Blow-up formula for quantum cohomology

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IHES
1. General framework for quantum products

**Algebraic version over** \( \mathbb{C} \): \( X \) - a smooth connected complex quasi-projective variety, with an ample line bundle \( \mathcal{L} \) (e.g. the pullback of \( \mathcal{O}(1) \)).

**Assumption**: *(the minimal amount of "convexity at infinity")*:
for any compact subset \( K \subset X \) and any homology class \( \beta \in H_2(X, \mathbb{Z}) \), there exists a larger compact set \( K' \), \( K \subset K' \subset X \) such that for any compact semistable map \( \phi : C \to X \) of genus zero, with class \( \phi_*[C] = \beta \) and touching \( K \) (i.e. \( \phi(C) \cap K \neq \emptyset \)) we have \( \phi(C) \subset K' \).

A sufficient condition: there exists a proper morphism \( X \to B \) where \( B \) is affine (a typical situation in GIT).

One can mostly have in mind the basic case: \( X \) is projective.
Under the above convexity-at-infinity assumption, we have a well-defined Gromov-Witten invariant for any \( \beta \in H_2(X, \mathbb{Z}) \) and \( n \geq 1 \):

\[
\langle \delta_1, \ldots, \delta_{n-1}; \delta(c) \rangle_\beta := \int_{[\overline{M}_{g,n}(X,\beta)]_{\text{virt}}} \prod_{i=1}^{n-1} ev^*_i(\delta_i) \cdot ev_n^*(\delta(c))
\]

where \( \delta_1, \ldots, \delta_{n-1} \in H := H^\bullet(X, \mathbb{Q}), \quad \delta(c) \in H^\vee = H^\bullet_c(X, \mathbb{Q}). \)

This gives a symmetric polylinear map of supervector spaces \( \text{Sym}^{n-1} H \to H \).

Choose a \( \mathbb{Z} \)-graded basis \( (\Delta_a)_{a=1, \dim H} \) of \( H \) containing \( 1 \in H^0(X, \mathbb{Q}) \) and \( c_1(\mathcal{L}) \in H^2(X, \mathbb{Q}) \), define the quantum product \( \star : \text{Sym}^2 H \to H[[[q, (t_a)]]] \) by

\[
(\delta_1 \star \delta_2, \delta(c)) := \sum_{\beta} \sum_{a_1, \ldots, a_m} q^{\int_\beta c_1(\mathcal{L})} \frac{\prod_{i=1}^m t_{a_i}}{m!} \langle \delta_1, \delta_2, \Delta_{a_1}, \ldots, \Delta_{a_m}; \delta(c) \rangle_\beta
\]
\[ \nabla \frac{ud}{du} = \frac{ud}{du} + \frac{1}{u}K + G \]

\[ \nabla \frac{d}{dt_a} = \frac{d}{dt_a} + \frac{1}{u}A_a, \quad \nabla \frac{qd}{dq} = \frac{qd}{dq} + \frac{1}{u}A_a: \Delta_a = c_1(\mathcal{L}) \]

where

- \( K = \) quantum product with \( c_1(T_X) + \sum_a (2 - \deg \Delta_a) t_a \Delta_a, \)
- \( A_a = \) quantum product with \( \Delta_a, \)
- \( G|_{\mathcal{H}^i(X)} = \frac{i-\dim X}{2} \cdot \text{id}_{\mathcal{H}^i(X)} \quad \forall i = 0, \ldots, 2 \dim X. \)

This connection is flat, has poles at hyperplanes \( u = 0 \) and \( q = 0. \)
Variables \( t_a \) corresponding to \( 1 \in H^0(X, \mathbb{Q}) \) and \( c_1(\mathcal{L}) \in H^2(X, \mathbb{Q}) \) are special: the dependence of the quantum product \( \star \) on the variable (say, \( t_1 \)) corresponding to \( 1 \) is trivial, whereas the variable (say, \( t_2 \)) corresponding to \( c_1(\mathcal{L}) \) can be ignored, as it is equivalent to \( \log(q) \).

We will be interested only in the restriction of the above flat connection to the purely even vector subspace for \( (t_a) \)-coordinates which we denote by \( H_{alg}(X) \subset H \). It is the subspace spanned by the classes of closed algebraic subvarieties in \( X \). Let us choose a graded complement \( H'_{alg}(X) \) to \( \mathbb{Q} \cdot c_1(\mathcal{L}) \).

