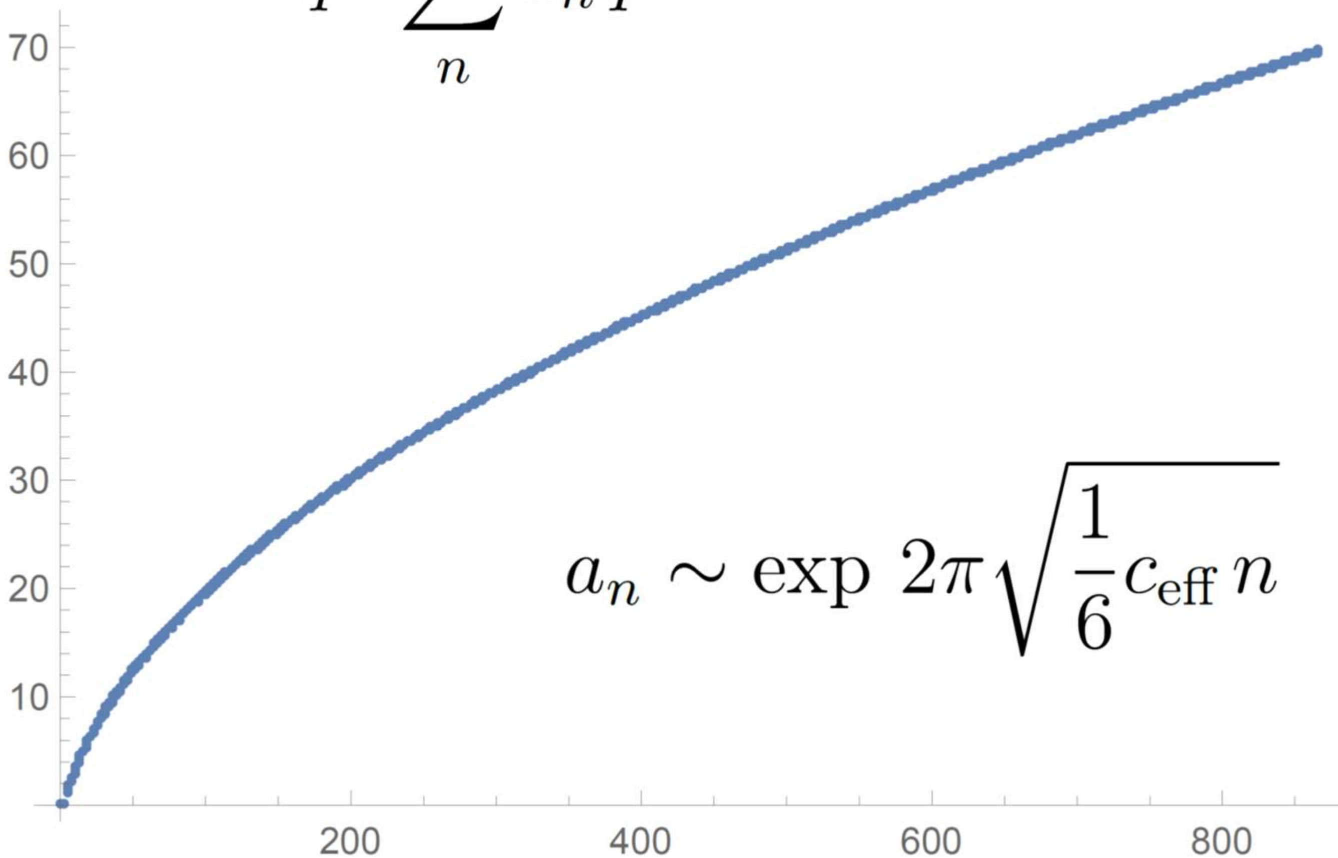
$$\sum_{n=0}^{\infty} P_n(a,q)x^n = \sum_{d_1,\ldots,d_m \geq 0} q^{\frac{1}{2}\sum_{i,j} C_{i,j} d_i d_j} \prod_{i=1}^{m} \frac{(-1)^{t_i d_i} q^{l_i d_i} a^{a_i d_i} x^{d_i}}{(q;q)_{d_i}}$$

P.Kucharski, M.Reineke, M.Stosic, P.Sulkowski
M.Stosic, P.Wedrich
T.Ekholm, P.Kucharski, P.Longhi
T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, P.Sulkowski
P.Kucharski
J.Jankowski, P.Kucharski, H.Larraguível, D.Noshchenko, P.Sulkowski
:

# Surprise #2:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

$$= q^{\Delta} \sum_n a_n q^n$$




$$a_n \sim \exp 2\pi\sqrt{\frac{1}{6}c_{\text{eff}}\, n}$$

John Cardy

Conjecture:

"conformal weight"

$$\chi_b(q) = \widehat{Z}_b(q) = q^{\Delta_b} \sum_n a_n q^n$$

Character of a logarithmic Vertex Algebra
(that depends on $M_3$, but not on $b$)

Corollary:

$$a_n \sim \exp 2\pi \sqrt{\frac{1}{6} c_{\text{eff}}\, n}$$

M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G., S.Harrison, D.Passaro
:

Conjecture:

"conformal weight"

$$\chi_b(q) \;=\; \widehat{Z}_b(q) \;=\; q^{\Delta_b} \sum_n a_n q^n$$

Character of a logarithmic Vertex Algebra
(that depends on $M_3$, but not on $b$)

$$\widehat{Z}_b(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

Problem: Construct spectral sequence to Heegaard Floer homology.

