Equivariant birational types

joint with Hassett, Kontsevich, Kresch, Pestun

Equivariant birational geometry

Main problem: study G-actions, modulo equivariant birational transformations, in particular, embeddings of G into the Cremona group

$$\mathrm{Cr}_n = \mathrm{BirAut}(\mathbb{P}^n).$$

Equivariant birational geometry

Main problem: study G-actions, modulo equivariant birational transformations, in particular, embeddings of G into the Cremona group

$$\mathrm{Cr}_n = \mathrm{BirAut}(\mathbb{P}^n).$$

- k – ground field, of characteristic 0 and algebraically closed
- G – finite group
- X – smooth projective G-variety, (mostly) rational over k, i.e., birational to $\mathbb{P}^n$
- X^G – fixed point locus

Basic facts

- If X is rational and G is cyclic, then $X^G \neq \emptyset$.
- If $X \dashrightarrow Y$ is a G-equivariant birational map between smooth projective G-varieties, and G is abelian, then

$$X^G \neq \emptyset \Leftrightarrow Y^G \neq \emptyset.$$

- If X and Y are smooth projective G-equivariantly (stably) birational algebraic varieties then

$$\mathrm{H}^1(G', \mathrm{Pic}(X)) = \mathrm{H}^1(G', \mathrm{Pic}(Y)),$$

for all subgroups $G' \subseteq G$ (H^1-triviality).

$\mathrm{H}^1(G, \mathrm{Pic}(X))$

- Bogomolov-Prokhorov (2013): If G is cyclic of order p, acting on a smooth rational surface X and fixing a curve of genus $g \geq 1$, then
$$\mathrm{H}^1(G, \mathrm{Pic}(X)) = (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

$\mathrm{H}^1(G, \mathrm{Pic}(X))$

- Bogomolov-Prokhorov (2013): If G is cyclic of order p, acting on a smooth rational surface X and fixing a curve of genus $g \geq 1$, then
$$\mathrm{H}^1(G, \mathrm{Pic}(X)) = (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

- Shinder (2016): If G is cyclic, acting on a smooth rational surface X, and such that all stabilizers are either trivial or equal to G, then
$$\mathrm{H}^1(G, \mathrm{Pic}(X)) = \bigoplus_{C \subset X^G} \mathrm{H}^1(C, \mathbb{Z}) \otimes \mathbb{Z}/m\mathbb{Z}.$$

Reichstein–Youssin (2002)

Let V and W be d-dimensional faithful representations of an abelian group G of rank $r \leq d$, and

$$\chi_1, \ldots, \chi_d, \quad \text{respectively} \quad \eta_1, \ldots, \eta_d,$$

the characters of G appearing in V, respectively W. Then V and W are G-equivariantly birational if and only if

$$\chi_1 \wedge \cdots \wedge \chi_d = \pm\, \eta_1 \wedge \cdots \wedge \eta_d$$

(This condition is meaningful only when $r = d$.)

Reichstein–Youssin (2002)

Let V and W be d-dimensional faithful representations of an abelian group G of rank $r \leq d$, and

$$\chi_1, \ldots, \chi_d, \quad \text{respectively} \quad \eta_1, \ldots, \eta_d,$$

the characters of G appearing in V, respectively W. Then V and W are G-equivariantly birational if and only if

$$\chi_1 \wedge \cdots \wedge \chi_d = \pm\, \eta_1 \wedge \cdots \wedge \eta_d$$

(This condition is meaningful only when $r = d$.)

- Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.

Reichstein–Youssin (2002)

Let V and W be d-dimensional faithful representations of an abelian group G of rank $r \leq d$, and

$$\chi_1, \ldots, \chi_d, \qquad \text{respectively} \qquad \eta_1, \ldots, \eta_d,$$

the characters of G appearing in V, respectively W. Then V and W are G-equivariantly birational if and only if

$$\chi_1 \wedge \cdots \wedge \chi_d = \pm\, \eta_1 \wedge \cdots \wedge \eta_d$$

(This condition is meaningful only when $r = d$.)

- Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.
- Note that any two faithful representations of G are equivariantly stably birational.

Birational types $\mathcal{B}_n(G)$

Let G be a finite abelian group, and $A = G^\vee$ its group of characters. Consider the $\mathbb{Z}$-module

$$\mathcal{B}_n(G)$$

generated by unordered tupels $[a_1, \ldots, a_n], \quad a_i \in A$, such that

(G) $\sum_i \mathbb{Z}a_i = A$, and

(B) for all $a_1, a_2, b_1, \ldots, b_{n-2} \in A$ we have

$$[a_1, a_2, b_1, \ldots b_{n-2}] =$$

$$[a_1 - a_2, a_2, b_1, \ldots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \ldots, b_{n-2}] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_1, \ldots, b_{n-2}] \qquad \text{if } a_1 = a_2.$$

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_3(G)) \stackrel{?}{=} \frac{(p-5)(p-7)}{24} = \frac{p^2 - 1}{24} + 1 - \frac{p-1}{2}$$

Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2-1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.

