Outline

- joint with A. Harder and L. Katzarkov.

- Review of the $\mathcal{P} = \mathcal{W}$ conjecture in non-abelian Hodge theory.

- The mirror $\mathcal{P} = \mathcal{W}$ conjecture and HMS for Fano varieties.

- Topological SYZ mirrors and a refinement of the mirror $\mathcal{P} = \mathcal{W}$ conjecture. cf. Harder's talk

- [Sukjoo Lee, 21] applications of nc gluing in relative HMS to:
  - the mirror $\mathcal{P} = \mathcal{W}$ conjecture;
  - Hodge and deformation theory of hybrid Landau-Ginzburg models.
1. The $P=W$ conjecture in non-abelian Hodge theory.

$C$ - smooth compact curve

$G$ - complex reductive group

$\Rightarrow$ $M_B$ - moduli of $G$ local systems on $C$

$M_{Dol}$ - moduli of $G$, Higgs bundles

$(E, \mathcal{O} \in H^0(C, \Omega^i \otimes E^*)_C)$

**Thm (Simpson)** The two spaces are homeomorphic

$M_B \cong M_{Dol}$

nah o RH
However:

- $M_\mathbb{B}$ and $M_{\text{Dol}}$ are not isomorphic as $\mathbb{C}$-algebraic or analytic varieties:
  - $M_\mathbb{B}$ is an affine variety
  - $M_{\text{Dol}}$ contains compact subvarieties passing through every point.

- The cohomology rings

  \[ H^\ast(M_\mathbb{B}, \mathbb{C}) \cong H^\ast(M_{\text{Dol}}, \mathbb{C}) \]

are isomorphic but the isomorphism does not respect Hodge and weight filtrations.

P=W Conjecture [dCMW'12]: $M_\mathbb{B} \cong M_{\text{Dol}}$

identifies $W_\mathbb{B}, H^\ast(M_\mathbb{B})$ with a filtration of different geometric origin on $H^\ast(M_{\text{Dol}})$. 
Perverse filtrations: Suppose $B$ is a smooth variety. The full subcategories

$$\mathcal{D}_{\leq 0}(B) = \{ E \in \mathcal{D}^b_c(B) \mid \dim \supp H^{i}(E) \leq -i \}$$

$$\mathcal{D}_{\geq 0}(B) = \{ E \in \mathcal{D}^b_c(B) \mid \dim \supp H^{i}(\mathcal{D}E) \leq i \}$$

compact supported constructible sheaves of $\mathbb{Q}$-vector spaces

define a $t$-structure on $\mathcal{D}^b_c(B)$ with heart

$$\text{Perf}(B) = \mathcal{D}^b_c(B)$$

the abelian category of perverse sheaves on $B$. 

Verdier duality
For $k \in \mathbb{Z}$ let

$$\text{per}_{\leq k} : \mathcal{D}_c^b (B) \to$$

be the truncation functor associated with this $t$-structure.

For $\mathcal{F} \in \mathcal{D}_c^b (B)$ we get a perverse filtration on $H^\ast (B, \mathcal{F})$ with steps defined by

$$\text{im} \left( H^\ast (B, \text{per}_{\leq k} \mathcal{F}) \to H^\ast (B, \mathcal{F}) \right)$$

for all $k$. 

**Def.** Suppose $h : M \to B$ is a proper morphism of smooth varieties. The perverse Leray filtration on $H^i(M, \mathbb{Q})$ is the filtration (concentrated in $[0, 2(\dim M - c)]$) given by

\[
P_k H^i(M, \mathbb{Q}) = \mathrm{im} \left[ H^i \left( B, \mathbb{Q} \right)^{\perp} \right] \cap \left[ H^i \left( \bigcup_{k \leq k} \text{R}h^* \mathbb{Q}[c] \right) \to H^i(M, \mathbb{Q}) \right]
\]

where $c = 2 \dim M - \dim M \times M_B$

**Remark:** $M_{\text{Dol}}$ comes with an intrinsically defined (proper) affinization map

\[h : M_{\text{Dol}} \to B \]

\[
\text{Hitchin map } \quad h(\mathcal{E}, \theta) = (p_{d_1}(\theta), \ldots, p_{d_r}(\theta))
\]

\[
\text{Spec} \left( \Gamma \left( M_{\text{Dol}}, \mathcal{O} \right) \right)
\]
**Conjecture** (de Cataldo-Hausel-Migliorini ’12)

