# P = W in HMS, part 1

## Outline

- joint with A. Harder and L. Katzarkov.
- review of the P = W conjecture in non-abelian Hodge theory.
- the mirror P = W conjecture and HMS for Fano varieties.
- topological SYZ mirrors and a refinement of the mirror P = W conjecture. cf Harder's talk
- [Sukjoo Lee '21] applications of nc gluing in relative HMS to:
  - the mirror P = W conjecture;
  - Hodge and deformation theory of hybrid Landau-Ginzburg models.

## 1. The $P=W$ conjecture in non-abelian Hodge theory:

$C$ - smooth compact curve

$G$ - complex reductive group

$\Rightarrow$

$M_B$ - moduli of $G$ local systems on $C$

$M_{Dol}$ - moduli of $G$ Higgs bundles $(E, \theta \in H^0(C, \mathrm{ad}E \otimes \Omega^1_C))$

**Thm** (Simpson) The two spaces are homeomorphic

$$M_B \underset{\uparrow}{\cong} M_{Dol}$$

nah $\circ$ RH

However:

- $M_B$ and $M_{Dol}$ are not isomorphic as $\mathbb{C}$-algebraic or analytic varieties:
  - $M_B$ is an affine variety
  - $M_{Dol}$ contains compact subvarieties passing through every point.
- The cohomology rings

$$H^\bullet(M_B, \mathbb{C}) \cong H^\bullet(M_{G}, \mathbb{C})$$

are isomorphic but the isomorphism does not respect Hodge + weight filtrations.

P = W Conjecture [dCHM'12]: $M_B \cong M_{Dol}$ identifies $W_\bullet H^\bullet(M_B)$ with a filtration of different geometric origin on $H^\bullet(M_{Dol})$.

**Perverse filtrations:** . Suppose $B$ is a smooth variety. The full complex algebraic subcategories

$$ {}^{per}D_{\leq 0}(B) = \{ E \in D^b_c(B) \mid \dim \operatorname{supp} \mathcal{H}^i(E) \leq -i \} $$

$$ {}^{per}D_{\geq 0}(B) = \{ E \in D^b_c(B) \mid \dim \operatorname{supp} \mathcal{H}^i(\mathbb{D} E) \leq -i \} $$

compactly supported constructible sheaves of $\mathbb{Q}$-vector spaces

Verdier duality

define a t-structure on $D^b_c(B)$ with heart

$$ \mathrm{Perv}(B) \subset D^b_c(B) $$

the abelian category of perverse sheaves on $B$.

For $k \in \mathbb{Z}$ let

$$ {}^{per}\tau_{\leq k} : D^b_c(B) \circlearrowleft $$

be the truncation functor associated with this t-structure.

For $\mathcal{F} \in D^b_c(B)$ we get a **perverse filtration** on $H^\bullet(B, \mathcal{F})$ with steps defined by

$$ \boxed{\mathrm{im}\left( H^\bullet(B, {}^{per}\tau_{\leq k}\mathcal{F}) \to H^\bullet(B, \mathcal{F}) \right)} $$

for all $k$.

**Def:** Suppose $h : M \to B$ is a proper morphism of smooth varieties. The **perverse Leray filtration** on $H^{\cdot}(M, \mathbb{Q})$ is the filtration (concentrated in $[0, 2(\dim M - c)]$) given by

$$\boxed{P_k H^i(M, \mathbb{Q}) = \operatorname{im}\left[H^{i-c}(B, {}^{per}\tau_{\leq k} Rh_* \mathbb{Q}[c]) \to H^i(M, \mathbb{Q})\right]}$$

where $c = 2\dim M - \dim M \times_B M$

**Remark:** $M_{Dol}$ comes with an intrinsically defined (proper) affinization map

$$h : M_{Dol} \to B = \operatorname{Spec}(\Gamma(M_{Dol}, \mathbb{O}))$$

Hitchin map

$h(E, \theta) = (p_{d_1}(\theta), \ldots, p_{d_r}(\theta))$

## P = W Conjecture (de Cataldo-Hausel-Migliorini '12)

The homeomorphism $M_B \cong M_{Dol}$ identifies the weight and perverse Leray filtrations

$$\begin{array}{ccc} W_{2k} H^{\bullet}(M_B) & \cong & P_k H^{\bullet}(M_{Dol}) \\ \| & & \\ W_{2k+1} H^{\bullet}(M_B) & & \end{array}$$

