Blow-up formula for quantum cohomology

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1. General framework for quantum products

Algebraic version over $\mathbb{C}$: X - a smooth connected complex quasi-projective variety, with an ample line bundle $\mathcal{L}$ (e.g. the pullback of $\mathcal{O}(1)$).

Assumption: ***(the minimal amount of "convexity at infinity")***:
for any compact subset $K \subset X$ *and any homology class* $\beta \in H_2(X, \mathbb{Z})$*, there exists a larger compact set* K'*,* $K \subset K' \subset X$ *such that for any compact semistable map* $\phi : C \to X$ *of genus zero, with class* $\phi_*[C] = \beta$ *and touching* K *(i.e.* $\phi(C) \cap K \neq \emptyset$*) we have* $\phi(C) \subset K'$*.*

A sufficient condition: there exists a proper morphism $X \to B$ where B is affine (a typical situation in GIT).

One can mostly have in mind the basic case: X is projective.

Under the above convexity-at-infinity assumption, we have a well-defined Gromov-Witten invariant for any $\beta \in H_2(X,\mathbb{Z})$ and $n \geq 1$:

$$\langle \delta_1, \ldots, \delta_{n-1}; \delta_{(c)} \rangle_\beta := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]_{virt}} \prod_{i=1}^{n-1} ev_i^*(\delta_i) \cdot ev_n^*(\delta_{(c)})$$

where $\delta_1, \ldots, \delta_{n-1} \in H := H^\bullet(X,\mathbb{Q}), \quad \delta_{(c)} \in H^\vee = H_c^\bullet(X,\mathbb{Q})$.
This gives a symmetric polylinear map of supervector spaces $Sym^{n-1}H \to H$.

Choose a $\mathbb{Z}$-graded basis $(\Delta_a)_{a=1,\dim H}$ of H containing $\mathbf{1} \in H^0(X,\mathbb{Q})$ and $c_1(\mathcal{L}) \in H^2(X,\mathbb{Q})$, define the quantum product $\star : Sym^2 H \to H[[q,(t_a)]]$ by

$$(\delta_1 \star \delta_2, \delta_{(c)}) := \sum_\beta \sum_{\substack{m \geq 0 \\ a_1,\ldots,a_m}} q^{\int_\beta c_1(\mathcal{L})} \frac{\prod_{i=1}^m t_{a_i}}{m!} \langle \delta_1, \delta_2, \Delta_{a_1}, \ldots, \Delta_{a_m}; \delta_{(c)} \rangle_\beta$$

$\rightsquigarrow$ commutative associative products on H parametrized by $\mathsf{Specf}\,\mathbb{Q}[[q,(t_a)]]$.
Define a connection on the trivial bundle with fiber H over $\mathsf{Specf}\,\mathbb{Q}[[q,(t_a)]][u]$:

$$\nabla_{\frac{ud}{du}} = \frac{ud}{du} + \frac{1}{u}\mathbf{K} + \mathbf{G}$$
$$\nabla_{\frac{d}{dt_a}} = \frac{d}{dt_a} + \frac{1}{u}\mathbf{A}_a, \quad \nabla_{\frac{qd}{dq}} = \frac{qd}{dq} + \frac{1}{u}\mathbf{A}_{a:\Delta_a=c_1(\mathcal{L})}$$

where

- $\mathbf{K}$ = quantum product with $c_1(T_X) + \sum_a (2 - \deg \Delta_a) t_a \Delta_a$,
- $\mathbf{A}_a$ = quantum product with Δ_a,
- $\mathbf{G}_{|H^i(X)} = \frac{i - \dim X}{2} \cdot \mathrm{id}_{H^i(X)} \quad \forall\, i = 0, \ldots, 2\dim X$.

This connection is ***flat***, has poles at hyperplanes $u = 0$ and $q = 0$.