The homeomorphism \( MB \cong M_{Dol} \) identifies the weight and perverse Leray filtrations

\[
W_{2k} \, H^*(MB) \cong P_k \, H^*(M_{Dol}) \\
\downarrow \\
W_{2k+1} \, H^*(MB)
\]

**Results:**

- [de Cataldo-Hausel-Migliorini ’12] True for \( g(C) > 0 \) and \( C \) of type \( A_1 \).
- [Shen-Zhang ’18] True for \( C = \mathbb{E}/\Gamma \) with \( E \) - elliptic, \( \Gamma = \{1, 2, 2/1, 2/3, 2/4, 2/6, A_0, E_6, E_7, E_8 \} \) and \( C \) of type \( A_{11/1} \).
- [de Cataldo-Maulik-Shen ’19] True for \( g(C) = 2 \) and \( C \) of type \( A_n \).
Variants:

- [Zhang '18] $P = W$ holds for 2-dimensional cluster varieties:

\[(X - \text{cluster}) \leadsto (Y \twoheadrightarrow \Delta \text{ is an } I_6 \text{ or } I_6 + I_6)\]

so that $X \cong Y$ inducing $W_{2k} H^*(X) = P_k H^*(Y)$.

- [Szabo '18, Szabo-Nemethi '20] $P = W$ holds for irregular character varieties corresponding to Painlevé I–VI.

- [Dancso-McBreen-Shende '19] $P = W$ holds for Hessel–Broudefood varieties:

\[(T - \text{finite graph}) \leadsto \mathbb{A}(T) - \text{affine} \]

\[\mathbb{A}(T) - \text{integrable system}\]

So that $\mathbb{A}(T) \cong \mathcal{A}(T)$ inducing $W_{2k} H^*(\mathcal{A}(T)) \cong P_k H^*(\mathcal{A}(T))$.
Hyperplane section description of $\mathcal{H}^*(B, \neq)$

Suppose:

- $B$ is smooth, affine, $\dim B = n$;
- $B = \Lambda^n$;
- $\mathcal{P} = \Lambda_{-(N+1)} \subseteq \Lambda_{-N} \subseteq \ldots \subseteq \Lambda_{-1} \subseteq \Lambda_0 = \Lambda^N$ is a general complete linear flag;
- $\mathcal{O} = B_{-(n+1)} \subset B_{-n} \subset \ldots \subset B_{-1} \subset B_0 = B$

$B_i = B \cap \Lambda_i$, the induced flag of hyperplane sections of $B$.

Then the perverse filtration on the cohomology of constructible sheaves on $B$ coincides with the geometric filtration induced by $B_i$. 
Thm [de Cataldo-Migliorini, 10] let $\mathcal{F}$ be a constructible complex of $\mathbb{Q}$-vector spaces on $\mathcal{B}$. Then

$$\text{im} \left[ H^i(\mathcal{B}, \mathcal{F}_{\leq -k} \mathcal{F}) \to H^i(\mathcal{B}, \mathcal{F}) \right]$$

$$\text{ker} \left[ H^i(\mathcal{B}, \mathcal{F}) \to H^i(\mathcal{B}_{k+i-1}, \mathcal{F}|_{\mathcal{B}_{k+i-1}}) \right]$$
2. The mirror $P = W$ conjecture.

Observation (Katzarkov):

\[ P = W \text{ for character varieties} + \text{Mirror symmetry for character varieties} \]

suggest a $P = W$ matching for general pairs of mirror log CY varieties.
Indeed:

1. One expects $M_{B}(C, G) \xrightarrow{\text{Hecke}} M_{\text{Dol}}(C, \mathcal{L}_{G})$

2. One expects $M_{\text{Dol}}(C, G) \leftrightarrow M_{\text{Dol}}(C, \mathcal{L}_{G})$

$\xrightarrow{\text{T-dual}}$

$\xrightarrow{\text{Langlands dual}}$

Known away from

$\text{Disc} \subset B$ [Ponzi = ?]