$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_3(G)) \stackrel{?}{=} \frac{(p-5)(p-7)}{24} = \frac{p^2-1}{24} + 1 - \frac{p-1}{2}$$

Jumps at

$$p = 43, 59, 67, 83, ...$$

Birational types

Let X be smooth projective, of dimension n, with regular G-action. Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. Put

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

Birational types

Let X be smooth projective, of dimension n, with regular G-action. Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha}X$, at some $x_\alpha \in F_\alpha$. Put

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Birational types

Variant: introduce the quotient

$$\mu^{-} : \mathcal{B}_n(G) \to \mathcal{B}_n^{-}(G)$$

by an additional relation

$$[a_1, a_2, \ldots, a_n] = -[-a_1, a_2, \ldots, a_n].$$

Birational types

Variant: introduce the quotient

$$\mu^- : \mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by an additional relation

$$[a_1, a_2, \dots, a_n] = -[-a_1, a_2, \dots, a_n].$$

The class of $\mathbb{P}^n$, $n \geq 2$, with linear action of $G := \mathbb{Z}/N\mathbb{Z}$ is

- torsion in $\mathcal{B}_n(G)$ and
- trivial in $\mathcal{B}_n^-(G)$.

Equivariant Burnside group (Kresch-T. 2020)

- G is a finite group
- $H \subseteq G$ is an abelian subgroup, with character group

$$H^\vee = \mathrm{Hom}(H, k^\times)$$

- $\mathrm{Bir}_d(k)$ is the set birational equivalence classes of function fields of algebraic varieties of dimension d over k, we identify a field with its class

Equivariant Burnside group (Kresch-T. 2020)

- G is a finite group
- $H \subseteq G$ is an abelian subgroup, with character group

$$H^\vee = \mathrm{Hom}(H, k^\times)$$

- $\mathrm{Bir}_d(k)$ is the set birational equivalence classes of function fields of algebraic varieties of dimension d over k, we identify a field with its class
- $\mathrm{Alg}_N(K_0)$ is the set of isomorphism classes of Galois algebras over $K_0 \in \mathrm{Bir}_d(k)$ for the group

$$N := N_G(H)/H,$$

satisfying

Assumption 1: the composition

$$\mathrm{H}^1(N_G(H), K^\times) \to \mathrm{H}^1(H, K^\times)^N \to H^\vee$$

is surjective

Equivariant Burnside group

Let

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

be the $\mathbb{Z}$-module, generated by symbols

$$(H, N \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup,
- $K \in \mathrm{Alg}_N(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \leq n$,
- $\beta = (a_1, \ldots, a_{n-d})$, a sequence, up to order, of nonzero elements of $H^\vee$, that generate $H^\vee$.

The sequence of characters β determines a faithful representation of H over k of dimension $(n - d)$ with trivial space of invariants.

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, N \circlearrowright K, \beta) = (H', N' \circlearrowright K, \beta')$, when $H' = gHg^{-1}$ and $N' = N_G(H')/H'$, with $g \in G$, and β and β' are related by conjugation by g.

(B1): $(H, N \circlearrowright K, \beta) = 0$ when $a_1 + a_2 = 0$.

(B2): $(H, N \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } a_1 = a_2, \\ (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (a_1, a_2 - a_1, a_3, \dots, a_{n-d}), \quad \beta_2 := (a_1 - a_2, a_2, a_3, \dots, a_{n-d}),$$

and

(B2): $(H, N \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } a_1 = a_2, \\ (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (a_1, a_2 - a_1, a_3, \dots, a_{n-d}), \quad \beta_2 := (a_1 - a_2, a_2, a_3, \dots, a_{n-d}),$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } a_i \in \langle a_1 - a_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle a_1 - a_2 \rangle, \quad \bar{\beta} := (\bar{a}_2, \bar{a}_3, \dots, \bar{a}_{n-d}), \quad \bar{a}_i \in \overline{H}^\vee.$$

Equivariant Burnside group: relations

Model case: Blowing up an isolated point (with abelian stabilizer) on a surface.

It will explain the action of $\overline{N}$ on $\overline{K}$.

Equivariant Burnside group

The class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

of a G-variety is computed on a standard model X:

- X is smooth projective,
- there exists a Zariski open $U \subset X$ such that G acts freely on U,
- the complement $X \setminus U$ is a normal crossings divisor,
- for every $g \in G$ and every irreducible component D of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, N \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, all all $F \subset X$ with generic stabilizer H.

The symbols record the generic eigenvalues of H in the normal bundle along F, as well as the $N = N_G(H)/H$-action on the function field of F, respectively the orbit of F.

Equivariant Burnside group

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, N \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, all all $F \subset X$ with generic stabilizer H.

The symbols record the generic eigenvalues of H in the normal bundle along F, as well as the $N = N_G(H)/H$-action on the function field of F, respectively the orbit of F.

Note that, on a standard model, all stabilizers are abelian, and all symbols satisfy Assumption 1.

Equivariant Burnside group

Kresch-T. 2020

The class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

is a well-defined G-equivariant birational invariant.