The result is a meromorphic flat connection on a super vector bundle \( \mathcal{H} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd} \) on \( \mathbb{P}^1_u \times \mathcal{M}_{alg} \) where \( \mathcal{M}_{alg} \) is a formal scheme over \( \mathbb{Q} \) (equal in our case to \( \text{Specf} \mathbb{Q}[[[q, H'_{alg}(X)]]]) \). All this satisfies a bunch of properties (e.g. ensuring that \( \mathcal{H} \) is canonically trivialized). Flat coordinates on \( \mathcal{M}_{alg} \) can be extracted from this structure and element \( 1 \in H'_{even} = \Gamma(\mathcal{H}_{even}) \).
Generalizations:

- $X$ can be a smooth Deligne-Mumford stack (in this case replace $H^\bullet(X)$ by string cohomology $H^\bullet_{str}(X) := H^\bullet(\text{inertia stack of } X)$),
- $X$ can be also endowed with a torsion class in the Brauer group, giving a bundle of Azumaya algebras,
- class $c_1(L)$ of an ample bundle can be replaced by any functional $\text{deg} : H_2(X, \mathbb{Z}) \to \mathbb{Z}$ which is non-negative on classes of rational curves, and such that for given degree $\text{deg} \in \mathbb{Z}_{\geq 0}$ and given pairing $\in \mathbb{Z}$ with $c_1(T_X)$, there are only finitely many homology classes represented by rational curves.

**Sufficient condition:** $\text{deg}(\beta) = ([\omega], \beta) \quad \forall \beta \in H_2(X, \mathbb{Z})$ where cohomology class $[\omega] \in H^2(X, \mathbb{Z})$ is non-negative, and there exists constant $C \in \mathbb{Q}$ such that cohomology class $[\omega] + C \cdot c_1(T_X)$ is strictly positive.

Different choices of $[\omega]$ give the same information, can be recalculated.
There are further deformations of the quantum product:

- by adding gravitational descendants,
- by adding a multiplicative characteristic class of $R\Gamma(C, \phi^* E)$, where
  $\phi : C \to X$ is the universal stable map (depending on a point in $\overline{M}_{g,n}(X, \beta)$)
  and $E$ is an algebraic vector bundle on $X$.

By Coates-Givental formalism, these deformations can be recalculated, by some universal formulas, from the original small quantum product.
Finally, GW-theory can be formulated for varieties definitely over *arbitrary* field $\mathbf{k}$ of characteristic zero (hypothetically also in positive characteristic).

We can assume safely that $\mathbf{k} \subset \mathbb{C}$.

**Definition 1:** $H^\bullet_{\text{alg}}(X) := \mathbb{Q}$-subspace in $H^\bullet_{\text{Betti}}(X) := H^\bullet(X(\mathbb{C})_{\text{an}}, \mathbb{Q})$ spanned by classes $[Z]$ of closed subvarieties defined over $\mathbf{k}$.

It is a finite-dimensional (even) vector space over $\mathbb{Q}$.

**Definition 2:** $\text{End}_{\text{alg}}(X) := \mathbb{Q}$-subalgebra in $\text{End}(H^\bullet_{\text{Betti}}(X))$ generated by the grading operator and by classes $[Z] \in H^\bullet_c(X(\mathbb{C})_{\text{an}}, \mathbb{Q}) \otimes H^\bullet(X(\mathbb{C})_{\text{an}}, \mathbb{Q})$ of subvarieties $Z \subset X \times X$ defined over $\mathbf{k}$ and proper over the first factor $X$.

It is just a finite-dimensional (even) algebra over $\mathbb{Q}$ containing commuting projectors $pr_i$ to graded components, $i = 0, \ldots, 2 \dim X$. Space $H^\bullet_{\text{alg}}(X)$ is a module over (the even part) of this algebra. By comparison isomorphisms, both algebra $\text{End}_{\text{alg}}(X)$ and module $H^\bullet_{\text{alg}}(X)$ do not depend on the embedding to $\mathbb{C}$.
2. Quantum spectrum and Blow-up conjecture

Operator $\mathbf{K}$ (the quantum product with $c_1(T_X) + \ldots$) is an even endomorphism of super vector space $H = H^*(X)$ parametrized by the formal polydisc $\mathcal{M}^{alg} = \text{Specf } \mathbb{Q}[\![q, H^{'}_{alg}(X)]\!]$. The (generic) quantum spectrum $\text{Spec}_X$ is the spectrum of $\mathbf{K}$ at the generic point of $\mathcal{M}^{alg}$.