3. One expects $R_{h} \otimes \mathbb{Q} \leftrightarrow R(\mu_{h}) \otimes \mathbb{Q}$

Thus: one expects

$W_{\mu}(M_{B}(C, G)) \leftrightarrow \mathbb{P} \cdot H^{\ast}(M_{\text{Dol}}(C, \mathcal{L}_{G}))$

up to relabeling

This motivates the mirror $\mathbb{P} = W$ conjecture which I will state next.
Setup: \( X \) - smooth projective, \( \dim X = d \)
\( Y \subset X \) strict nc divisor with \( Y \in |-K_X| \)
\( U = X - Y \).

\( H^*(U, \mathbb{C}) \) has canonical Hodge and weight filtrations \( F^* \) and \( W \).

Set
\[
HH_{d+1}^i(U) := \bigoplus_{q-p=i} \bigoplus_{F} H^{p+q}(U)
\]
\[
= \bigoplus_{q-p=i} H^q(X, R^p_X(\log Y))
\]

Note: The weight filtration on \( H^{p+q}(U) \) induces a weight filtration on each \( H^q(X, R^p_X(\log Y)) \).
Explicitly: We have a codimension 1 residue map

\[ \text{Res}_i : \mathcal{H}^p_{\mathcal{X}}(\log \mathcal{Y}) \to \mathcal{H}^{p-i}_{\mathcal{Y}^\nu} \left( \log \mathcal{Y}_{\mathcal{C}i} \right) \]

where

\[ \mathcal{Y}_{\mathcal{C}i} = \mathcal{Y} - \text{set of points of multiplicity } \geq i \]

\[ \mathcal{Y}^\nu_{\mathcal{C}i} = \text{the normalization of } \mathcal{Y}_{\mathcal{C}i} \]

\[ \mathcal{Y}^\nu_{\mathcal{C}i} = \text{the preimage of } \mathcal{Y}_{\mathcal{C}i+1} \subset \mathcal{Y}_{\mathcal{C}i} \]

Then

\[ W_i \mathcal{H}^p_{\mathcal{X}}(\log \mathcal{Y}) = \ker(\text{Res}_i) \]

and we have

\[ W_i \mathcal{H}^q_{\mathcal{X}}(\mathcal{Y}) = \im \left[ \mathcal{H}^q\left( W_i - \mathcal{p} - q \mathcal{H}^p_{\mathcal{X}}(\log \mathcal{Y}) \right) \to \mathcal{H}^q\left( \mathcal{H}^p_{\mathcal{X}}(\log \mathcal{Y}) \right) \right] \]
For the purposes of the mirror \( P = W \) conjecture it is better to relabel that so we set

\[
\tilde{\omega}_j : H^q \left( \mathcal{M}^p \left( \log Y \right) \right)
\]

\[
:= \text{im} \left[ H^q \left( \mathcal{M}^p \left( \log Y \right) \right) \to H^q \left( \mathcal{M}^p \left( \log Y \right) \right) \right]
\]

and

\[
\tilde{\omega}_j : \mathcal{H}_{\alpha^j \tilde{\omega}^i} (\mathcal{O}) = \bigoplus \tilde{\omega}_j : H^q \left( \mathcal{M}^p \left( \log Y \right) \right)
\]

\( q - p = i \)
let $\tilde{U}$ be the mirror of the non-compact Calabi-Yau variety $U$ \ ($\dim \tilde{U} = d$).

Let $h : \tilde{U} \to B = \text{Spec} \left( \Gamma (\tilde{U}, \omega) \right)$ be the affinization map of $\tilde{U}$.

**Terminology:** $h : \tilde{U} \to B$ is the hybrid Landau-Ginzburg model mirror to $(X, \Sigma)$.
Auroux's mirror construction for $(X, \mathcal{Y})$ describes the mirror as a pair $(\mathcal{U}, f)$, where

- $\mathcal{U}$ is an SYZ mirror of $\mathcal{U}$;
- $f : \mathcal{U} \to \mathbb{A}^1$ is a holomorphic superpotential counting pseudo-holomorphic disks in $X$ with boundaries in SYZ fibers in $\mathcal{U}$ and order of contact 1 with $\mathcal{Y}$.