Equivariant Burnside group: properties

- Let $\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^G(G)$ be the quotient by the subgroup generated by all symbols with $H \subsetneq G$. Then

$$\mathrm{Burn}_n^G(G) \twoheadrightarrow \mathcal{B}_n(G).$$

Equivariant Burnside group: properties

- Let $\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^G(G)$ be the quotient by the subgroup generated by all symbols with $H \subsetneq G$. Then

$$\mathrm{Burn}_n^G(G) \twoheadrightarrow \mathcal{B}_n(G).$$

- For $n = 2$ and G cyclic, we recover Blanc's theory of normalized fixed curves with action (NFCA).

Equivariant Burnside group: properties

- Let $\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^G(G)$ be the quotient by the subgroup generated by all symbols with $H \subsetneq G$. Then

$$\mathrm{Burn}_n^G(G) \twoheadrightarrow \mathcal{B}_n(G).$$

- For $n = 2$ and G cyclic, we recover Blanc's theory of normalized fixed curves with action (NFCA).
- For $n = 2$ and G cyclic of prime order, $[X \circlearrowleft G]$ encodes

$$\mathrm{H}^1(G, \mathrm{Pic}(X)).$$

Abelian actions on surfaces

- If there is no curve of genus ≥ 1 in the fixed locus X^G, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Abelian actions on surfaces

- If there is no curve of genus ≥ 1 in the fixed locus X^G, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.
- When there is a curve of genus ≥ 1 in X^G, it will appear on every equivariantly birational model.

Abelian actions on surfaces

- If there is no curve of genus ≥ 1 in the fixed locus X^G, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.
- When there is a curve of genus ≥ 1 in X^G, it will appear on every equivariantly birational model.

In particular, $\mathcal{B}_2(G)$ does not give anything new in dimension 2.

Abelian actions on surfaces

- If there is no curve of genus ≥ 1 in the fixed locus X^G, then all actions are linear, with the exception of one fixed-point free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.
- When there is a curve of genus ≥ 1 in X^G, it will appear on every equivariantly birational model.

In particular, $\mathcal{B}_2(G)$ does not give anything new in dimension 2.

However, it enters as coefficient group in higher dimensions and can contribute nontrivially to $\mathrm{Burn}_n(G)$.

Nonabelian actions on surfaces

Consider the action of $G = C_2 \times \mathfrak{S}_3 = W(\mathsf{G}_2)$ on the corresponding torus T and its Lie algebra $\mathfrak{t}$.

- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)

Nonabelian actions on surfaces

Consider the action of $G = C_2 \times \mathfrak{S}_3 = W(\mathsf{G}_2)$ on the corresponding torus T and its Lie algebra $\mathfrak{t}$.

- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)
- They are not equivariantly birational (Iskovskikh 2005)

Nonabelian actions on surfaces

These actions can be realized via:

- the action on $y_1y_2y_3 = 1$ via permutation of variables and taking inverses, with model DP6

Nonabelian actions on surfaces

These actions can be realized via:

- the action on $y_1y_2y_3 = 1$ via permutation of variables and taking inverses, with model DP6
- the action on $x_1 + x_2 + x_3$ via permutation and reversing signs, with model $\mathbb{P}^2$

Nonabelian actions on surfaces

The action on $\mathbb{P}^2 = \mathbb{P}(I \oplus V_2)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The action on $\mathbb{P}^2 = \mathbb{P}(I \oplus V_2)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

There is one fixed point, $(1 : 0 : 0)$; after blowing up, the exceptional curve is stabilized by the central involution ι, and comes with a nontrivial $\mathfrak{S}_3$-action, contributing the symbol

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \in [X \circlearrowleft G].$$

Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol.

Nonabelian actions on surfaces

The action on $\mathbb{P}^2 = \mathbb{P}(I \oplus V_2)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

There is one fixed point, $(1 : 0 : 0)$; after blowing up, the exceptional curve is stabilized by the central involution ι, and comes with a nontrivial $\mathfrak{S}_3$-action, contributing the symbol

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \in [X \circlearrowleft G].$$

Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol. ... There are also other terms.

A better model for the second action is the quadric

$$v_0v_1 + v_1v_2 + v_2v_0 = 3w^2,$$

where $\mathfrak{S}_3$ permutes the coordinates $(v_0 : v_1 : v_2)$ and the central involution exchanges the sign on w. There are no G-fixed points, but a conic $R_0 := \{w = 0\}$ with stabilizer the central C_2 and a nontrivial action of $\mathfrak{S}_3$, A better model for the second action is the quadric

$$v_0v_1 + v_1v_2 + v_2v_0 = 3w^2,$$

where $\mathfrak{S}_3$ permutes the coordinates $(v_0 : v_1 : v_2)$ and the central involution exchanges the sign on w. There are no G-fixed points, but a conic $R_0 := \{w = 0\}$ with stabilizer the central C_2 and a nontrivial action of $\mathfrak{S}_3$, ... and some other terms.