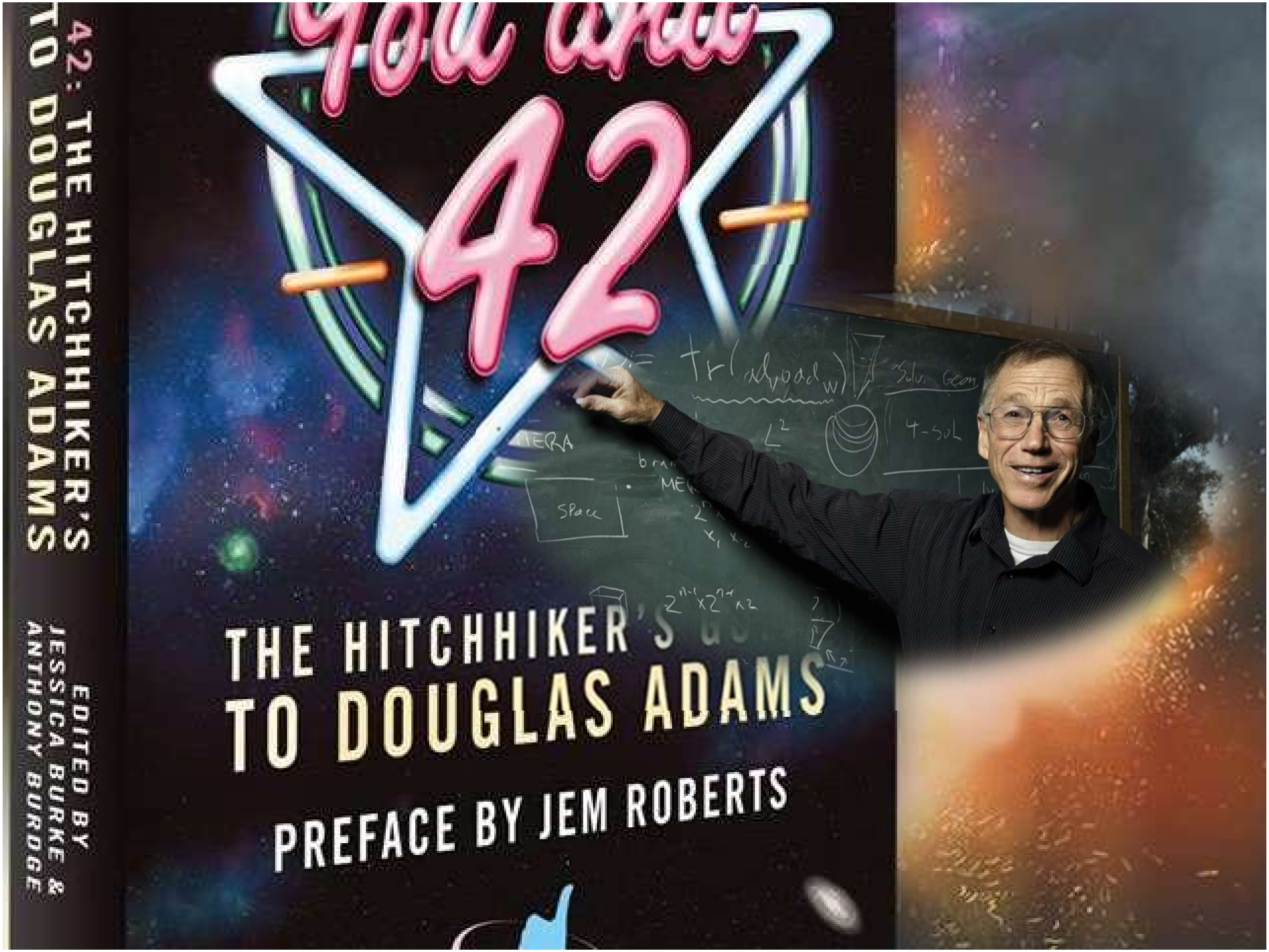
$M_3 = S^3_{-1}($  $) = S^3_{+1}($  $)$

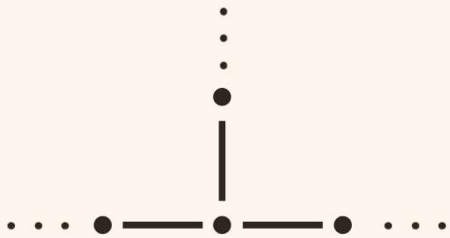
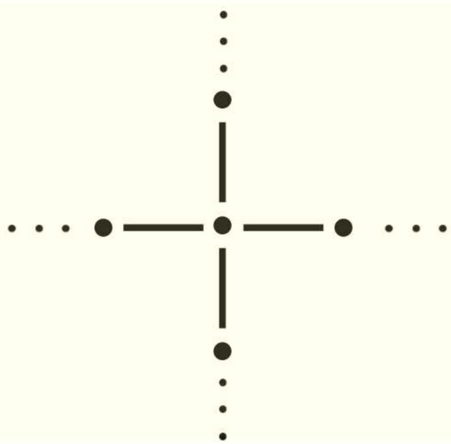
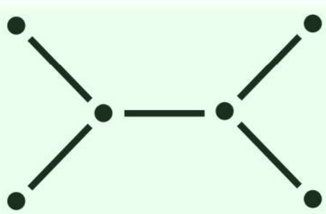
"scaling dimension"

$$\widehat{Z}(q) = q^{1/2}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + \cdots)$$

$$= q^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1};q)_n}$$

$=$ character of (1,p) "singlet" log-VOA with p = 42


42
42: THE HITCHHIKER'S
TO DOUGLAS ADAMS
EDITED BY
JESSICA BURKE &
ANTHONY BURDGE
THE HITCHHIKER'S
TO DOUGLAS ADAMS
PREFACE BY JEM ROBERTS

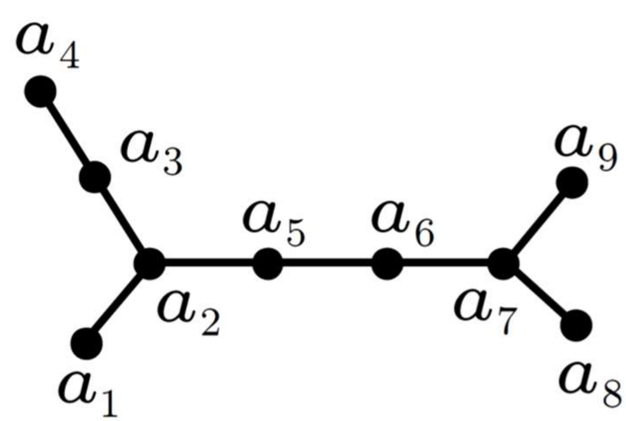
| 3-manifold | $\chi_{\mathrm{VOA}[M_3]}$ | Algebra |
|---|---|---|
|  | weight 1/2 mock | (1,p) "singlet" log-VOA |
|  | K.Bringmann, K.Mahlburg, A.Milas (2018)<br>weight 3/2 mock<br>M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison (2018) | New log-VOA |
|  | Higher depth<br>K.Bringmann, K.Mahlburg, A.Milas (2019) | ? |
| $S^3_{-1/2}($$)$ | $q^{-\frac{1}{2}}(1-q+2q^3-2q^6+\dots$<br>$\dots-15040q^{500}+\dots)$ | Non C2-cofinite log-VOA ? |