By construction $f = b_1 t + \cdots + b_n$, where

- $b_k$ counts disks having order of contact with the $k$-th component of $\mathcal{Y}$.

Then we get a proper hybrid Landau-Ginzburg potential

$$(b_1, \ldots, b_n) : \mathcal{U} \to \mathbb{A}^n$$

with image $\text{Spec} \left( \mathcal{O}(\mathcal{U}, \mathcal{U}) \right)$. 
Now choose a generic linear flag
\[ \emptyset = \Lambda_{h+1} \subset \Lambda_h \subset \ldots \subset \Lambda_1 \subset \Lambda_0 = \Lambda^n \]
and define

\[ \widetilde{P}_j \colon H^k(\tilde{\mathcal{U}}) := \ker \left[ H^k(\tilde{\mathcal{U}}) \to H^k(\Lambda^{-1}_{j-k+1}) \right] \]

Then we have the following

**Mirror $P=W$ conjecture**: Under the $\text{HMIS identification}$, we have
\[ HH_k(\mathcal{U}) \simeq H^k(\tilde{\mathcal{U}}) \]

we have
\[ \tilde{w}_j \colon HH_k(\mathcal{U}) = \widetilde{P}_j H^k(\tilde{\mathcal{U}}). \]
**Results and examples:** Harder, Katzarkov, Przyjalkowski, ...

details in Andrew Harder's talk.

**Refined version:** Let

\[
\begin{array}{c}
U \\
\downarrow \pi \\
S \\
\downarrow \pi \\
U^o
\end{array}
\]

be dual Lagrangian torus fibrations.

**Notation:**

\begin{itemize}
  \item \( \Delta = \text{Discr}(\pi) = \text{Discr}(\pi^\vee) \subset S \)
  \item \( S^o = S - \Delta \), \( U^o = \pi^{-1}(S^o) \), \( U^o = \pi^{-1}(S^o) \)
\end{itemize}
\[ \pi^0 = \bar{\pi}^0 | U^0, \quad \bar{\pi}^0 = \bar{\pi} | U^0. \]

\[ j : S^0 \rightarrow S. \]

**Assume:** \( \pi \) and \( \bar{\pi} \) are simple, i.e.

\[ R^\bullet \pi^0 \mathcal{O}_Q = j_* R^\bullet \pi^0 \mathcal{O}_Q, \]

\[ R^\bullet \bar{\pi}^0 \mathcal{O}_Q = j_* R^\bullet \bar{\pi}^0 \mathcal{O}_Q. \]

The Leray shears for \( \pi \) and \( \bar{\pi} \) degenerate at \( E_2 \):

\[ H^k(U^0, \mathcal{O}) = \bigoplus_{p+q=k} H^q(S^0, j_* R^p \bar{\pi}^0 \mathcal{O}_Q) \]

\[ H^k(U^0, \mathcal{O}) = \bigoplus_{p+q=k} H^q(S^0, j_* R^p \bar{\pi}^0 \mathcal{O}_Q). \]
Verdier duality then implies

\[ j_* R^p \tau \, \Omega \cong j_* R^{d-p} \tau \, \Omega \]

and in particular

\[ H^q (\mathcal{X}, j_* R^p \tau \, \Omega) \cong H^q (\mathcal{Y}, j_* R^{d-p} \tau \, \Omega) \]

Also note that by analogy with M. Gross' analysis of Hodge filtrations on compact CYs we expect

\[ H^q (\mathcal{X}, R^q (\log V)) = H^q (\mathcal{Y}, j_* R^p \tau \, \Omega) \]
SYZ refined mirror $P = W$ conjecture:

There are natural weight and perverse filtrations on $H^q(S, R^p \pi_* \mathbb{Q})$ and the under the topological SYZ mirror isomorphism

$$H^q(S, R^p \pi_* \mathbb{Q}) \cong H^q(S, R^{d-p} \pi_* \mathbb{Q})$$

we have

$$\tilde{W}_j H^q(S, R^p \pi_* \mathbb{Q}) = \tilde{P}_j H^q(S, R^{d-p} \pi_* \mathbb{Q}).$$

Detailed definitions of $\tilde{W}_j, \tilde{P}_j$ and strategy of proof in Andrew Harder's talk.
3. The mirror $P=W$ conjecture from HM$^S$.