The crucial difference is that the summand

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1))$$

appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model.

The crucial difference is that the summand

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1))$$

appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model. No relations can eliminate this symbol.

This $\mathbb{P}^1$, with $\mathfrak{S}_3$-action, should be viewed as an analog of a curve of genus ≥ 1 in the fixed locus – it will appear on every equivariantly birational model.

Nonabelian actions on surfaces

Similar situations (Bannai–Tokunaga 2007):

- $\mathfrak{S}_4$-action on $\mathbb{P}^2 = \mathbb{P}(V_3)$ and DP6:

$$\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \lambda_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similar situations (Bannai–Tokunaga 2007):

- $\mathfrak{S}_4$-action on $\mathbb{P}^2 = \mathbb{P}(V_3)$ and DP6:

$$\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \lambda_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $\mathfrak{A}_5$-action on $\mathbb{P}(W_3)$ and on DP5

Abelian actions in higher dimensions

Abelian actions in dimension 3 are not fully settled, but should be, in principle, accessible.

Abelian actions in higher dimensions

Abelian actions in dimension 3 are not fully settled, but should be, in principle, accessible.

The following examples focus on dimension 4, where we currently do not know how to systematically factor birational maps.

Birational types: using $\mathrm{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0, 0, 0, 1, 3, 4)$. This X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0$$

Birational types: using $\mathrm{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0,0,0,1,3,4)$. This X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0$$

There is a cubic surface $S \subset X$, with $\mathbb{Z}/3\mathbb{Z}$-stabilizer, $\mathbb{Z}/2\mathbb{Z}$ fixes an elliptic curve, and this S is not stably $\mathbb{Z}/2\mathbb{Z}$-equivariantly rational; the corresponding symbol

$$[\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \circlearrowright k(S), \beta] \neq 0 \in \mathrm{Burn}_4(\mathbb{Z}/6\mathbb{Z}),$$

does not interact with any other symbols in $[X \circlearrowleft G]$.

Birational types: using $\mathrm{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0,0,0,1,3,4)$. This X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0$$

There is a cubic surface $S \subset X$, with $\mathbb{Z}/3\mathbb{Z}$-stabilizer, $\mathbb{Z}/2\mathbb{Z}$ fixes an elliptic curve, and this S is not stably $\mathbb{Z}/2\mathbb{Z}$-equivariantly rational; the corresponding symbol

$$[\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \circlearrowright k(S), \beta] \neq 0 \in \mathrm{Burn}_4(\mathbb{Z}/6\mathbb{Z}),$$

does not interact with any other symbols in $[X \circlearrowleft G]$. Thus X is not G-equivariantly birational to $\mathbb{P}^4$ with linear action.

Nonabelian actions in higher dimensions

Consider the action of $G = C_2 \times \mathfrak{A}_5$ on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ (with C_2 acting diagonally with -1 on W_4) and on

$$x_1^2 + \cdots + x_5^2 = 5x_0^2 \quad \subset \mathbb{P}^5,$$

with C_2 acting by $x_0 \to \pm x_0$ and $\mathfrak{A}_5 \subset \mathfrak{S}_5$ via permutations of the indices.

Nonabelian actions in higher dimensions

Consider the action of $G = C_2 \times \mathfrak{A}_5$ on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ (with C_2 acting diagonally with -1 on W_4) and on

$$x_1^2 + \cdots + x_5^2 = 5x_0^2 \quad \subset \mathbb{P}^5,$$

with C_2 acting by $x_0 \to \pm x_0$ and $\mathfrak{A}_5 \subset \mathfrak{S}_5$ via permutations of the indices.

As before, look for symbols with C_2-stabilizers:

$$(C_2, \mathfrak{A}_5 \circlearrowright K, (1)),$$

with $K = k(Q)$, where Q is the quadric given by $x_0 = 0$;

Consider the action of $G = C_2 \times \mathfrak{A}_5$ on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ (with C_2 acting diagonally with -1 on W_4) and on

$$x_1^2 + \cdots + x_5^2 = 5x_0^2 \quad \subset \mathbb{P}^5,$$

with C_2 acting by $x_0 \to \pm x_0$ and $\mathfrak{A}_5 \subset \mathfrak{S}_5$ via permutations of the indices.

As before, look for symbols with C_2-stabilizers:

$$(C_2, \mathfrak{A}_5 \circlearrowright K, (1)),$$

with $K = k(Q)$, where Q is the quadric given by $x_0 = 0$; and

$$2(C_2, \mathfrak{A}_5 \circlearrowright \mathbb{P}(W_4), (1)),$$

for the action on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ – one from the fixed point, and the other from the hyperplane at infinity.

Nonabelian actions in higher dimensions

Consider the action of $G = C_2 \times \mathfrak{A}_5$ on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ (with C_2 acting diagonally with -1 on W_4) and on

$$x_1^2 + \cdots + x_5^2 = 5x_0^2 \quad \subset \mathbb{P}^5,$$

with C_2 acting by $x_0 \to \pm x_0$ and $\mathfrak{A}_5 \subset \mathfrak{S}_5$ via permutations of the indices.