Variables t_a corresponding to $\mathbf{1} \in H^0(X, \mathbb{Q})$ and $c_1(\mathcal{L}) \in H^2(X, \mathbb{Q})$ are special: the dependence of the quantum product $\star$ on the variable (say, t_1) corresponding to $\mathbf{1}$ is *trivial*, whereas the variable (say, t_2) corresponding to $c_1(\mathcal{L})$ can be *ignored*, as it is equivalent to $\log(q)$.

We will be interested only in the restriction of the above flat connection to the purely even vector subspace for (t_a)-coordinates which we denote by $H_{alg}(X) \subset H$. It is the subspace spanned by the classes of closed algebraic subvarieties in X. Let us choose a graded complement $H'_{alg}(X)$ to $\mathbb{Q} \cdot c_1(\mathcal{L})$.

The result is a meromorphic flat connection on a super vector bundle $\mathcal{H} = \mathcal{H}^{even} \oplus \mathcal{H}^{odd}$ on $\mathbb{P}^1_u \times \mathcal{M}^{alg}$ where $\mathcal{M}^{alg}$ is a formal scheme over $\mathbb{Q}$ (equal in our case to $\mathsf{Specf}\, \mathbb{Q}[[q, H'_{alg}(X)]]$). All this satisfies a bunch of properties (e.g. ensuring that $\mathcal{H}$ is canonically *trivialized*). Flat coordinates on $\mathcal{M}^{alg}$ can be extracted from this structure and element $\mathbf{1} \in H^{even} = \Gamma(\mathcal{H}^{even})$.

Generalizations:

- X can be a smooth Deligne-Mumford stack (in this case replace $H^\bullet(X)$ by *string cohomology* $H^\bullet_{str}(X) := H^\bullet(\text{inertia stack of } X)$),
- X can be also endowed with a torsion class in the Brauer group, giving a bundle of Azumaya algebras,
- class $c_1(\mathcal{L})$ of an ample bundle can be replaced by any functional $\deg : H_2(X,\mathbb{Z}) \to \mathbb{Z}$ which is non-negative on classes of rational curves, and such that for given degree $\deg \in \mathbb{Z}_{\geq 0}$ and given pairing $\in \mathbb{Z}$ with $c_1(T_X)$, there are only *finitely many* homology classes represented by rational curves. **Sufficient condition**: $\deg(\beta) = ([\omega], \beta) \quad \forall \beta \in H_2(X,\mathbb{Z})$ where cohomology class $[\omega] \in H^2(X,\mathbb{Z})$ is *non-negative*, and there exists constant $C \in \mathbb{Q}$ such that cohomology class $[\omega] + C \cdot c_1(T_X)$ is strictly positive.

Different choices of $[\omega]$ give the *same* information, can be recalculated.

There are further deformations of the quantum product:

- by adding gravitational descendants,
- by adding a multiplicative characteristic class of $R\Gamma(C, \phi^* E)$, where $\phi : C \to X$ is the universal stable map (depending on a point in $\overline{\mathcal{M}}_{g,n}(X, \beta)$) and E is an algebraic vector bundle on X.

By Coates-Givental formalism, these deformations can be recalculated, by some universal formulas, from the original small quantum product.

Finally, GW-theory can be formulated for varieties definitely over *arbitrary* field $\mathbf{k}$ of characteristic zero (hypothetically also in positive characteristic.).

We can assume safely that $\mathbf{k} \subset \mathbb{C}$.
Definition 1: $H^\bullet_{alg}(X) :=$ $\mathbb{Q}$*-subspace in* $H^\bullet_{Betti}(X) := H^\bullet(X(\mathbb{C})_{an}, \mathbb{Q})$ *spanned by classes* $[Z]$ *of closed subvarieties defined over* $\mathbf{k}$.
It is a finite-dimensional (even) vector space over $\mathbb{Q}$.

