"The Hodge filtration by age and the spectrum of Du Val singularities"

(Joint work with L. Katzarkov and E. Lupercio)

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Plan of the talk:

1. The Ito-Reid filtration
2. The cyclic homology of a finite group.
3. The Hodge filtration by age.
4. The spectrum of the Du Val singularities.

The Ito-Reid filtration

Let $\mu_n := \langle e^{\frac{2\pi i}{n}} \rangle \subset S^1 \subset \mathbb{C}$ be the cyclic group of n-roots of unity and $\mu := \lim_n \mu_n \cong \hat{\mathbb{Z}}$ the pro-cyclic group of roots of unity. For any finite sub-group $G \leq SL_n(\mathbb{C})$, there is a G-action on $Hom(\mu, G)$ by conjugation and there is a canonical identification:

$$\Gamma := Hom(\mu, G)/G \cong \{\text{Conjugancy classes of } G\}$$

Note that there is a compatible μ-action on Γ by "reparametrization of loops".

Given an element $g \in G$ of order r, fix a basis of $\mathbb{C}^n$ such that $g = diag(e^{\frac{2\pi a_1 i}{r}}, \dots, e^{\frac{2\pi a_n i}{r}})$ for some integers $a_1, \dots, a_n \in [0, r)$. This allows us to define the age of g by means of:

$$age(g) := \frac{1}{r} \sum_{k=1}^{n} a_k$$

Clearly, the age is invariant under conjugation. In fact, we have:

Theorem (Ito-Reid) The age induces a well-defined grading:

$$\Gamma = \coprod_{k=1}^{n} \Gamma_k \quad \text{where } \Gamma_k = \{\text{Conjugancy classes in } \Gamma \text{ of age } k\}$$

Moreover, the grading by age on Γ is compatible with its μ-action.

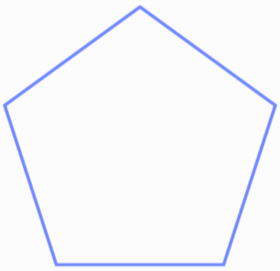
Within the realm of McKay correspondence, the age filtration acquires the following geometric meaning. Let $G \leq SL_n(\mathbb{C})$ be a finite sub-group and $X := \mathbb{C}^n/G$ the corresponding singular quotient variety. Thus, for each resolution of singularities $\pi : Y \longrightarrow X$ with exceptional divisor $\{\mathcal{E}\}$ we have the discrepancy divisor $\sum_{\mathcal{E}} a_{\mathcal{E}} \mathcal{E} = K_Y - \pi^*(K_X)$ with positive discrepancies $a_{\mathcal{E}} \geq 0$.

Theorem (Ito-Reid) There is a one-to-one correspondence between the elements of Γ_k and the components of $\{\mathcal{E}\}$ of discrepancy $a_{\mathcal{E}} = k-1$. In particular, the junior classes $(\text{age}=1)$ corresponds with the crepant components $(\text{discrepancy}=0)$.

There is a well-known classification (Klein) of the finite sub-groups of $SL_2(\mathbb{C})$. Basically, we have two families, namely; cyclic A_n and dihedral D_n, together with the exceptional tetrahedral $\mathcal{E}_6$, octahedral $\mathcal{E}_7$, and icosahedral $\mathcal{E}_8$ sub-groups corresponding with the platonic solids.

Examples:

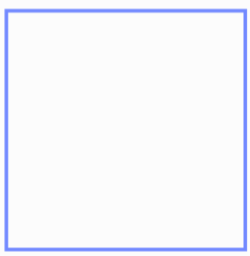
- Cyclic A_5: $\mathrm{Rot}(⬠) \cong \mathbb{Z}/5\mathbb{Z}$



Conjugancy classes $= \{1, 2, 3, 4, 5\} \circlearrowleft \mu$

Cyclic permutation action : $\{1, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$

- Dihedral D_4: $\mathrm{Sym}(\square) \cong \mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}/2\mathbb{Z} \cong \langle x, y \mid x^4 = y^2 = e, (xy)^2 = e \rangle$