The goal of my lectures is to formulate several conjectures concerning the quantum spectrum and its behavior under blow-ups. In particular, the number of elements in the spectrum should be additive in an appropriate sense, giving a motivic measure.

An additional invariant ("dimension") will be introduced in the next lecture, giving a new criterion for non-rationality, which seems to be surprisingly close to the optimal one (see the talk by Ludmil Katzarkov later today).
There is a very optimistic conjecture, for which I do not have a really solid evidence (and which is completely out of reach now).

To simplify life, let us assume that the quantum connection is given by a convergent series.

**Conjecture:** for any point in $\mathcal{M}^{alg}$ and a choice of disjoint paths from $-\infty$ to points of the corresponding spectrum (Gabrielov paths):

we obtain a semi-orthogonal decomposition $D^b(Coh(X)) = \langle C_1, \ldots, C_r \rangle$ where $r$ is the number of elements of the spectrum.

For $X$ being a DM stack with a gerbe, modify $D^b(Coh(X))$ appropriately.
If $X$ is compact, all categories $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are saturated (i.e. smooth and proper), equal to local Fukaya-Seidel categories for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists. In general, I expect that all $\mathcal{C}_i$ are of finite type (in particular, they are homologically smooth).

Notice that one can choose not a generic point in $\mathcal{M}^{alg}$, then the number $r$ of elements of the spectrum will be strictly smaller relative with the generic case. The semi-orthogonal decomposition associated with the non-generic point, is obtained form the generic one by combining several subsequent subcategories into one larger saturated subcategory.

In this conjecture all subcategories $\mathcal{C}_i$ are not phantoms, its Hochschild homology (which are $\mathbb{Z}$-graded vector spaces over the $k \subset \mathbb{C}$) are non-zero.

One can omit the assumption of convergence, working over the field of Puiseaux series in an auxiliary variable which can be thought of as a small positive number.
I will not assume the over-optimistic conjecture on semi-orthogonal decompositions, but still try to extract more accessible corollaries.

Let $Y \subset X$ be a smooth closed subvariety of codimension $m \geq 2$, and denote by $\pi : \widetilde{X} \to X$ the blow-up of $X$ with center $Y$. We have the following basic facts:

- if $X$ is "convex-at-infinity", then the same is true for $Y$ and $\widetilde{X}$,
- if $[\omega] \in H^2(X, \mathbb{Z})$ is an ample class, then $[\omega]|_Y$ is ample, $\pi^*([\omega]) \geq 0$ and for sufficiently small $\epsilon > 0$ class $\pi^*[\omega] + \epsilon c_1(T_{\widetilde{X}}) \in H^2(\widetilde{X}, \mathbb{Q})$ is ample,
- $H^\bullet(\widetilde{X}) \simeq H^\bullet(X) \oplus \bigoplus_{(m-1)} \text{copies } H^\bullet(Y),$
- $\dim \mathcal{M}^{alg}(\widetilde{X}) = \dim \mathcal{M}^{alg}(X) + (m-1) \dim \mathcal{M}^{alg}(Y),$
- $D^b(Coh(\widetilde{X})) = \langle D^b(Coh(X)), D^b(Coh(Y)), \ldots, D^b(Coh(Y)) \rangle.$
Informal version of the Blow-up conjecture: the spectrum $\operatorname{Spec}_X$ is close to

\[
\cdots \cdots
\]

with $(m - 1)$ shifted copies of $\operatorname{Spec}_Y$ around one copy of $\operatorname{Spec}_X$.

One year ago in Miami I talked already about Blow-up conjecture via certain "gluing", see notes of my lecture 2 on the webpage of the collaboration

https://schms.math.berkeley.edu/events/miami2020/#schedule

I'll sketch below a reformulation of the gluing in a slightly different way.
Let us endow $X, Y, \widetilde{X}$ with semi-ample classes

$$[\omega], \quad [\omega]_Y = (Y \to X)^*[\omega], \quad \pi^*([\omega]) = (\widetilde{X} \to X)^*[\omega]$$

respectively. The first two classes are in fact ample, and the third one still gives a well-defined series for the quantum product.

Operator $K_{\widetilde{X},0}$, which is $K_\widetilde{X}$ at point $0 \in M^\text{alg}_{\widetilde{X}}$, has spectrum

$$\text{Spec}_{\widetilde{X},0} = \{0\} \cup \{z \in \mathbb{C} | z = (m - 1)^{m-1}\sqrt{1}\}$$

Meromorphic connection $\frac{ud}{du} + \frac{1}{u} K_{\widetilde{X},0} + G_{\widetilde{X}}$ over $\mathbb{C}[[u]]$ can be explicitly identified with the sum of connections corresponding to elements of $\text{Spec}_{\widetilde{X},0}$.