**Results:**

- [de Cataldo-Hausel-Migliorini '12] True for $g(C) > 0$ and $G$ of type $A_1$.
- [Shen-Zhang '18] True for $C = [E/\Gamma]$ with $E$ - elliptic, $\Gamma = \underset{\tilde{A}_0}{\{1\}}, \underset{\tilde{D}_4}{\mathbb{Z}/2}, \underset{\tilde{E}_6}{\mathbb{Z}/3}, \underset{\tilde{E}_7}{\mathbb{Z}/4}, \underset{\tilde{E}_8}{\mathbb{Z}/6}$ and $G$ of type $A_{n|\Gamma|}$.
- [de Cataldo-Maulik-Shen '19] True for $g(C) = 2$ and $G$ of type $A_n$.

**Variants:** • [Zhang'18] $P=W$ holds for 2-dimensional cluster varieties:

$$\begin{pmatrix} X\text{-cluster} \\ \text{surface} \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y \xrightarrow{h} \Delta \text{ is an } I_b \text{ or } I_b + I_b \\ \text{elliptic surface} \end{pmatrix}$$

so that $X \underset{C^\infty}{\cong} Y$ inducing $W_{2k} H^\bullet(X) = P_k H^\bullet(Y)$.

• [Szabo'18, Szabo-Nemethi'20] $P=W$ holds for irregular character varieties corresponding to Painlevé I–VI.

• [Dancso-McBreen-Shende'19] $P=W$ holds for Hausel-Proudfoot varieties:

$$(T\text{ - finite graph}) \begin{array}{l} \rightsquigarrow \mathcal{B}(T) \text{ - affine} \\ \rightsquigarrow \mathcal{D}(T) \text{ - integrable system} \end{array}$$

So that $\mathcal{B}(T) \underset{C^\infty}{\cong} \mathcal{D}(T)$ inducing

$$W_{2k} H^\bullet(\mathcal{B}(T)) \cong P_k H^\bullet(\mathcal{D}(T))$$

## Hyperplane section description of $P_k H^\bullet(B, F)$

Suppose:

- $B$ - smooth, affine, $\dim B = n$;
- $B \subset \mathbb{A}^N$;
- $\emptyset = \Lambda_{-(N+1)} \subset \Lambda_{-N} \subset \cdots \subset \Lambda_{-1} \subset \Lambda_0 = \mathbb{A}^N$
  a general complete linear flag;
- $\emptyset = B_{-(n+1)} \subset B_{-n} \subset \cdots \subset B_{-1} \subset B_0 = B$
  $B_i = B \cap \Lambda_i$
  the induced flag of hyperplane sections of $B$.

Then the perverse filtration on the cohomology of constructible sheaves on $B$ coincides with the geometric filtration induced by $B_i$:

**Thm** [de Cataldo–Migliorini '10] Let $\mathcal{F}$ be a constructible complex of $\mathbb{Q}$-vector spaces on $B$. Then

$$\boxed{\begin{array}{c} \mathrm{im}\left[ H^i(B, {}^{per}\tau_{\leq -k}\mathcal{F}) \to H^i(B, \mathcal{F}) \right] \\ \| \\ \ker\left[ H^i(B, \mathcal{F}) \longrightarrow H^i(B_{k+i-1}, \mathcal{F}|_{B_{k+i-1}}) \right] \end{array}}$$

## 2. The mirror $P = W$ conjecture.

Observation (Katzarkov):

(P = W for character varieties) + (Mirror symmetry for character varieties)

suggest a $P = W$ matching for general pairs of mirror log CY varieties.