# Questions:

- Prove or disprove $a_n \sim \exp 2\pi\sqrt{\frac{1}{6} c_{\text{eff}}\, n}$ for surgeries on links

- What combination of 3-manifold invariants is $c_{\text{eff}}$ ?

- Construct a family of (logarithmic) VOAs labeled by plumbing graphs

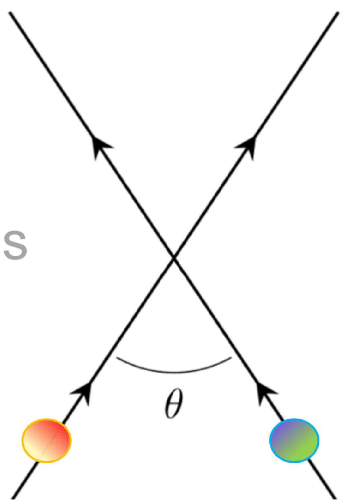


# Surprise #3:

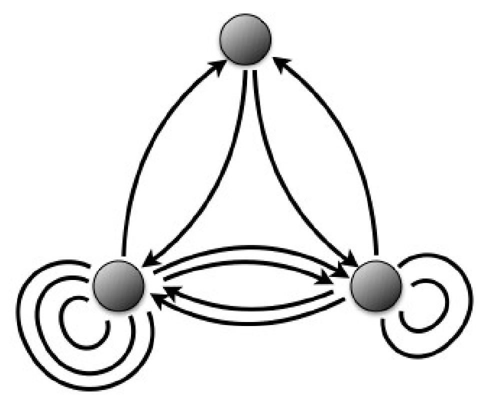
$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\widehat{Z}_b(M_3, q) = \sum_{d_i \geq 0} \frac{1}{(q)_{\boldsymbol{d}}} q^{\frac{1}{2}\boldsymbol{d}\cdot C\cdot\boldsymbol{d} + (\text{terms linear in } \boldsymbol{d})}$$

quasiparticles




BPS quiver




P.Kucharski

T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski

# Quiver form = Fermionic form of VOA characters

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

$$\sum_{d_i \geq 0} \frac{1}{(q)_{\boldsymbol{d}}} q^{\frac{1}{2}\boldsymbol{d}\cdot C\cdot\boldsymbol{d} + (\text{terms linear in } \boldsymbol{d})}$$

$$C = \begin{pmatrix} 4 & 3 & 4 & 4 & 3 & 4 \\ 3 & 4 & 4 & 4 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 5 & 4 & 4 \\ 3 & 4 & 4 & 4 & 5 & 5 \\ 4 & 5 & 4 & 4 & 5 & 5 \end{pmatrix}$$