Conjugancy classes $= \{e, x^2, x, y, xy\}$

Cyclic permutation action between the same age : $\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$

The periodic cyclic homology of a finite group

Now let us consider an arbitrary finite group G. From a geometric viewpoint, we have defined the purelly ineffective orbifold given by the classifying stack $\mathcal{X} := BG$. Thus, one has a natural identification:

$$\mathrm{Vect}_{\mathbb{C}}(\mathcal{X}) \simeq \mathrm{Rep}_{\mathbb{C}}(G)$$

However, note that the regular representation $R_G \in \mathrm{Rep}_{\mathbb{C}}(G)$ is a projective generator, and we have an identification

$$\mathrm{Rep}_{\mathbb{C}}(G) \simeq \mathrm{Mod}\left(\mathrm{End}(R_G)\right)$$

where $\mathrm{End}(R_G) \cong \mathbb{C}[G]$ is nothing but the complex group algebra of the group G. This means that, from the perspective of non-commutative geometry, one can think of $A := \mathbb{C}[G]$ as the "algebra of functions" on the stack $\mathcal{X}$.

Let us recall that the (reduced) Hochschild complex of the $\mathbb{C}$-algebra A is defined as the negatively graded complex $(C^{red}_\bullet(A), \partial)$ given by:

$$\cdots \xrightarrow{\partial} A/(\mathbb{C}\cdot 1) \otimes A/(\mathbb{C}\cdot 1) \otimes A \xrightarrow{\partial} A/(\mathbb{C}\cdot 1) \otimes A \xrightarrow{\partial} A$$

where the differential is given by:

$$\partial(a_0 \otimes \cdots \otimes a_n) := \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

This complex computes the Hochschild homology $HH_\bullet(A)$. Moreover, we also have the Connes differential:

$$\begin{tikzcd} \cdots \arrow[r, shift right, "\partial"'] & A/(\mathbb{C}\cdot 1) \otimes A/(\mathbb{C}\cdot 1) \otimes A \arrow[l, shift right, "B"'] \arrow[r, shift right, "\partial"'] & A/(\mathbb{C}\cdot 1) \otimes A \arrow[l, shift right, "B"'] \arrow[r, shift right, "\partial"'] & A \arrow[l, shift right, "B"'] \end{tikzcd}$$

given by:

$$B(a_0 \otimes \cdots \otimes a_n) := \sum_{\sigma \in \mathbb{Z}/(n+1)\mathbb{Z}} (-1)^\sigma \, 1 \otimes a_{\sigma(0)} \otimes \cdots \otimes a_{\sigma(n)}$$

It turns out that $B^2 = 0$, $B\partial + \partial B = 0$, and $\partial^2 = 0$.

Now, by introducing a formal parameter u of degree 2, we can define the periodic complex:

$$C^{per}_{\bullet}(A) := \left(C^{red}_{\bullet}(A)((u)),\ \partial + uB \right)$$

By means of the above complex we can compute the even $HP_{even}(A)$ and odd $HP_{odd}(A)$ homology groups, respectively. We can think of this as the periodic cyclic cohomology of the stack $HP^{\bullet}(\mathcal{X})$.

Remark: It turns out that the cyclic permutations on the periodic complex induces a well-defined μ-action on $HP_{\bullet}(A)$, which can be thought as a sort of monodromy action on the cohomology of the "loop space" of $\mathcal{X}$.

In our particular situation, we have:

Theorem (Burghelea) For the group algebra $A = \mathbb{C}[G]$ there is an identification $HP_{even}(A) \cong Class(G, \mathbb{C})$ with the space of class functions on G.

There is a more "geometric" way to look at the above result. In fact, one has a non-commutative Chern character (Connes):

$$ch : K_0(A) \to HP_{even}(A)$$

which induces an isomorphism after complexification $K_0(A) \otimes \mathbb{C} \cong HP_{even}(A)$.

Note that under the identifications $K_0(A) \cong K^0(Rep(G))$ together with $HP_{even}(A) \cong Class(G, \mathbb{C})$ this is nothing but the bijection between the representation algebra of G and the space of class functions on G given by taking the character of a representation.