**Indeed:**

- One expects $M_B(C, G) \overset{HMS}{\longleftrightarrow} M_{Dol}(C, {}^LG)$ <- Langlands dual

- One expects
$$\begin{tikzcd} M_{Dol}(C, G) \arrow[rr, leftrightarrow, "T\text{-dual}"] \arrow[dr, "h"'] & & M_{Dol}(C, {}^LG) \arrow[dl, "{}^Lh"] \\ & B & \end{tikzcd}$$
Known away from $\mathrm{Discr} \subset B$ [Donagi–P]

- One expects $Rh_*\mathbb{Q} \overset{\mathbb{D}}{\longleftrightarrow} R({}^Lh)_*\mathbb{Q}$

Thus: one expects

$$W_\bullet\, H^\bullet(M_B(C, G)) \overset{P=W}{\longleftrightarrow} P_\bullet\, H^\bullet(M_{Dol}(C, {}^LG))$$

up to relabeling

This motivates the mirror $P = W$ conjecture which I will state next.

**Setup:** $X$ - smooth projective, $\dim X = d$

$Y \subset X$ strict nc divisor with $Y \in |-K_X|$

$U = X - Y$.

$H^\bullet(U, \mathbb{C})$ has canonical Hodge and weight filtrations $F^\bullet$ and $W_\bullet$

Set

$$HH_{d+i}(U) := \bigoplus_{q-p=i} gr_F^p H^{p+q}(U)$$

$$= \bigoplus_{q-p=i} H^q(X, \Omega_X^p(\log Y))$$

**Note:** The weight filtration on $H^{p+q}(U)$ induces a weight filtration on each $H^q(\Omega_X^p(\log Y))$

**Explicitly:** We have a codimension $i$ residue map

$$\mathrm{Res}_i : \Omega^p_X(\log Y) \to \Omega^{p-i}_{Y^\vee_{[i]}}(\log \partial Y^\vee_{[i]})$$

where

$Y_{[i]} \subset Y$ - set of points of multiplicity $\geq i$,

$Y^\vee_{[i]}$ - the normalization of $Y_{[i]}$,

$\partial Y^\vee_{[i]}$ - the preimage of $Y_{[i+1]} \subset Y_{[i]}$.

Then $W_i \Omega^p_X(\log Y) = \mathrm{Ker}(\mathrm{Res}_i)$ and we have

$$W_i H^q(\Omega^p_X(\log Y)) = \mathrm{im}\left[H^q(W_{i-p-q}\Omega^p_X(\log Y)) \to H^q(\Omega^p_X(\log Y))\right]$$

For the purposes of the mirror $P = W$ conjecture it is better to relabel that so we set

$$\boxed{\begin{aligned}\widetilde{W}_j H^q(\Omega^p_X(\log Y)) \\ := \operatorname{im}\left[H^q(W_j \Omega^p_X(\log Y)) \to H^q(\Omega^p_X(\log Y))\right]\end{aligned}}$$

and

$$\boxed{\widetilde{W}_j HH_{d+i}(U) = \bigoplus_{q-p=i} \widetilde{W}_j H^q(\Omega^p_X(\log Y))}$$

Let $\check{U}$ be the mirror of the non-compact Calabi-Yau variety $U$ ($\dim \check{U} = d$).

Let

$$h : \check{U} \longrightarrow B = \operatorname{Spec}(\Gamma(\check{U}, \mathcal{O}))$$

be the affinization map of $\check{U}$.

**Terminology:** $h : \check{U} \longrightarrow B$ is the hybrid Landau-Ginzburg model mirror to $(X, Y)$.

Explanation: If $Y$ has $n$-components, then Auroux's mirror construction for $(X, Y)$ describes the mirror as a pair $(\check{U}, f)$, where

- $\check{U}$ is an SYZ mirror of $U$;
- $f : \check{U} \to \mathbb{A}^1$ is a holomorphic superpotential counting pseudo holomorphic disks in $X$ with boundaries in SYZ fibers in $U$ and order of contact $1$ with $Y$.

By construction $f = h_1 + \cdots + h_n$, where

$h_k$ counts disks having order of contact with the $k$-th component of $Y$.

Then we get a proper hybrid Landau-Ginzburg potential

$$(h_1, \ldots, h_n) : \check{U} \to \mathbb{A}^n$$

with image $\operatorname{Spec}(\Gamma(\check{U}, \mathcal{O}))$.