T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G., S.Harrison, D.Passaro

# LATTICE COHOMOLOGY AND $q$-SERIES INVARIANTS OF 3-MANIFOLDS

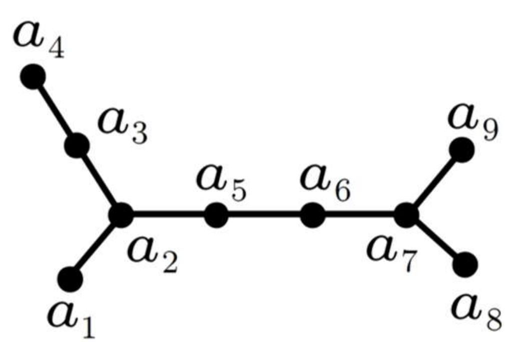
ROSTISLAV AKHMECHET, PETER K. JOHNSON, AND VYACHESLAV KRUSHKAL



**Definition 4.1.** Fix a commutative ring $\mathcal{R}$. A family of functions $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n \geq 0}$ is *admissible* if

(A1) $F_2(0) = 1$ and $F_2(r) = 0$ for all $r \neq 0$.
(A2) For all $n \geq 1$ and $r \in \mathbb{Z}$,

$$F_n(r+1) - F_n(r-1) = F_{n-1}(r).$$




**Theorem 5.10.** *For any admissible family of functions $F$, the weighted graded root is an invariant of the 3-manifold $Y(\Gamma)$ equipped with the* spin$^c$ *structure $[k]$.*

cf. A.Nemethi

## Conjecture ("mirror symmetry"):

$$\widehat{Z}(M_3, q) = \chi(q) \quad \longleftrightarrow \quad \widehat{Z}(-M_3, q) = \chi(q^{-1})$$