As before, look for symbols with C_2-stabilizers:

$$(C_2, \mathfrak{A}_5 \circlearrowright K, (1)),$$

with $K = k(Q)$, where Q is the quadric given by $x_0 = 0$; and

$$2(C_2, \mathfrak{A}_5 \circlearrowright \mathbb{P}(W_4), (1)),$$

for the action on $\mathbb{P}^4 = \mathbb{P}(I \oplus W_4)$ – one from the fixed point, and the other from the hyperplane at infinity. These actions are not equivariantly birational.

Algebraic tori

An algebraic torus of dimension n over a field k is a linear algebraic group T which is a k-form of $\mathbb{G}_m^n$. The absolute Galois group $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ acts on its geometric character group

$$M := \mathfrak{X}^*(T_{\bar{k}})$$

via a finite subgroup $G \subset \mathrm{GL}_n(\mathbb{Z})$, we have:

$$\rho := \Gamma_k \to G \subset \mathrm{GL}_n(\mathbb{Z}).$$

A torus T over k is uniquely determined by this representation.

Rationality of tori over nonclosed fields k has been extensively studied by Voskresenskii, Endo–Miyata, Colliot-Thélène–Sansuc, ... The Zariski problem for algebraic tori, i.e., the question of whether or not stably rational tori over k are rational over k is still open.

Algebraic tori

Rationality of tori over nonclosed fields k has been extensively studied by Voskresenskii, Endo–Miyata, Colliot-Thélène–Sansuc, ... The Zariski problem for algebraic tori, i.e., the question of whether or not stably rational tori over k are rational over k is still open.

The categorical approach to rationality of tori, following Kuznetsov, has been explored by Ballard–Duncan–Lamarche–McFaddin (2020).

A relevant cohomological obstruction comes from the exact sequence (of Galois modules)

$$0 \to M \to \Pi \to \mathrm{Pic}(X) \to 0,$$

where Π is a permutation module, spanned by geometric components of the boundary $X \setminus T$, in some equivariant projective compactification X of T.

A relevant cohomological obstruction comes from the exact sequence (of Galois modules)

$$0 \to M \to \Pi \to \mathrm{Pic}(X) \to 0,$$

where Π is a permutation module, spanned by geometric components of the boundary $X \setminus T$, in some equivariant projective compactification X of T. An obstruction to stable k-rationality is nontriviality of

$$\mathrm{H}^1(G', \mathrm{Pic}(X))$$

for some subgroup $G' \subset G$.

Kunyavskii proved this is the only obstruction in dimensions ≤ 3.

Kunyavskii proved this is the only obstruction in dimensions ≤ 3.

However, there are 10 conjugacy classes of subgroups of

- $C_2 \times \mathfrak{A}_5$
- $C_2 \times \mathfrak{S}_4$

for which stable rationality is known but rationality of the corresponding algebraic tori is unknown.

Focus on $G := C_2 \times \mathfrak{A}_5 \subset \mathrm{GL}_4(\mathbb{Z})$. The action of $\mathfrak{A}_5$ is via W_4, the central C_2 acts diagonally via (-1).

Focus on $G := C_2 \times \mathfrak{A}_5 \subset \mathrm{GL}_4(\mathbb{Z})$. The action of $\mathfrak{A}_5$ is via W_4, the central C_2 acts diagonally via (-1). A compactification is given by

$$x_1x_2x_3x_4x_5 = y_1y_2y_3y_4y_5 \subset (\mathbb{P}^1)^5,$$

where $\mathfrak{A}_5$ permutes the indices and C_2 acts via $x_i \to y_i$.

Focus on $G := C_2 \times \mathfrak{A}_5 \subset \mathrm{GL}_4(\mathbb{Z})$. The action of $\mathfrak{A}_5$ is via W_4, the central C_2 acts diagonally via (-1). A compactification is given by

$$x_1x_2x_3x_4x_5 = y_1y_2y_3y_4y_5 \subset (\mathbb{P}^1)^5,$$

where $\mathfrak{A}_5$ permutes the indices and C_2 acts via $x_i \to y_i$.

The only G-fixed point is the origin. After blowing up have the symbols

$$(C_2, \mathfrak{A}_5 \circlearrowright \mathbb{P}(W_4), (1)).$$

Focus on $G := C_2 \times \mathfrak{A}_5 \subset \mathrm{GL}_4(\mathbb{Z})$. The action of $\mathfrak{A}_5$ is via W_4, the central C_2 acts diagonally via (-1). A compactification is given by

$$x_1x_2x_3x_4x_5 = y_1y_2y_3y_4y_5 \subset (\mathbb{P}^1)^5,$$

where $\mathfrak{A}_5$ permutes the indices and C_2 acts via $x_i \to y_i$.

The only G-fixed point is the origin. After blowing up have the symbols

$$(C_2, \mathfrak{A}_5 \circlearrowright \mathbb{P}(W_4), (1)).$$

On the other hand, the linear representation of G given by $\mathbb{P}(I \oplus W_4)$, with C_2 acting diagonally -1 on the 4-dim piece contributes two such symbols.