Definition 2: $End_{alg}(X) :=$ $\mathbb{Q}$*-subalgebra in* $End(H^\bullet_{Betti}(X))$ *generated by the grading operator and by classes* $[Z] \in H^\bullet_c(X(\mathbb{C})_{an}, \mathbb{Q}) \otimes H^\bullet(X(\mathbb{C})_{an}, \mathbb{Q})$ *of subvarieties* $Z \subset X \times X$ *defined over* $\mathbf{k}$ *and proper over the first factor* X.
It is just a finite-dimensional (even) algebra over $\mathbb{Q}$ containing commuting projectors pr_i to graded components, $i = 0, \ldots, 2\dim X$. Space $H^\bullet_{alg}(X)$ is a module over (the even part) of this algebra. By comparison isomorphisms, both algebra $End_{alg}(X)$ and module $H^\bullet_{alg}(X)$ do not depend on the embedding to $\mathbb{C}$.

2. Quantum spectrum and Blow-up conjecture

Operator $\mathbf{K}$ (the quantum product with $c_1(T_X) + \ldots$) is an even endomorphism of super vector space $H = H^\bullet(X)$ parametrized by the formal polydisc $\mathcal{M}^{alg} = \mathsf{Specf}\, \mathbb{Q}[[q, H'_{alg}(X)]]$. The (generic) **quantum spectrum Spec_X** is the spectrum of $\mathbf{K}$ at the *generic* point of $\mathcal{M}^{alg}$.

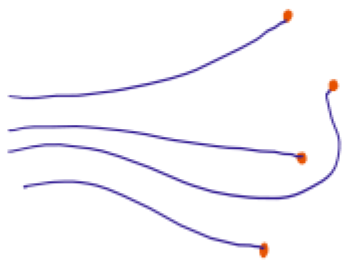
The goal of my lectures is to formulate several conjectures concerning the quantum spectrum and its behavior under blow-ups. In particular, the number of elements in the spectrum should be additive in an appropriate sense, giving a motivic measure.

An additional invariant ("dimension") will be introduced in the next lecture, giving a new criterion for non-rationality, which seems to be surprisingly close to the optimal one (see the talk by Ludmil Katzarkov later today).

There is a very optimistic conjecture, for which I do not have a really solid evidence (and which is completely out of reach now).

To simplify life, let us assume that the quantum connection is given by a *convergent series*.

Conjecture: *for any point in* $\mathcal{M}^{alg}$ *and a choice of disjoint paths from* $-\infty$ *to points of the corresponding spectrum (Gabrielov paths):*



we obtain a semi-orthogonal decomposition $D^b(Coh(X)) = \langle \mathcal{C}_1, \ldots, \mathcal{C}_r \rangle$ *where* r *is the number of elements of the spectrum.*

For X being a DM stack with a gerbe, modify $D^b(Coh(X))$ appropriately.

If X is compact, all categories $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are *saturated* (i.e. smooth and proper), equal to *local Fukaya-Seidel categories* for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists. In general, I expect that all $\mathcal{C}_i$ are of finite type (in particular, they are homologically smooth).

Notice that one can choose *not a generic* point in $\mathcal{M}^{alg}$, then the number r of elements of the spectrum will be strictly smaller relative with the generic case. The semi-orthogonal decomposition associated with the non-generic point, is obtained form the generic one by combining several subsequent subcategories into one larger saturated subcategory.

In this conjecture all subcategories $\mathcal{C}_i$ are not phantoms, its Hochschild homology (which are $\mathbb{Z}$-graded vector spaces over the $\mathbf{k} \subset \mathbb{C}$) are *non-zero*.

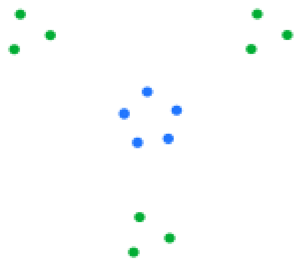
One can omit the assumption of convergence, working over the field of Puiseaux series in an auxiliary variable which can be thought of as a small positive number.

I *will not* assume the over-optimistic conjecture on semi-orthogonal decompositions, but still try to extract more accessible corollaries.