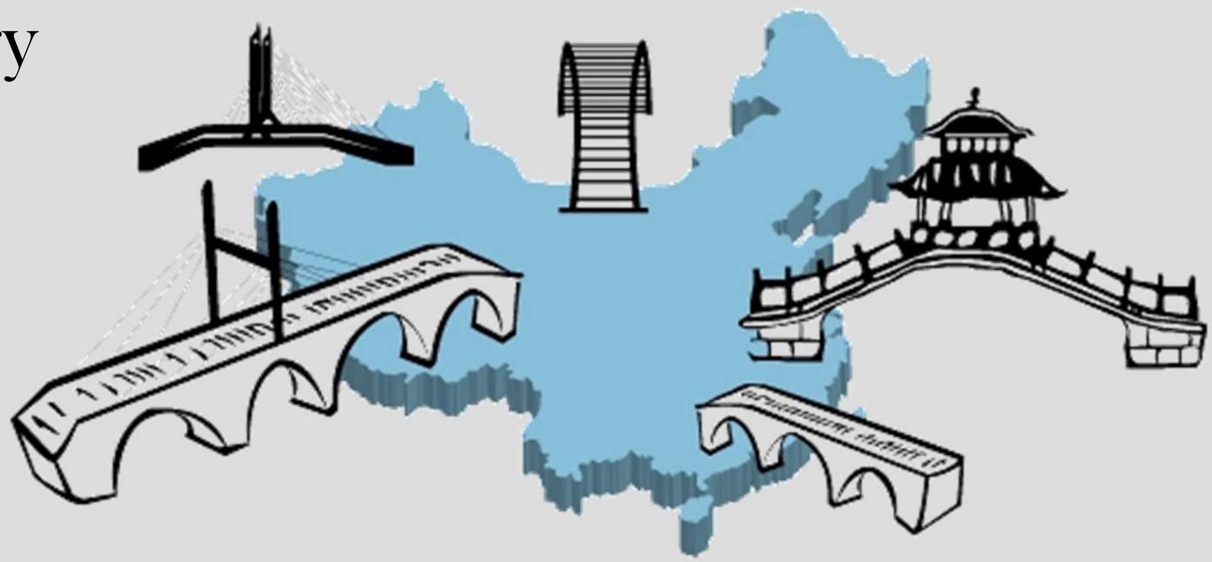
Character of a log-VOA

Character of a "mirror" log-VOA

topology

enumerative
geometry

mathematical
physics



gauge
theory

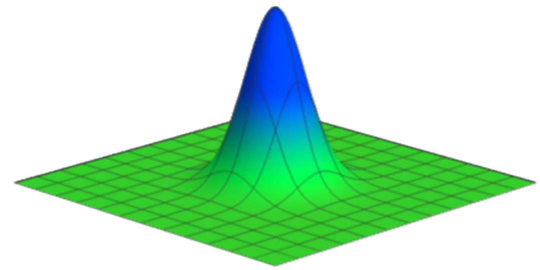
vertex algebra

quantum groups

$$\widehat{Z}(q) = \text{ partition function on } S^1 \times_q D^2$$

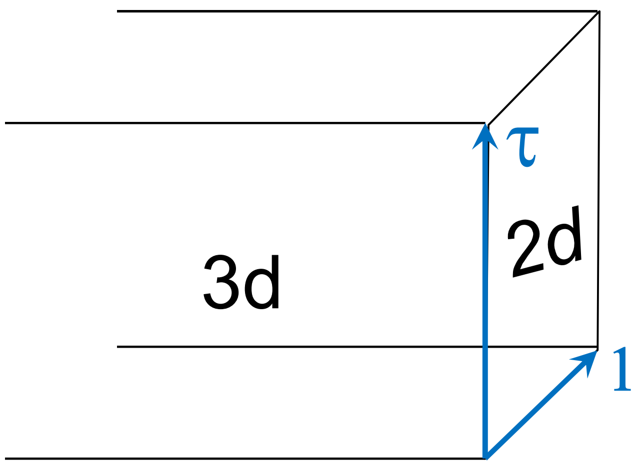
$$= \mathrm{Tr}_{\mathcal{H}_{D^2}} (-1)^F q^{R/2+J_3}$$

Counting BPS states



3d $\mathcal{N}=2$

+ 2d (0,2) boundary condition $\mathcal{B}_b$




$b$: background momentum/charge sectors of 2d boundary theory

3d "distorts" modular symmetry

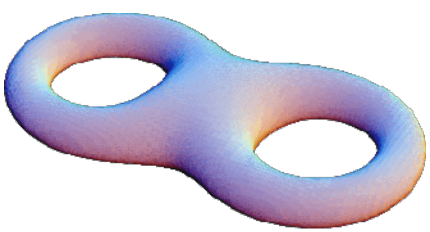
A.Gadde, S.G., P.Putrov (2013)

# Quantization of Algebraic Curves

$\mathbb{C}^* \times \mathbb{C}^*$ $\qquad$ $\widehat{x}\,,\ \widehat{y} \qquad q = e^{\hbar}$

$$\frac{dy}{y} \wedge \frac{dx}{x} \quad \rightsquigarrow \quad \widehat{y}\widehat{x} = q\,\widehat{x}\widehat{y}$$

Lagrangian submanifold

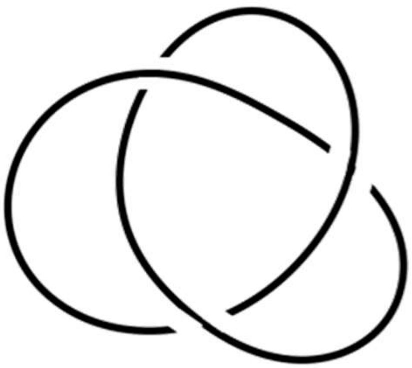


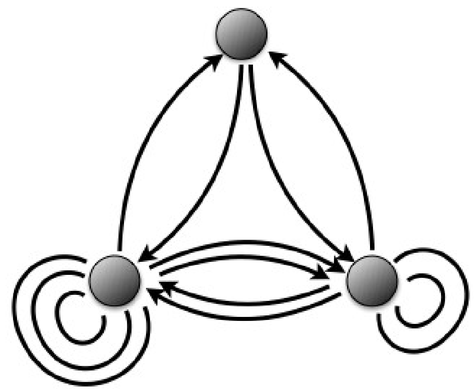
$A(x,y) = 0$

$\rightsquigarrow$ "wave-function"

