Dimension theory

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Let X be a smooth *projective* variety over $\mathbb{C}$, $[\omega] \in H^2(X, \mathbb{Z})$ be an ample class.

Gromov-Witten theory in genus zero gives operators $\mathbf{K}$, $\mathbf{G}$ acting on supervector space $H^\bullet(X)$, with coefficients in $\mathbb{Q}[[q, (t_a)]]$ where (t_a) are coordinates in $H^\bullet(X)$ (or in a complement to $\mathbb{Q} \cdot [\omega]$).

The main hero is the meromorphic connection:

$$\nabla_{\frac{ud}{du}} = \frac{ud}{du} + \frac{1}{u}\mathbf{K} + \mathbf{G}$$

on the trivial bundle with fiber $H^\bullet(X)$ over the u-plane $\mathbb{A}^1_u$.

We have an isomondromic deformation of this connection over the parameter space $\mathsf{Specf}\,\mathbb{Q}[[q, (t_a)]]$ (or its formal subschemes corresponding to deformations by the $\mathbb{Q}$-span of the image of the total Chow group of X in $H^\bullet(X, \mathbb{Q})$).

Eigenvalues of $\mathbf{K}$ form the *quantum spectrum* $\{(z_i)\}$ of X, it depends in general on parameters. If X is Calabi-Yau or of general type, the spectrum consists of only one point, independently on the value of parameters of deformation.

Quantum spectrum at $q = 0$ consists always of only one point $z = z_1 = t_1$.
The reason is that at $q = 0$ quantum product coincides with the classical one, and the ring $H^\bullet(X)$ is local (for connected X).

Hypothetically, elements of the quantum spectrum parametrize terms in a semi-orthogonal decomposition of $D^b(Coh(X))$ (*elementary pieces*). Unconditionally, there is a decomposition of $H^\bullet(X)$ into a direct sum.

We would like to associate with each point of the spectrum a rational (?non-negative) number called the **dimension**.

Expected properties of dimensions:

1. for $q = 0$ the spectrum consists of only one point, and the dimension associated to it is $\dim X$,
2. the dimension is upper semi-continuous: if after a deformation, an eigenvalue of $\mathbf{K}$ splits into a cluster of several ($\geqslant 2$) eigenvalues, then the new dimensions are $\leqslant$ the dimension of the initial eigenvalue,
3. (facultative) if there exists a semi-orthogonal decomposition of $D^b(Coh(X))$ corresponding to the spectrum, and each component $\mathcal{C}_i$ is indecomposable (not a direct sum of two proper subcategories), then the dimension associated to the corresponding eigenvalue z_i coincides with the *Serre dimension* of category $\mathcal{C}_i$.

Reminder: Serre dimensions (after A.Elagin and V.Lunts)

$$\dim_{\mathrm{Serre}} \mathcal{C} := \lim_{|k|\to+\infty} \left\{ \frac{i}{k} | Ext^i(\mathrm{Id}_{\mathcal{C}}, S_{\mathcal{C}}^k) \neq 0 \right\} \subset \mathbb{R}$$

Here $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the *Serre functor* as defined by A.Bondal and M.Kapranov:

$$\mathrm{Hom}_{\mathcal{C}}(E, F)^* = \mathrm{Hom}_{\mathcal{C}}(F, S_{\mathcal{C}} E), \qquad \forall\, E, F \in Ob(\mathcal{C})$$

In general, Serre dimension could be an empty set, or an interval.

For categories $D^b(Coh(X))$ it is exactly the dimension $\dim X \in \mathbb{Z}_{\geqslant 0}$.
For a fractional Calabi-Yau category $S_{\mathcal{C}}^k \sim [n]$ the Serre dimension is equal to Calabi-Yau dimension $\frac{n}{k}$, hence *fractional*. **Example**: Fukaya-Seidel category of $Y = \mathbb{C}_x, W = x^d, d \geqslant 2 : \quad \dim_{\mathrm{Serre}} \mathcal{F}S(Y, W) = 1 - \frac{2}{d}$.

Applications to non-rationality

Just properties 1,2 *together* with the Blow-up conjecture imply the following criterion for non-rationality (assuming dimension $n \geqslant 2$):

if for n-dimensional variety X and a generic parameter in $\mathcal{M}_X^{alg}$, there exists a point in the quantum spectrum of dimension $> (n-2)$, then X is not rational.

This criterion is arithmetic (not purely geometric): the generic decomposition depends on the non-algebraically closed field of definition of X (as $\mathcal{M}_X^{alg}$ depends on the field).

Ludmil will talk more about examples. The criterion seems to be extremely close to be adequate.