The summand corresponding to $z = 0$ can be explicitly identified with $\frac{ud}{du} + \frac{1}{u} K_{X,0} + G_X$, and with $\frac{ud}{du} + \frac{1}{u} K_{Y,0} + \frac{z}{u} + G_Y$ for $z = (m - 1)^{m-1}\sqrt{1}$. 
Meromorphic connection of the form \( \frac{ud}{du} + \frac{1}{u} K + G \) where \( K, G \) are operators in a finite-dimensional (super) vector space, can be understood in certain sense as a connection with second order pole over \( \mathbb{C}[[u]] \) and connection with first order pole on \( \mathbb{C}[u^{-1}] \) glued along an identification on \( \mathbb{C}(u) \) in such a way that the resulting super vector bundle over \( \mathbb{C}P^1 \) is trivial.

Now, let us deform by an isomonodromic deformations (parametrized by \( \mathcal{M}^{alg}_X \) and by copies of \( \mathcal{M}^{alg}_Y \)) connections over \( \mathbb{C}[[u]] \) given by \( \frac{ud}{du} + \frac{1}{u} K_{X,0} + G_X \) and \((m - 1)\) copies of \( \frac{ud}{du} + \frac{1}{u} K_{Y,0} + \frac{z}{u} + G_Y \). Gluing to the same connection \( \frac{ud}{du} + \frac{1}{u} K_{\bar{X},0} + G_{\bar{X}} \) over \( \mathbb{C}[u^{-1}] \) we obtain again a trivial bundle over \( \mathbb{C}P^1 \).

One can read flat coordinates in a canonical way, and obtain a non-linear map

\[
\mathcal{M}^{alg}_{\bar{X}} \to \mathcal{M}^{alg}_X \times (\mathcal{M}^{alg}_Y)^{m-1}
\]
**Conjecture**: the pullback of the flat connection on $\mathbb{P}^1_u \times \mathcal{M}^{alg}_{\tilde{X}} \times (\mathcal{M}_{Y}^{alg})^{m-1}$ to $\mathbb{P}^1_u \times \mathcal{M}^{alg}_{\tilde{X}}$ coincides with those given by $GW$-invariants of $\tilde{X}$.

This is a bit non-explicit description of the quantum product of $\tilde{X}$ in terms of those for $X$ and $Y$, and some data from the classical topology (restriction morphisms, cup-products on cohomology, and characteristic classes of normal/tangent bundles).

The Blow-up conjecture is still not proven (and not refuted).

I will finish this talk with the description of the strategy, which (I hope) can work. The main statement which I will try to prove is that the genus zero GW-invariants of $\tilde{X}$ are canonically determined in terms of those for $X$ and $Y$ and the classical data. Then the Blow-up conjecture will be reduced to certain formal identity.
Main idea: introduce a new manifold

\[ \widetilde{X} := Bl_{Y \times \{0\}}(X \times \mathbb{P}^1) \]

It carries \( \mathbb{C}^\times \)-action by rescaling the canonical coordinate \( t \) on \( \mathbb{P}^1 \). The locus of fixed points consists of 3 components \( \widetilde{X} \times \{0\}, Y \times \{0\} \) and \( X \times \{\infty\} \).
Now consider moduli spaces of genus zero curves on $\widehat{X}$ of all possible degrees $\hat{\beta} \in H_2(\widehat{X}, 0)$ such that the image in $H_2(\mathbb{P}^1, \mathbb{Z})$ vanishes ("vertical curves").

The locus of fixed points in $\overline{M}_{0,n}(\widehat{X}, \hat{\beta})$ consists either of curves in $X \times \{\infty\}$, or of trees of curves in $\widehat{X} \times \{0\}$ and $Y \times \{0\}$ joined by cyclic covers of orbits of $\mathbb{C}^\times$-action connecting points of $\widehat{X} \times \{0\}$ and $Y \times \{0\}$.
The sum of contributions of the fixed loci (by Bott formula) should have *vanishing* coefficients for *strictly negative* powers of the equivariant parameter. This gives an infinite bunch of identities, and there are good signs that these identities determine genus zero GW invariants of $\widetilde{X}$ *uniquely*. 