$$F_K(x,q) = \exp\left(\frac{1}{\hbar}\int \log y \frac{dx}{x} + \ldots\right)$$

$\hat{A}$ = quantum Hamiltonian



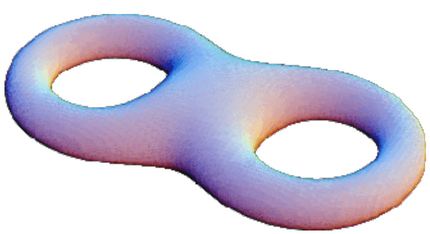


$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

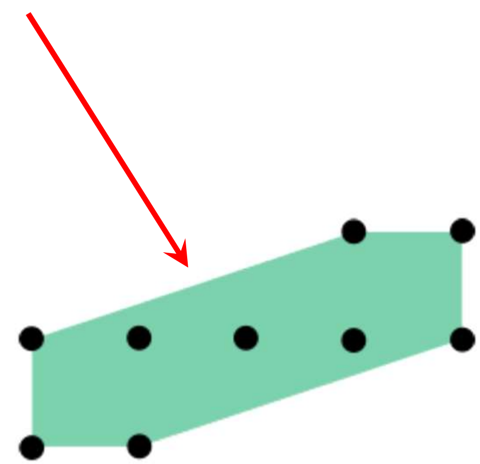
$A(x,y) = 0$

quantization

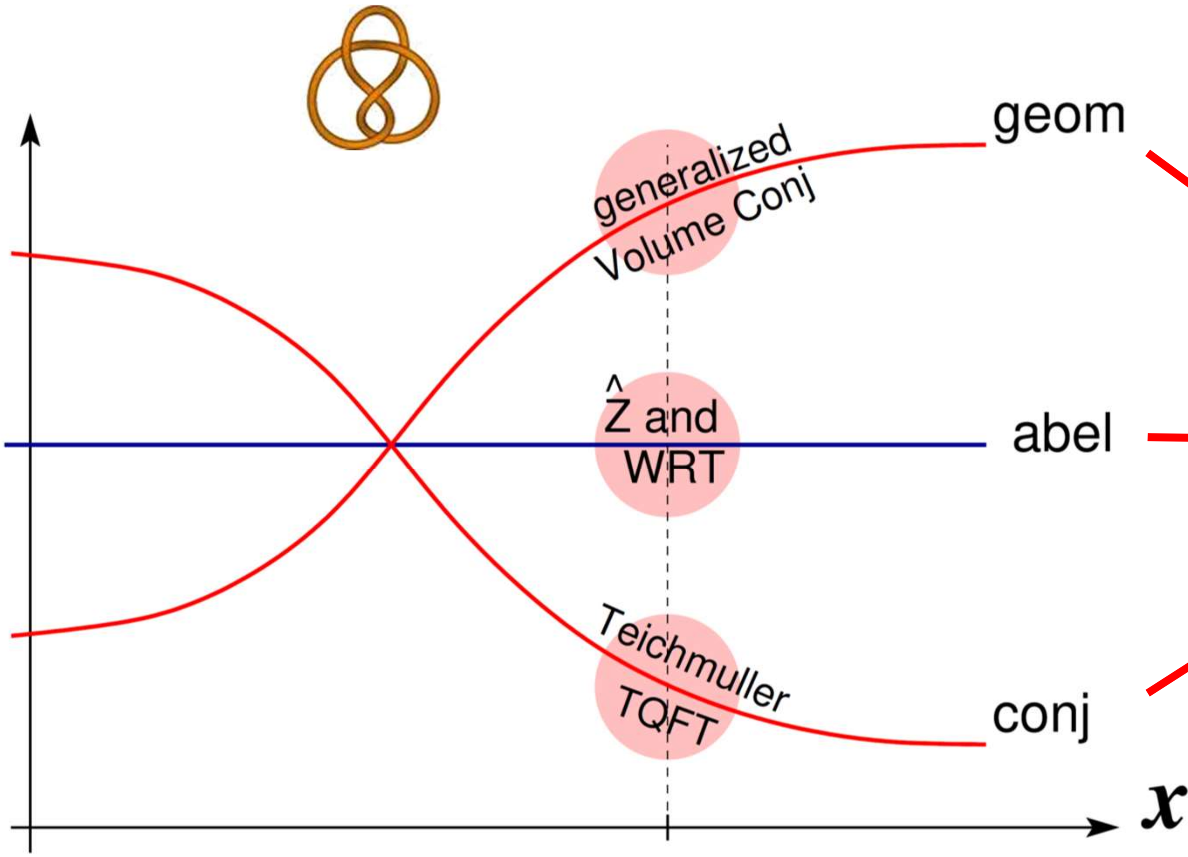
$$F_K^{(\alpha)}(x,q) = \sum_{\boldsymbol{d} \geq 0} q^{\frac{1}{2}\boldsymbol{d}\cdot\boldsymbol{C}\cdot\boldsymbol{d}} \frac{\boldsymbol{x}^{\boldsymbol{d}}}{(q)_{\boldsymbol{d}}}$$

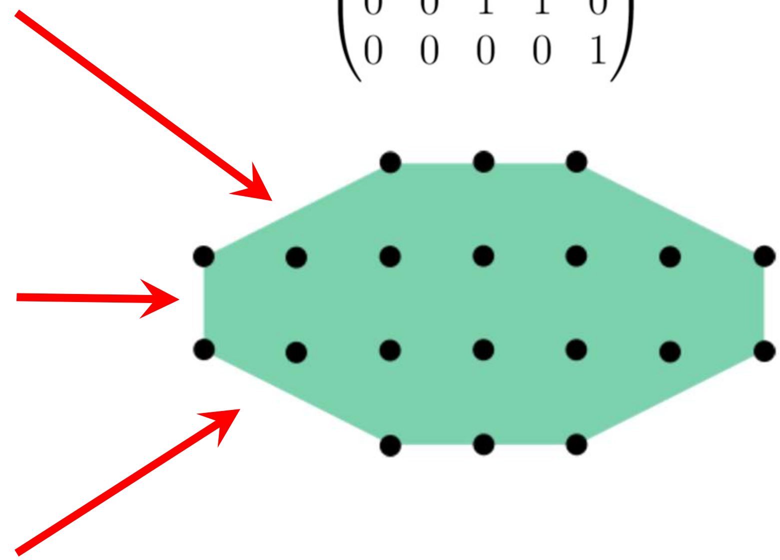


“wave-function”