Now choose a generic linear flag

$$\emptyset = \Lambda_{n+1} \subset \Lambda_n \subset \cdots \subset \Lambda_1 \subset \Lambda_0 = \mathbb{A}^n$$

and define

$$\boxed{\tilde{P}_j H^k(\check{U}) := \operatorname{Ker}\left[H^k(\check{U}) \to H^k(h^{-1}(\Lambda_{j-k+1}))\right]}$$

Then we have the following

**Mirror P = W conjecture:** Under the HMS identification

$$HH_k(U) \cong H^k(\check{U})$$

we have

$$\tilde{W}_j HH_k(U) = \tilde{P}_j H^k(\check{U}).$$

<u>Results and examples:</u> Harder, Katzarkov, Przyjalkowski, ...

details in Andrew Harder's talk

<u>Refined version:</u> Let

$$\begin{tikzcd} U \arrow[dr, "\pi"'] & & \check{U} \arrow[dl, "\check{\pi}"] \\ & S & \end{tikzcd}$$

be dual Lagrangian torus fibrations.

<u>Notation:</u>

- $\Delta = \mathrm{Discr}(\pi) = \mathrm{Discr}(\check{\pi}) \subset S$.
- $S^o = S - \Delta$, $U^o = \pi^{-1}(S^o)$, $\check{U}^o = \check{\pi}^{-1}(S^o)$.

- $\pi^0 = \pi|_{U^0}$, $\check{\pi}^0 = \check{\pi}|_{\check{U}^0}$.
- $j : S^0 \hookrightarrow S$.

<u>Assume:</u> • $\pi$ and $\check{\pi}$ are simple, i.e.

$$R^\bullet \pi_* \mathbb{Q} = j_* R^\bullet \pi^0_* \mathbb{Q},$$

$$R^\bullet \check{\pi}_* \mathbb{Q} = j_* R^\bullet \check{\pi}^0_* \mathbb{Q}.$$

• The Leray ss for $\pi$ and $\check{\pi}$ degenerate at $E_2$:

$$H^k(U, \mathbb{Q}) = \bigoplus_{p+q=k} H^q(S, j_* R^p \pi^0_* \mathbb{Q})$$

$$H^k(\check{U}, \mathbb{Q}) = \bigoplus_{p+q=k} H^q(S, j_* R^p \check{\pi}^0_* \mathbb{Q})$$

Verdier duality then implies

$$j_* R^p \pi^o_* \mathbb{Q} \cong j_* R^{d-p} \check{\pi}^o_* \mathbb{Q}$$

and in particular

$$H^q(S, j_* R^p \pi^o_* \mathbb{Q}) \cong H^q(S, j_* R^{d-p} \check{\pi}^o_* \mathbb{Q})$$

Also note that by analogy with M. Gross' analysis of Hodge filtrations on compact CY's we expect

$$H^q(X, \Omega^q(\log Y)) = H^q(S, j_* R^p \pi^o_* \mathbb{Q}).$$

# SYZ refined mirror $P=W$ conjecture:

There are natural weight and perverse filtrations on $H^q(\mathcal{S}, R^p\pi_*\mathbb{Q})$ and the under the topological SYZ mirror isomorphism

$$H^q(\mathcal{S}, R^p\pi_*\mathbb{Q}) \cong H^q(\check{\mathcal{S}}, R^{d-p}\check{\pi}_*\mathbb{Q})$$

we have

$$\tilde{W}_j\, H^q(\mathcal{S}, R^p\pi_*\mathbb{Q}) = \tilde{P}_j\, H^q(\check{\mathcal{S}}, R^{d-p}\check{\pi}_*\mathbb{Q}).$$

Detailed definitions of $\tilde{W}_j$, $\tilde{P}_j$ and strategy of proof in Andrew Harder's talk.