Specialization

To understand specialization, we introduce invariants of quasi-projective varieties:

$$[U \circlearrowleft G]^{\text{naive}} := \sum_{H} \sum_{V \subset U} (H, N_G(H)/H \circlearrowright k(V), \beta_V(U)) \in \mathrm{Burn}_n(G)$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subset G$, V has generic stabilizer H, an abelian subgroup of G.

This is a G-birational invariant.

To understand specialization, we introduce invariants of quasi-projective varieties:

$$[U \circlearrowleft G]^{\text{naive}} := \sum_{H} \sum_{V \subset U} (H, N_G(H)/H \circlearrowright k(V), \beta_V(U)) \in \mathrm{Burn}_n(G)$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subset G$, V has generic stabilizer H, an abelian subgroup of G.

This is a G-birational invariant.

However, with this definition, the boundary does not carry enough information about $U \circlearrowleft G$.

Specialization

To rectify this, consider

$$U = X \setminus D, \quad D = \cup_{i \in \mathcal{I}} D_i, \quad D_I := \cap_{i \in I} D_i, \quad I \subseteq \mathcal{I},$$

where U has generically free G-action, D_i are G-invariant.

Specialization

To rectify this, consider

$$U = X \setminus D, \quad D = \cup_{i\in\mathcal{I}} D_i, \quad D_I := \cap_{i\in I} D_i, \quad I \subseteq \mathcal{I},$$

where U has generically free G-action, D_i are G-invariant.

Put

$$[U \circlearrowleft G] := [X \circlearrowleft G] + \sum_{\emptyset\neq I\subseteq\mathcal{I}} (-1)^{|I|}[\mathcal{N}_{D_I/X} \circlearrowleft G]^{\mathrm{naive}}.$$

Specialization

To rectify this, consider

$$U = X \setminus D, \quad D = \cup_{i \in \mathcal{I}} D_i, \quad D_I := \cap_{i \in I} D_i, \quad I \subseteq \mathcal{I},$$

where U has generically free G-action, D_i are G-invariant.

Put

$$[U \circlearrowleft G] := [X \circlearrowleft G] + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}_{D_I / X} \circlearrowleft G]^{\mathrm{naive}}.$$

Note that the classes $[U \circlearrowleft G]$ generate $\mathrm{Burn}_n(G)$.

Equivariant Burnside volume

Imitating the above construction in the relative setting, we have:

Theorem (Kresch-T. 2020)

Let $\mathfrak{o}$ be a DVR with fraction field K and residue field k, of characteristic zero. There exists a well-defined homomorphism (depending on the choice of uniformizer π)

$$\rho_\pi^G : \mathrm{Burn}_{n,K}(G) \to \mathrm{Burn}_{n,k}(G).$$

Major recent progress in birational geometry, using failure of (stable) rationality via specialization:

- Voisin (2013): integral decomposition of Δ (Bloch–Srinivas)
- Colliot-Thélène–Pirutka (2015): universal CH_0-triviality
- Nicaise–Shinder (2017): $\mathrm{K}_0(Var_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

Specialization of equivariant birational types

Theorem (Kresch-T. 2020)

Let X and X' be smooth projective varieties over K with generically free G-actions, admitting regular models $\mathcal{X}$, respectively $\mathcal{X}'$, smooth and projective over $\mathfrak{o}$, to which the G-action extends. If X and X' are G-equivariantly birational over K then so are the special fibers of $\mathcal{X}$ and $\mathcal{X}'$.

There is also a notion of mild singularities allowing to understand the equivariant birational type of special fibers:

Specialization of equivariant birational types

There is also a notion of mild singularities allowing to understand the equivariant birational type of special fibers:

Definition

We say that X_0 has BG-rational singularities if for every projective model $\mathcal{X}$ over $\mathfrak{o}$, with G-action, smooth generic fiber X, and special fiber G-equivariantly isomorphic to X_0 we have

$$\rho_\pi^G([X \circlearrowleft G]) = [X_0 \circlearrowleft G].$$

There is also a notion of mild singularities allowing to understand the equivariant birational type of special fibers:

Definition

We say that X_0 has *BG-rational singularities* if for every projective model $\mathcal{X}$ over $\mathfrak{o}$, with G-action, smooth generic fiber X, and special fiber G-equivariantly isomorphic to X_0 we have

$$\rho_\pi^G([X \circlearrowleft G]) = [X_0 \circlearrowleft G].$$

For example, if the singular locus of X_0 is an orbit of rational double points, on which G acts simply transitively, then X_0 has BG-rational singularities.