$HM^S$ in the complement of an anti-canonical divisor:

\[(X, D) \leftrightarrow (Y, Y \xrightarrow{\pi} A', Y_{sm})\]

- smooth
- Fano
- divisor in $\mid K_X \mid$
- smooth
- proper
- smooth
- fibration

and predicts matching of cohomology and categories:

<table>
<thead>
<tr>
<th>B side</th>
<th>A side</th>
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<tbody>
<tr>
<td>$H^i(X), H^i(D)$</td>
<td>$H^i(Y, Y_{sm}), H^i(Y_{sm})$</td>
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<tr>
<td>$h^{p,q}(X), h^{p,q}(D)$</td>
<td>$h^{d-p, q}(Y, Y_{sm}), h^{d-1-q}(Y_{sm})$</td>
</tr>
<tr>
<td>$D^b_{coh}(X), D^b_{coh}(D)$</td>
<td>$FS(Y, W), Fuk(Y_{sm})$</td>
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</table>
More precisely, relative HMS for \((X, D)\) predicts that we have a commutative diagram of functors:

\[
\begin{array}{ccc}
\mathcal{D}_{\text{coh}}^b(D) & \xrightarrow{i_*} & \mathcal{D}_{\text{coh}}^b(X) & \xrightarrow{j^*} & \mathcal{D}_{\text{coh}}^b(U) \\
\end{array}
\]

where:

- \(\cap\) is the intersection with a fiber
- \(U\) is the Orlov functor (Abouzaid-Ganatra)
- \((i_*, i^*)\) and \((U, \cap)\) are compatible adjoint pairs
- the horizontal sequences are localizations
Passing to Hochschild homology we get

$$\bigoplus H^q(D, S^p_D) \rightarrow \bigoplus H^q(X_c, S^p_{X_c})$$

$q - p = a$.

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$$H^{a+d-1}(Y_{sm}) \rightarrow H^0(Y, Y_{sm})$$

Observation:

- the top row is a map in the $E_1$ page of the weight $ss$.
- the bottom row is a map in the $E_1$ page of the perverse Leray $ss$.

in the de Cataldo-Migliorini-hyperplane version
**Thm:** [Harder-Katzarkov-Przyjalkowski: '20]

Under the above assumptions \( \text{HMS} \) induces isomorphisms for \( i = 0, 1, \)

\[
\begin{align*}
\text{gr}_F^p \text{gr}_p^W H^{p+q}(U) & \cong \text{gr}_d^p H^{d+a}(Y) \\
q-p & = a
\end{align*}
\]

Moreover the monodromy weight filtrations associated to the \( \text{Serre} \) functors refine the isomorphisms to matching of summands

\[
\begin{align*}
\text{gr}_F^p \text{gr}_{p+q+i}^W H^{p+q}(U) & \cong \text{gr}_W^W \text{gr}_{2(a-p)}^d H^{d+a}(Y).
\end{align*}
\]

**Corollary:** [Harder-Katzarkov-Przyjalkowski: '20]

If the \( \text{HMS} \) on \( H^*(Y) \) is of Tate type, then relative \( \text{HMS} \) for \( (\mathcal{X}, \mathcal{D}) \) implies the mirror \( \mathcal{I} = W \) conjecture for the mirror pair \( \mathcal{U} / Y \).
Recently Sukjoo Lee explained how one can extend these results to the case of normal crossing divisors.