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



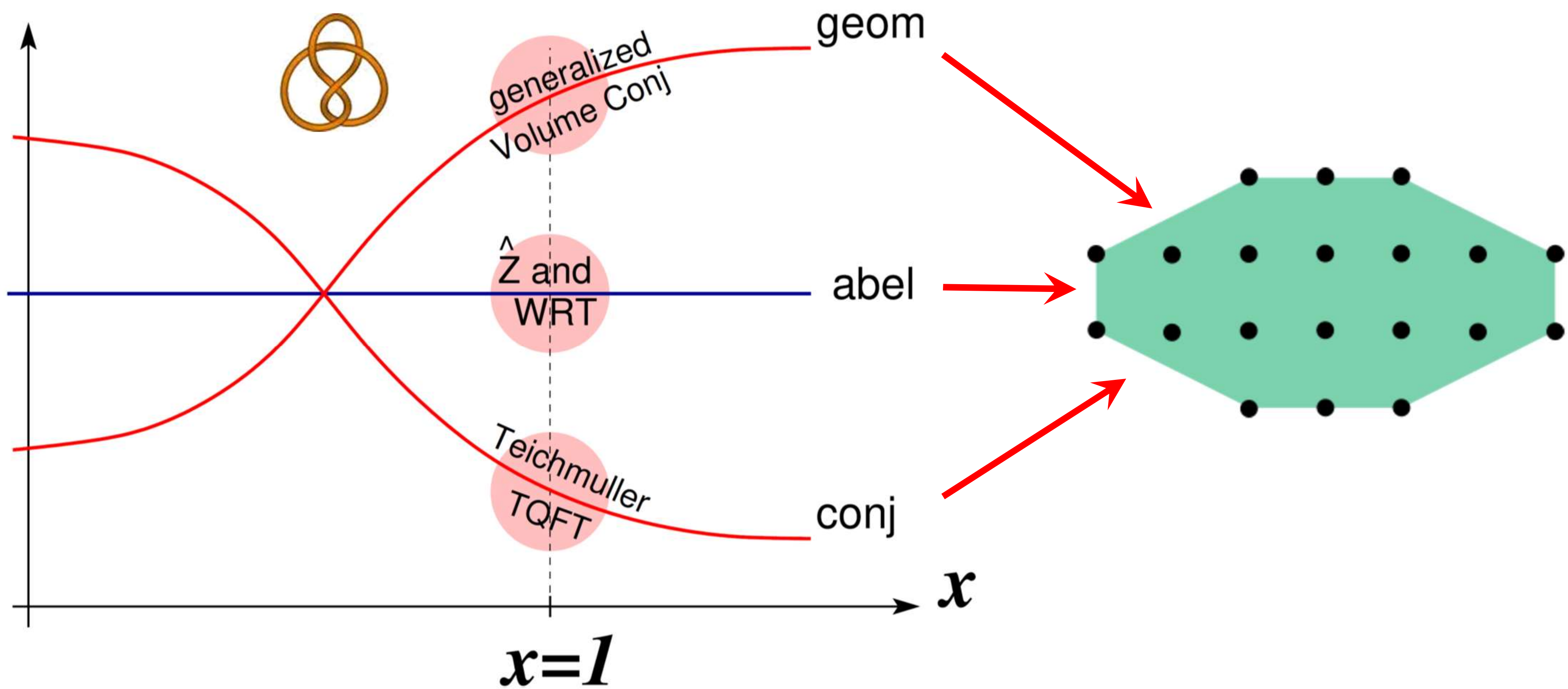
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(x,y) = (y-1)\left(1-(x^{-2}-x^{-1}-2-x+x^2)y+y^2\right)$$


P.Kucharski

T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski

$$A(x,y) = (y-1)\left(1-(x^{-2}-x^{-1}-2-x+x^{2})y+y^{2}\right)$$



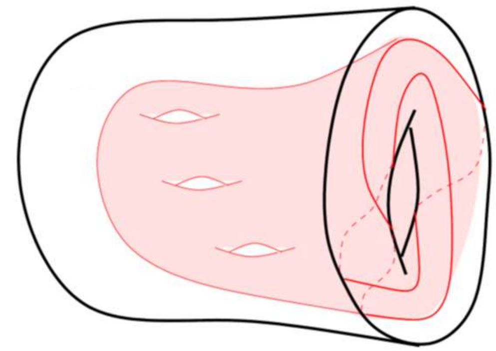

**Proposition 1.** *For any choice of branch $\alpha$, there is a left-edge $e_\alpha$ of the Newton polygon with slope $\frac{n_y}{n_x}$, such that*

$$\lim_{x\to 0} y^{(\alpha)}(x)x^{\frac{n_x}{n_y}} = \text{const} \tag{29}$$

*Moreover, the map $E : \alpha \mapsto e_\alpha$ is a surjective map onto the set of left-edges.*

boundary slope

$$\lim_{q\to 1} \frac{F_K^{(\alpha)}(qx,q)}{F_K^{(\alpha)}(x,q)} = y^{(\alpha)}(x) \sim x^{-\frac{n_x}{n_y}}$$