## 3. The mirror $P = W$ conjecture from HMS.

HMS in the complement of an anti-canonical divisor:

$$(X, D) \overset{MS}{\longleftrightarrow} (Y,\ Y \xrightarrow{w} \mathbb{A}^1,\ Y_{sm}),$$

smooth Fano    smooth divisor in $|-K_X|$    smooth    proper    smooth fiber

and predicts matching of cohomology and categories:

| B side | A side |
|---|---|
| $H^\bullet(X)$, $H^\bullet(D)$ | $H^\bullet(Y, Y_{sm})$, $H^\bullet(Y_{sm})$ |
| $h^{p,q}(X)$, $h^{p,q}(D)$ | $h^{d-p,q}(Y, Y_{sm})$, $h^{d-1-p,q}(Y_{sm})$ |
| $D^b_{coh}(X)$, $D^b_{coh}(D)$ | $FS(Y, w)$, $Fuk(Y_{sm})$. |

More precisely relative HMS for $(X, D)$ predicts that we have a commutative diagram of functors

$$U = X - D$$

$$\begin{tikzcd}
D^b_{coh}(D) \arrow[r, shift left, "i_*"] \arrow[d, "\text{HMS}"', "\cong"] & D^b_{coh}(X) \arrow[l, shift left, "i^*"] \arrow[r, "j^*"] \arrow[d, "\cong"', "\text{HMS}"] & D^b_{coh}(U) \arrow[d, "\cong"', "\text{HMS}"] \\
\mathrm{Fuk}(Y_{sm}) \arrow[r, shift left, "\cup"] & FS(Y, w) \arrow[l, shift left, "\cap"] \arrow[r, "\iota"] & Wf(Y)
\end{tikzcd}$$

where:

- $\cap$ is the intersection with a fiber
  $\cup$ is the Orlov functor (Abouzaid-Ganatra);
- $(i_*, i^*)$ and $(\cup, \cap)$ are compatible adjoint pairs;
- the horizontal sequences are localizations

Passing to Hochschild homology we get

$$\begin{array}{ccc}
\bigoplus_{q-p=a} H^q(D, \Omega^p_D) & \xrightarrow{i_*} & \bigoplus_{q-p=a} H^q(\underline{X}, \Omega^p_{\underline{X}}) \\
\| \wr & & \| \wr \\
H^{a+d-1}(Y_{sm}) & \longrightarrow & H^{a+d}(Y, Y_{sm})
\end{array}$$

<u>Observation:</u>

- the top row is a map in the $E_1$ page of the weight ss.

- the bottom row is a map in the $E_1$ page of the perverse Leray ss.

in the de Cataldo - Migliorini hyperplane version

**Thm:** [Harder-Katzarkov-Przyjalkowski '20]
Under the above assumptions HMS induces isomorphisms for $i=0,1$:

$$\bigoplus_{q-p=a} gr_F^p \, gr^W_{p+q+i} H^{p+q}(U) \cong gr^{\mathcal{P}}_{d+a+i} H^{d+a}(Y)$$

Moreover the monodromy weight filtrations associated to the Serre functors refine the isomorphisms to matching of summands

$$gr_F^p \, gr^W_{p+q+i} H^{p+q}(U) \cong gr^W_{2(d-p)} \, gr^{\mathcal{P}}_{d+a+i} H^{d+a}(Y).$$

**Corollary:** [Harder-Katzarkov-Przyjalkowski '20]
If the MHS on $H^{\cdot}(Y)$ is of Tate type, then relative HMS for $(\mathbb{X}, \mathbb{D})$ implies the mirror $P=W$ conjecture for the mirror pair $U|Y$.

Recently Sukjoo Lee explained how one can extend these results to the case of normal crossing divisors.

<u>Setup:</u> Suppose $(X, D)$ is a log CY pair with

- $X$ - smooth fano, $\dim X = d$.
- $D = D_1 \cup D_2$ snc divisor in $|-K_X|$ with $D_{12} = D_1 \cap D_2$ connected.
- $D_1, D_2$ are smooth Fanos and $(D_1, D_{12})$ are $(D_2, D_{12})$ are log CY pairs.

**Recall:** A hybrid LG model mirroring $(X, D)$ is a triple

$$(Y, \; Y \xrightarrow{h} \mathbb{A}^2, \; \omega_Y)$$

quasi-projective    proper    symplectic form

satisfying:

- If $h = (h_1, h_2)$ and $Y_1, Y_2$ are general fibers of $h_1, h_2$, then

$$\left.\begin{array}{ccc} (D_1, D_{12}) & \Big| & (Y_1, \; Y_1 \xrightarrow{h_2} \mathbb{A}^1, \; \omega_{Y|Y_1}) \\ (D_2, D_{12}) & \Big| & (Y_2, \; Y_2 \xrightarrow{h_1} \mathbb{A}^1, \; \omega_{Y|Y_2}) \end{array}\right\| \leftarrow \text{Fano mirror pairs}$$

- $D_{12}$ is mirror to $Y_{12}$ - general fiber of $h$.