Modular/motivic types $\mathcal{M}_n(G)$, $n \geq 2$

Let G be an abelian group. Consider the $\mathbb{Z}$-module

$$\mathcal{M}_n(G)$$

generated by unordered tupels $\langle a_1, \ldots, a_n \rangle$, $\quad a_i \in A$, such that

(G) $\sum_i \mathbb{Z} a_i = A$, and

(M) for all $a_1, a_2, b_1, \ldots, b_{n-2} \in A$ we have

$$\langle a_1, a_2, b_1, \ldots b_{n-2} \rangle =$$

$$\langle a_1 - a_2, a_2, b_1, \ldots, b_{n-2} \rangle + \langle a_1, a_2 - a_1, b_1, \ldots, b_{n-2} \rangle.$$

Modular/motivic types $\mathcal{M}_n(G)$, $n \geq 2$

Let G be an abelian group. Consider the $\mathbb{Z}$-module

$$\mathcal{M}_n(G)$$

generated by unordered tupels $\langle a_1, \dots, a_n \rangle$, $\quad a_i \in A$, such that

(G) $\sum_i \mathbb{Z} a_i = A$, and

(M) for all $a_1, a_2, b_1, \dots, b_{n-2} \in A$ we have

$$\langle a_1, a_2, b_1, \dots b_{n-2} \rangle =$$

$$\langle a_1 - a_2, a_2, b_1, \dots, b_{n-2} \rangle + \langle a_1, a_2 - a_1, b_1, \dots, b_{n-2} \rangle.$$

The only difference with $\mathcal{B}_n(G)$: $\quad [a, a] = [a, 0], \quad \langle a, a \rangle = 2\langle a, 0 \rangle.$

Consider the map

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G)$$

$$(\boldsymbol{\mu}_0) \qquad [a_1, \dots, a_n] \mapsto \langle a_1, \dots, a_n \rangle, \qquad \text{if all } a_1, \dots, a_n \neq 0,$$

$$(\boldsymbol{\mu}_1) \qquad [0, a_2, \dots, a_n] \mapsto 2\langle 0, a_2, \dots, a_n \rangle, \quad \text{if all } a_2, \dots, a_n \neq 0,$$

$$(\boldsymbol{\mu}_2) \qquad [0, 0, a_3, \dots, a_n] \mapsto 0, \qquad \text{for all } a_3, \dots, a_n,$$

and extended by $\mathbb{Z}$-linearity.

Birational types → modular types

Theorem

- μ is a well-defined homomorphism; surjective, modulo 2-torsion (Kontsevich-Pestun-T. 2019)

Birational types $\to$ modular types

Theorem

- μ is a well-defined homomorphism; surjective, modulo 2-torsion (Kontsevich-Pestun-T. 2019)
- μ is an isomorphism, $\otimes\mathbb{Q}$ (Hassett-Kresch-T. 2020)

Modular types – lattice theory

Consider the free abelian group $\mathcal{S}_n(G)$, generated by symbols

$$\beta = [a_1, \dots, a_n] = [a_{\sigma(1)}, \dots, a_{\sigma(n)}], \quad \forall \sigma \in \mathfrak{S}_n,$$

where β is an n-dimensional faithful representation of G, i.e., a collection of characters $a_1, \dots, a_n$ of G, up to permutation, spanning $G^\vee$.

Consider the free abelian group $\mathcal{S}_n(G)$, generated by symbols

$$\beta = [a_1, \ldots, a_n] = [a_{\sigma(1)}, \ldots, a_{\sigma(n)}], \quad \forall \sigma \in \mathfrak{S}_n,$$

where β is an n-dimensional faithful representation of G, i.e., a collection of characters $a_1, \ldots, a_n$ of G, up to permutation, spanning $G^\vee$.

We have a diagram

$$\begin{array}{ccc} \mathcal{S}_n(G) & \xrightarrow{\mathsf{b}} & \mathcal{B}_n(G) \\ & & \downarrow \mu \\ \mathcal{S}_n(G) & \xrightarrow{\mathsf{m}} & \mathcal{M}_n(G) \end{array}$$

Modular types – lattice theory

Consider the free abelian group on triples

$$(\mathbf{L}, \chi, \Lambda),$$

where

- $\mathbf{L} \simeq \mathbb{Z}^n$ is an n-dimensional lattice,
- $\chi \in \mathbf{L} \otimes A$ is an element inducing, by duality, a surjection $\mathbf{L}^\vee \to A$,
- Λ is a basic cone, i.e., a simplicial cone spanned by a basis of $\mathbf{L}$.

Let $\mathbf{T}$ be the quotient by $\mathrm{GL}_n(\mathbb{Z})$-equivalence. There is a natural map

$$\begin{array}{ccc} \mathbf{T} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda) & \mapsto & [a_1, \ldots, a_n], \end{array}$$

defined by decomposing

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \quad a_i \in A,$$

where $\{e_1, \ldots, e_n\}$ is a basis of Λ.

Modular types – lattice theory

Let $\mathbf{T}$ be the quotient by $\mathrm{GL}_n(\mathbb{Z})$-equivalence. There is a natural map

$$\begin{array}{rcl} \mathbf{T} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda) & \mapsto & [a_1, \ldots, a_n], \end{array}$$

defined by decomposing

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \quad a_i \in A,$$

where $\{e_1, \ldots, e_n\}$ is a basis of Λ.