Numerical example

Let X be a smooth 3-dimensional cubic in $\mathbb{P}^4$, $[\omega] := c_1(\mathcal{O}(1))$. All parameters (t_a) are set to be zero, so we are left with only one parameter q for the qunatum product. Operators $\mathbf{K}, \mathbf{G}$ restricted to the even part, which is 4-dimensional space $H^{even}(X) = \oplus_{i=0}^{3} H^{2i}(X)$, basis = powers of the hyperplane section, are:

$$\mathbf{K} = 2 \cdot \begin{pmatrix} 0 & 6q & 0 & 36q^2 \\ 1 & 0 & 15q & 0 \\ 0 & 1 & 0 & 6q \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

Solutions of the equation

$$\left(\frac{ud}{du} + \frac{1}{u}\mathbf{K} + \mathbf{G}\right)\psi(u) = 0$$

grow at $u \to 0$ as

$$\sim e^{\frac{6\sqrt{3q}}{u}}, \sim e^{-\frac{6\sqrt{3q}}{u}}, \sim u^{-\frac{1}{6}}, \sim u^{-\frac{5}{6}} \quad (q \neq 0)$$

$$\sim u^{-\frac{3}{2}}, u^{-\frac{3}{2}} \cdot \log u, u^{-\frac{3}{2}} \cdot (\log u)^2, u^{-\frac{3}{2}} \cdot (\log u)^3 \quad (q = 0)$$

Claim: $D^b(Coh(X)) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{C} \rangle$ where $\mathcal{C}$ is a fractional Calabi-Yau category of dimension $\frac{5}{3}$.

Definition of dimension via the solution growth

Conjecture : $\dim_{\mathrm{Serre}} C_{z_i} = -2\min\{s \in \mathbb{Q}_{\leqslant 0} | \ \exists \text{ solution } \sim u^s \log(u)^k e^{\frac{z_i}{u}} + \dots\}$

Evidence: if X is a Fano complete intersection of hypersurfaces of degrees $d_1, \dots, d_r$ in $\mathbb{P}^n$, then the corresponding semi-orthogonal decomposition is

$$D^b(Coh(X)) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n - d_{sum}), \text{Kuznetsov component} \rangle$$

where $d_{sum} := \sum_i d_i$, the spectrum is $\{0\} \cup \mu_{n+1-d_{sum}}$. The predicted Serre dimension for the Kuznetsov component $\mathcal{C}_{z=0}$ is

$$\dim_{\mathrm{Serre}} C_{z=0} = (n-r) - 2\,\frac{n+1-\sum_i d_i}{\max_i d_i} = \dim X - 2\,\frac{\overbrace{n+1-d_{\mathsf{sum}}}^{\text{Fano index of } X}}{d_{\mathsf{max}}}$$

Moreover, calculations show that there is always a very striking *equality*

$$\max\{i \in 2\mathbb{Z} + \dim X \,|\, i \leqslant \dim_{\text{Serre}} C_{z=0}\} \overset{!}{=}$$

$$\overset{!}{=} \max\{k \in \mathbb{Z} |\, HH_k(\mathcal{C}_{z=0}) \neq 0\} := \max\{q - p |\, H^{p,q}(X) \neq 0\}$$

Analogy: for any smooth projective variety Y we have:

$$\dim Y \geqslant \max\{k \in \mathbb{Z} |\, HH_k(\mathsf{Perf}\, Y) \neq 0\} = \max\{q - p |\, H^{p,q}(Y) \neq 0\}$$

The formula for the growth exponent (and hence hypothetically for Serre dimension) generalizes to so-called smooth *well-formed* complete intersections in *weighted* projective spaces (main tool: Givental's hypergeometric equation):

$$\dim_{\text{Serre}} C_{z=0} \overset{?}{=} \dim X - 2\,\frac{w_{\mathsf{sum}} - d_{\mathsf{sum}}}{d_{\mathsf{max}}}, \quad w_{\mathsf{sum}} := \sum_j w_j \text{ for } \mathbb{P}^{w_0,\dots,w_n}$$

Relation to Reconstruction theorem

In general, for Fano variety X, the meromorphic connection $\frac{ud}{du} + \frac{1}{u}\mathbf{K} + \mathbf{G}$ is nice (algebro-geometric, or known explicitly) *only* for all parameters (t_a) equal to zero, except those corresponding to $H^0(X)$ and $H^2(X)$.

It seems to be a hard question in GW theory to understand whether deformations corresponding to classes from $H^4(X), H^6(X), \dots$ will lead to an additional splitting of eigenvalues, or not.

I expect that if $H^{even}(X)$ is generated as a ring (with the classical multiplication) by $H^2(X)$, then this additional splitting will not happen. The rationale is the Reconstruction theorem (M.K, Yu.Manin 1994): in this case all genus zero GW invariants restricted to $H^{even}(X)$ are **uniquely** determined by the quantum product deformed only in $H^2(X)$-direction (and only classes $[\overline{\mathcal{M}}_{0,2}(X, \beta)]_{virt}$ are needed, besides the classical product).

Landau-Ginzburg model perspective

Let Y be a *noncompact* complex manifold of dimension $n \geq 0$, and $W : Y \to \mathbb{C}$ be a holomorphic map. Denote by $\mathsf{Crit}(W) \subset Y$ the critical locus considered as a closed analytic subspace (possibly non-reduced).