- $(Y, \; W = h_1 + h_2, \; \omega_Y)$ is the LG mirror of $(X, D)$.

**Thm:** [Sukjoo Lee '21] Suppose $X$ is a smooth Fano complete intersection and $D = D_1 \cup D_2$ is as above. Then there exists a hybrid LG mirror

$$(X, D) \mid (Y, Y \xrightarrow{h} \mathbb{A}^2, \omega_Y)$$

so that if $Y_{sm}$ is a general fiber of $w$, then

$$h|_{Y_{sm}} : Y_{sm} \to \mathbb{A}^1 \quad \text{is a gluing of}$$

$$h_2 : Y_1 \to \mathbb{A}^1 \text{ and } h_1 : Y_2 \to \mathbb{A}^1.$$

**Note:** The proof combines the Hori-Vafa construction, fiberwise compactification, Przyjalkowski-Shramov crepant resolution, deformation of $Y_{sm}$.

**Corollary:** [Sukjoo Lee '21] If relative HMS holds for $(D_1, D_{12})$ and $(D_2, D_{12})$, then relative HMS holds for $(X, D)$ i.e. we have equivalences

$$\begin{tikzcd}[column sep=small]
 & D^b_{coh}(D_{12}) \arrow[dl, shift left] \arrow[dr, shift left] & \\
D^b_{coh}(D_1) \arrow[ur, shift left] \arrow[dr, shift left] & & D^b_{coh}(D_2) \arrow[ul, shift left] \arrow[dl, shift left] \\
 & D^b_{coh}(X) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}
\quad \overset{HMS}{\cong} \quad
\begin{tikzcd}[column sep=small]
 & Fuk(Y_{12}) \arrow[dl, shift left] \arrow[dr, shift left] & \\
FS(Y_1, h_2) \arrow[ur, shift left] \arrow[dr, shift left] & & FS(Y_2, h_1) \arrow[ul, shift left] \arrow[dl, shift left] \\
 & WFS(Y, w) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

**Corollary:** [Sukjoo Lee '21] In this setting we have matching for all $i = 0, 1, 2$:

$$\bigoplus_{q-p=a} gr_F^p \, gr^W_{p+q+i} H^{p+q}(U) \cong gr^P_{d+a+i} H^{d+a}(Y)$$

and refined matching

$$gr_F^p \, gr^W_{p+q+i} H^{p+q}(U) \cong gr^W_{2(d-p)} \, gr^P_{d+a+i} H^{d+a}(Y)$$

**Corollary:** [Sukjoo Lee '21] If the MHS on $H^\bullet(Y)$ is of Tate type, then relative HMS for $(X, D)$ implies the mirror $\mathcal{P} = W$ conjecture for the pair $(U \mid Y)$.

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The careful analysis of these filtrations also leads to full control of the Hodge and deformation theory of hybrid LG models.

Suppose $Y \xrightarrow{h} \mathbb{A}^2$ is a hybrid LG model as above and let

$$Z \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1$$

be a tame compactification.

Then one can study the deformations of $(Z, f)$ anchored at $(\infty \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \infty)$.

**Thm:** [Sukjoo Lee '21] The space of deformations of $(Z, f)$ anchored at $\infty$ is unobstructed.

The proof again follows by checking that the Hodge-to-de Rham spectral sequence for the complex

$$\left( \Omega^{\bullet}_{Z}(\log D, f_1, f_2), d \right)$$

degenerates at $E_1$.

**Conjecture:** The KKP conjecture holds in this setting, i.e.

$$h^q(Z, \Omega^p(\log D, f)) = \dim \operatorname{gr}_q^{W(N)} H^{p+q}(Y, Y_{12})$$

where $W(N)$ is the monodromy weight filtration associated with any monodromy around $\infty$.