Remark: We can rephrase this by saying that the non-commutative Chern character realizes Segal's localization formula

$$H^{orb}(X, \mathbb{C}) \xrightarrow{\simeq} H^{\bullet}(\Lambda X, \mathbb{C})$$

identifying the "orbifold cohomology", $H^{orb}(X, \mathbb{C}) := K^0(X) \otimes \mathbb{C}$, with the cohomology of the "twisted sectors", encoded by the inertia ΛX.

The Hodge filtration by age

There is a filtration $F^{\bullet}HP_{even}(A)$ on the even part of the periodic cyclic homology defined by means of:

$$F^{n}HP_{even}(A) = \{ [\sum_{k} \gamma_{2k} u^{k}] \in HP_{even}(A) \mid \gamma_{2k} \in C^{red}_{2k}(A),\ k \geq n \}$$

The induced filtration on $Class(G,\mathbb{C}) \cong HP_{even}(A)$ will be called the Hodge filtration by age. It turns out that this filtration is compatible with the monodromic μ-action on $HP_{even}(A)$.

Theorem (B., Katzarkov, Lupercio) For any finite sub-group $G \leq SL_n(\mathbb{C})$, the Hodge filtration by age on $Class(G,\mathbb{C})$ coincides with the filtration induced by the Ito-Reid grading by age.

Remark: In fact, as follows from the work of Kontsevich, Katzarkov, Kaledin, Pantev, Soibelman, etc, there is a Hodge filtration $F^{\bullet}HP_{\bullet}(A)$ inducing a Hodge structure for the lattice $ch: K_{\bullet}(A) \to HP_{\bullet}(A)$.

On the other hand, there is a well-defined spectrum map:

$$Sp : K_0^{\hat{\mu}}(HS) \longrightarrow K_0(Rep(\mu))$$

from the Grothendieck group of monodromic Hodge structures onto the representation ring of μ. However, since the Pontryagin dual of the procyclic group of roots of unity is given by

$$\hat{\mu} = Hom(\mu, \mathbb{C}^*) \cong \mathbb{Q}/\mathbb{Z}$$

we have $K_0(Rep(\mu)) \cong \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. For any finite group G, we define the stringy spectrum of the corresponding classifying stack $\mathcal{X} = BG$ the element of the group algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ determined by:

$$Sp^{orb}(\mathcal{X}) := Sp([HP(\mathcal{X})]^{\mu}) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

The spectrum of Du Val singularities

For finite sub-groups $G \leq SL_2(\mathbb{C})$, the corresponding quotient singular surfaces $X := \mathbb{C}^2/G$ can be realized as hypersurface singularities, known as Du Val singularities:

Kleinian quotient singularities	Du Val singularity	Steenbrink Spectrum
A_n: Cyclic	$x^2+y^2+z^{n+1}$	$1, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$
D_n: Dihedral	$x^2+y^2z+z^{n-1}$	$1, \frac{1}{2}, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}$
$\mathcal{E}_6$: Tetrahedral	$x^2+y^3+z^4$	$\frac{1}{12}, \frac{4}{12}, \frac{5}{12}, \frac{7}{12}, \frac{8}{12}, \frac{11}{12}$
$\mathcal{E}_7$: Octahedral	$x^2+y^3+yz^3$	$\frac{1}{18}, \frac{5}{18}, \frac{7}{18}, \frac{9}{18}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}$
$\mathcal{E}_8$: Icosahedral	$x^2+y^3+z^5$	$\frac{1}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{29}{30}$

Question: What is the relation between the Steenbrink spectrum of the Du Val singularities and the stringy spectrum of the corresponding Kleinian quotient orbifolds?

Answer: One can state a sort of "McKay correspondence" for the spectrum by means of motivic integration techniques and the associated motivic zeta functions (suitably extended onto the realm of Deligne-Mumford stacks).

In fact, we have the following:

Theorem (B., Katzarkov, Lupercio) The Steenbrink spectrum of of the Du Val singularities coincides with the stringy spectrum of the corresponding Kleinian orbifolds.