Assumptions:

1. (the most crucial) $\mathsf{Crit}(W)$ is compact
2. (also important) there exists a Kähler metric on Y (the choice is *not* a part of the structure, only existence is required)
3. (technical, for later convenience) $\mathsf{Crit}(W)$ is nonempty and connected, and moreover $f(\mathsf{Crit}(W)) = \{0\} \subset \mathbb{C}$.

What follows will not change if we replace Y by any open subset $Y' \subset Y$ containing $\mathsf{Crit}(W)$ (alternatively, one can consider Y as a *germ* at $\mathsf{Crit}(W)$).

Consider $\mathbb{Z}$-graded complex

$$(\Gamma(Y, \Omega^\bullet_{C^\infty}(Y)[[u]], \text{ differential } d_{\text{tot}} := \bar{\partial} + u\partial + dW \wedge \cdot)$$

It calculates hypercohomology $R\Gamma(Y, \Omega^\bullet_Y[[u]], ud + dW \wedge \cdot)$.
If $\alpha \in \Gamma(Y, \Omega^\bullet_{C^\infty}(Y))\{u\}$ (i.e. not only a formal series in u, but a germ of analytic [forms on Y]-valued function at $u = 0$ and near the compact set $\mathsf{Crit}(W)$) then $d_{\text{tot}}\alpha = 0$ means that

$$d\left(e^{\frac{W}{u}} u^{\mathbf{Gr}}(\alpha)\right) = 0, \qquad \mathbf{Gr}_{|\Omega^{p,q}_{C^\infty}} := \frac{q-p}{2}$$

Conjecture (Hodge-de Rham degeneration for LG models): *Cohomology of d_{tot} is a free finite rank (equivalently, flat) $\mathbb{C}[[u]]$-module*.

It is true in algebro-geometric situation by irregular Hodge theory.

Let us assume that Y is endowed with an everywhere non-vanishing holomorphic volume form $vol \in \Gamma(Y, \Omega^n), \quad n = \dim Y$.

Then in the case when $\mathsf{Crit}(W)$ is *connected and non-empty*, there is a *canonical* 1-dimensional subspace at the fiber at 0:

$$[vol] \in \mathbb{H}^n(Y, \Omega_Y^\bullet, dW \wedge \cdot) \to$$
$$\to \mathbb{H}^n(\mathsf{Crit}(W), (\Omega_Y^\bullet)_{|\mathsf{Crit}(W)}) \to$$
$$\to H^0(\mathsf{Crit}^{red}(W), \Omega_Y^n) = \mathbb{C} \ni 1$$

Conjecture: *the leading growth of solutions appears in the exactly one-dimensional subspace of cohomology of d_{tot}, its reduction at $u = 0$ is $\mathbb{C} \cdot [vol]$.*

Upper semicontinuity: was proven by A.Varchenko for isolated singularities, by J.Steenbrink in general. *Problem*: deformations of $(Y, W) \iff H^2(\mathrm{mirror})$.

Abstract semi-continuity conjecture (minimal version).

Let $G \in Mat(N \times N, \mathbb{C})$ be a semisimple operator with spectrum in $\mathbb{Z} \subset \mathbb{C}$ (i.e a $\mathbb{Z}$-grading on $\mathbb{C}^N$), and we have two formal series

$$A = A_0 + A_1 t + \ldots, K = K_0 + K_1 t + \cdots \in Mat(N \times N, \mathbb{C}[[t]])$$

satisfying equations

$$[A, K] = 0$$
$$t\partial_t K + [A, G] = 0$$

which means that we get a flat connection: $\left[t\partial_t + \frac{A}{u}, \partial_u + \frac{K}{u^2} + \frac{G}{u}\right] = 0$.

Then one has *special* connection over $\mathbb{C}$ given by $\partial_u + \frac{K_0}{u^2} + \frac{G}{u}$, and *generic* connection over $\mathbb{C}((t))$ given by $\partial_u + \frac{K}{u^2} + \frac{G}{u}$.

Conjecture: *Assume that* K_0 *is nilpotent and the special connection* $\partial_u + \frac{K_0}{u^2} + \frac{G}{u}$ *has regular singularities. Then the generic connection is of exponential type, and the exponent of the largest growth of a regular solution of the special connection is more singular than those of the general connection (over the base field* $\mathbb{C}((t))$*).*

Strictly speaking, the above conjecture does not make sense as the "largest growth" exponent for the base field $\mathbb{C}((t))$ is an element of its algebraic closure

$$\overline{\mathbb{C}((t))} = \lim_{N \to +\infty} \mathbb{C}((t^{\frac{1}{N}})) \quad \text{(Puiseux series)}$$

We expect that the "largest growth" exponent for the generic connection is in fact an element of $\mathbb{C} \subset \overline{\mathbb{C}((t))}$, and it differs from the largest growth exponent for the special connection by a *rational* number.