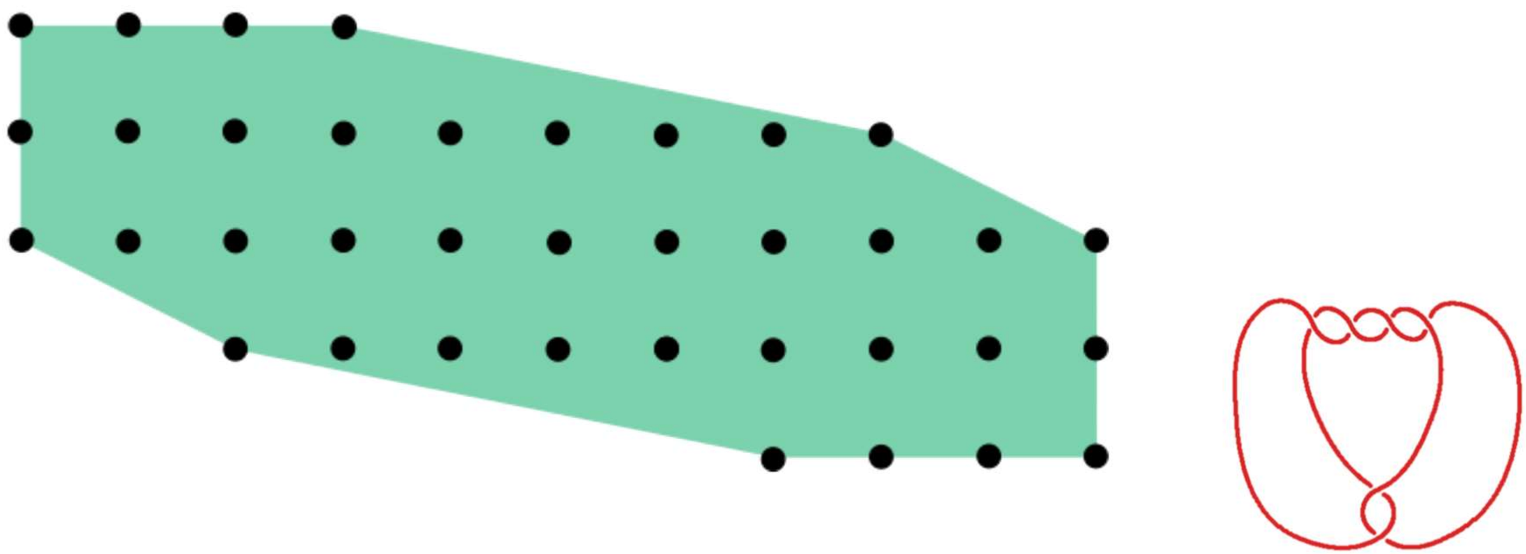
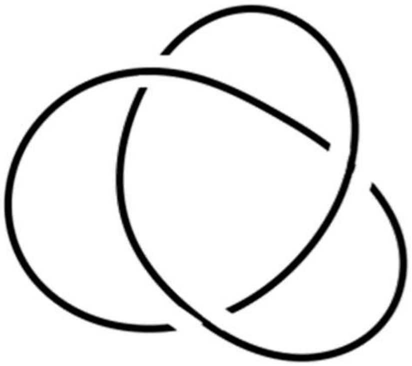


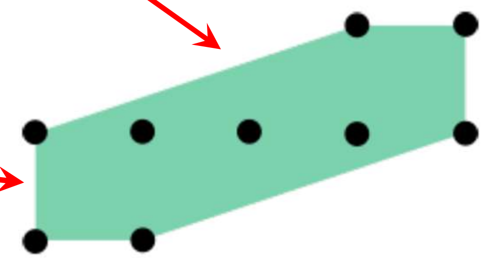
FIGURE 3. The Newton polygon of $A_{\mathbf{5}_2}$

**Remark 4.** When the $y$-height $n_y$ of $e_\alpha$ is 1, $y^{(\alpha)}(x)$ is a power series in $x$. In general, however, when $n_y > 1$, $y^{(\alpha)}(x)$ is a Puiseux series in $x$.



$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$A(x, y) = (y-1)(y+x^3)$$

$$F_K^{(\alpha)}(x, q) = \sum_{\boldsymbol{d} \geq 0} q^{\frac{1}{2}\boldsymbol{d} \cdot \boldsymbol{C} \cdot \boldsymbol{d}} \frac{\boldsymbol{x}^{\boldsymbol{d}}}{(q)_{\boldsymbol{d}}}$$