**Setup:** Suppose $(X, D)$ is a log CY pair with

- $X$ - smooth Fano, $\dim X = d$.
- $D = D_1 \cup D_2$ smooth divisor in $|\mathcal{K}_X|$ with $D_{12} = D_1 \cap D_2$ connected.
- $D_1, D_2$ are smooth Fanoos and $(D_1, D_{12})$ are $(D_2, D_{12})$ are log CY pairs.
Recall: A hybrid LG model mirroring \((X, D)\) is a triple

\[(Y_1, Y \rightarrow \mathbb{A}^2, \omega_Y)\]

quasi-projective proper symplectic form

satisfying:

- If \(h = (h_1, h_2)\) and \(Y_1, Y_2\) are general fibers of \(h_1, h_2\), then
  
  \[(D_1, D_{12}) \mid (Y_1, Y_1 \rightarrow \mathbb{A}^1, W_{Y_1 Y_1}) \]
  
  \[(D_2, D_{12}) \mid (Y_2, Y_2 \rightarrow \mathbb{A}^1, W_{Y_2 Y_2}) \]

  \(\left\|\right.\) Fan mirror pairs

  \(\left\|\right.\)

- \(D_{12}\) is mirror to \(Y_{12}\) - general fiber of \(h\).

- \((Y, W = h_1 + h_2, \omega_Y)\) is the LG mirror of \((X, D)\).
Thm: [Sukjoo Lee, '21] Suppose $X$ is a smooth Fano complete intersection and $D = D_1 \cup D_2$ is as above. Then there exists a hybrid LG mirror

$$(X, D) \mid (Y, Y \xrightarrow{h} \mathbb{A}^2, \omega_Y)$$

so that if $Y_{Sm}$ is a general fiber of $W$, then

$h_1|_{Y_{Sm}}: Y_{Sm} \to \mathbb{A}^1$ is a gluing of

$h_2: Y_1 \to \mathbb{A}^1$ and $h_1: Y_2 \to \mathbb{A}^1$.

Note: The proof combines the Hori-Vafa construction, fiberwise compactification, Przybalski-Shramov crepant resolution, deformation of $Y_{Sm}$. 
**Corollary:** [Sukjoo Lee '21] If relative HMs holds for $(D_1, D_2)$ and $(D_2, D_1)$, then relative HMs holds for $(X, D)$ i.e. we have equivalences

\[
\begin{array}{ccc}
\mathcal{D}^b_{coh}(D_1) & \xrightarrow{\sim} & \mathcal{D}^b_{coh}(D_2) \\
\mathcal{D}^b_{coh}(X) & \xrightarrow{\sim} & \mathcal{D}^b_{coh}(D_1) \\
\mathcal{D}^b_{coh}(D_2) & \xrightarrow{\sim} & \mathcal{D}^b_{coh}(X)
\end{array}
\]

**Corollary:** [Sukjoo Lee '21] In this setting we have matching for all $i = 0, 1, 2$:

\[
\Theta \quad \text{gr}^F \text{gr}^W H^{p+q}(\overline{U}) \cong \text{gr}^P H^{d+a}(Y)
\]

and refined matching

\[
\text{gr}^F \text{gr}^P \text{gr}^W H^{p+q+1}(\overline{U}) \cong \text{gr}^W H^{d+a}(Y)
\]
Corollary: [Bux, Lee ‘21] If the MHS on $H^*(Y)$ is of Tate type, then relative MHS for $(X, D)$ implies the mirror $P=W$ conjecture for the pair $(D/Y)$.

The careful analysis of these filtrations also leads to full control of the Hodge and deformation theory of hybrid LG models.

Suppose $Y \xrightarrow{h} \mathbb{A}^2$ is a hybrid LG model as above and let

$$z \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1$$

be a tame compactification.

Then one can study the deformations of $(Z, f)$ anchored at $(0 \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times 0)$.
**Theorem:** [Sukjoo Lee '21] The space of deformations of $(\mathbb{Z}, \mathbf{f})$ anchored at $\infty$ is unobstructed.

The proof again follows by checking that the Hodge-to-de Rham spectral sequence for the complex

$$
\mathcal{L}_{\mathbb{Z}} \left( \log D, f_1, f_2 \right), \mathbf{d}
$$

degenerates at $E_1$.

**Conjecture:** The KKL conjecture holds in this setting, i.e.

$$
h^{\mathbf{q}}(\mathbb{Z}, \mathcal{H}^p(\log D, f)) = \dim \mathfrak{F}_{\mathbf{q}} W(N) H^{p+q} (\mathbf{Y}, \mathbf{Y}_{1,2})
$$

where $W(N)$ is the monodromy weight filtration associated with any monodromy around $\infty$.  