The symmetry property is precisely the ambiguity in the order of generating elements of Λ.

Modular types – lattice theory

Imposing scissor-type relations on $\mathbf{T}$, via subdivision of cones, we obtain a diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\psi} & \mathcal{M}_n(G) \\ \Big\downarrow \mathsf{s} & \nearrow_{\sim} & \\ \mathbf{T}/(\text{scissor-type relations}) & & \end{array}$$

There is a similar group $\widetilde{\mathbf{T}}$, based on triples

$$(\mathbf{L}, \chi, \Lambda'),$$

where now Λ' is a smooth cone of arbitrary dimension (i.e., one spanned by part of a basis of $\mathbf{L}$), such that

- $$\chi \in \mathrm{Im}(\mathbf{L}' \otimes A \to \mathbf{L} \otimes A),$$

  where $\mathbf{L}' \subseteq \mathbf{L}$ is the sublattice spanned by Λ'.

Birational types – lattice theory

There is a similar group $\widetilde{\mathbf{T}}$, based on triples

$$(\mathbf{L}, \chi, \Lambda'),$$

where now Λ' is a smooth cone of arbitrary dimension (i.e., one spanned by part of a basis of $\mathbf{L}$), such that

- $$\chi \in \mathrm{Im}(\mathbf{L}' \otimes A \to \mathbf{L} \otimes A),$$

  where $\mathbf{L}' \subseteq \mathbf{L}$ is the sublattice spanned by Λ'.

Again, impose relations coming from the $\mathrm{GL}_n(\mathbb{Z})$-action.

There is a natural map

$$\begin{array}{rcl} \widetilde{\mathbf{T}} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda') & \mapsto & [a_1, \ldots, a_n]. \end{array}$$

For a face Λ'' of Λ' of dimension at least 2,

$$\Lambda'' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_r\rangle \subset \Lambda' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_s\rangle,$$

consider the star subdivision

$$\Sigma^*_{\Lambda'}(\Lambda''),$$

consisting of the $2^r - 1$ cones spanned by

$$e_1 + \cdots + e_r, e_{r+1}, \dots, e_s,$$

and all proper subsets of $\{e_1, \dots, e_r\}$.

We introduce **Subdivision relations** on $\widetilde{\mathbf{T}}$:

(S) Put

$$(\mathbf{L},\chi,\Lambda') = \sum_{\substack{\widetilde{\Lambda}' \in \Sigma^*_{\Lambda'}(\Lambda'') \\ \chi \in \mathrm{Im}(\widetilde{\mathbf{L}}' \otimes A \to \mathbf{L} \otimes A)}} (-1)^{\dim(\Lambda') - \dim(\widetilde{\Lambda}')} (\mathbf{L},\chi,\widetilde{\Lambda}'),$$

respectively,

$(\mathbf{L},\chi,\Lambda') = (\mathbf{L},\chi,\Lambda)$, for a basic cone Λ, having Λ' as a face.

Birational types – lattice theory

We have:

$$\begin{array}{ccc} \widetilde{\mathbf{T}} & \xrightarrow{\quad\tilde{\psi}\quad} & \mathcal{B}_n(G) \\ \Big\downarrow{\scriptstyle \tilde{\mathsf{s}}} & \nearrow{\scriptstyle \sim} & \\ \widetilde{\mathbf{T}}/(\text{subdivision relations}) & & \end{array}$$

Birational types – lattice theory

The definition of

$$\tilde{\psi}(\mathbf{L}, \chi, \Lambda')$$

extends to the case of a simplicial cone Λ' (satisfying the condition), with $\mathbf{L}' = \mathbf{L} \cap \Lambda'_{\mathbb{R}}$.

Birational types – lattice theory

The definition of

$$\tilde{\psi}(\mathbf{L}, \chi, \Lambda')$$

extends to the case of a simplicial cone Λ' (satisfying the condition), with $\mathbf{L}' = \mathbf{L} \cap \Lambda'_{\mathbb{R}}$.

We can define Hecke operators

$$T_{\ell,r} : \mathcal{B}_n(G) \to \mathcal{B}_n(G),$$

where $\ell \nmid |G|$ and $1 \leq r \leq n-1$, as a sum over certain overlattices:

$$T_{\ell,r}(\tilde{\psi}(\mathbf{L}, \chi, \Lambda')) := \sum_{\substack{\mathbf{L} \subset \widehat{\mathbf{L}} \subset \mathbf{L} \otimes \mathbb{Q} \\ \widehat{\mathbf{L}}/\mathbf{L} \simeq (\mathbb{Z}/\ell\mathbb{Z})^r}} \tilde{\psi}(\widehat{\mathbf{L}}, \chi, \Lambda').$$

Summary

- Birational symbols groups

$$\mathcal{B}_n(G), \quad \mathrm{Burn}_n(G)$$

and applications to equivariant rationality

Summary

- Birational symbols groups

$$\mathcal{B}_n(G), \quad \mathrm{Burn}_n(G)$$

  and applications to equivariant rationality
- Intricate internal structure