Let $Y \subset X$ be a smooth closed subvariety of codimension $m \geq 2$, and denote by $\pi : \widetilde{X} \to X$ the blow-up of X with center Y. We have the following basic facts:

- if X is "convex-at-infinity", then the same is true for Y and $\widetilde{X}$,
- if $[\omega] \in H^2(X, \mathbb{Z})$ is an ample class, then $[\omega]_{|Y}$ is ample, $\pi^*([\omega]) \geq 0$ and for sufficiently small $\epsilon > 0$ class $\pi^*[\omega] + \epsilon c_1(T_{\widetilde{X}}) \in H^2(\widetilde{X}, \mathbb{Q})$ is ample,
- $H^\bullet(\widetilde{X}) \simeq H^\bullet(X) \oplus \bigoplus_{(m-1) \text{ copies}} H^\bullet(Y)$,
- $\dim \mathcal{M}^{alg}(\widetilde{X}) = \dim \mathcal{M}^{alg}(X) + (m-1) \dim \mathcal{M}^{alg}(Y)$,
- $D^b(Coh(\widetilde{X})) = \langle D^b(Coh(X)), \underbrace{D^b(Coh(Y)), \ldots, D^b(Coh(Y))}_{(m-1) \text{ times}} \rangle$.

Informal version of the Blow-up conjecture:
the spectrum $\mathrm{Spec}_{\widetilde{X}}$ is close to



with $(m-1)$ shifted copies of Spec_Y around one copy of Spec_X.

One year ago in Miami I talked already about Blow-up conjecture via certain "gluing", see notes of my lecture 2 on the webpage of the collaboration

https://schms.math.berkeley.edu/events/miami2020/#schedule

I'll sketch below a reformulation of the gluing in a slightly different way.

Let us endow $X, Y, \widetilde{X}$ with semi-ample classes

$$[\omega], \quad [\omega]_{|Y} = (Y \to X)^*[\omega], \quad \pi^*([\omega]) = (\widetilde{X} \to X)^*[\omega]$$

respectively. The first two classes are in fact ample, and the third one still gives a well-defined series for the quantum product.

Operator $\mathbf{K}_{\widetilde{X},0}$, which is $\mathbf{K}_{\widetilde{X}}$ at point $0 \in \mathcal{M}^{alg}_{\widetilde{X}}$, has spectrum

$$\mathsf{Spec}_{\widetilde{X},0} = \{0\} \sqcup \{z \in \mathbb{C} | z = (m-1)\sqrt[m-1]{1}\}$$

Meromorphic connection $\frac{ud}{du} + \frac{1}{u}\mathbf{K}_{\widetilde{X},0} + \mathbf{G}_{\widetilde{X}}$ over $\mathbb{C}[[u]]$ can be explicitly identified with the sum of connections corresponding to elements of $\mathsf{Spec}_{\widetilde{X},0}$.

The summand corresponding to $z = 0$ can be explicitly identified with $\frac{ud}{du} + \frac{1}{u}\mathbf{K}_{X,0} + \mathbf{G}_X$, and with $\frac{ud}{du} + \frac{1}{u}\mathbf{K}_{Y,0} + \frac{z}{u} + \mathbf{G}_Y$ for $z = (m-1)\sqrt[m-1]{1}$.

Meromorphic connection of the form $\frac{ud}{du}+\frac{1}{u}\mathbf{K}+\mathbf{G}$ where $\mathbf{K},\mathbf{G}$ are operators in a finite-dimensional (super) vector space, can be understood in certain sense as a connection with second order pole over $\mathbb{C}[[u]]$ and connection with first order pole on $\mathbb{C}[u^{-1}]$ glued along an identification on $\mathbb{C}((u))$ in such a way that the resulting super vector bundle over $\mathbb{C}P^1$ is *trivial*.

Now, let us deform by an isomonodromic deformations (parametrized by $\mathcal{M}_X^{alg}$ and by copies of $\mathcal{M}_Y^{alg}$) connections over $\mathbb{C}[[u]]$ given by $\frac{ud}{du}+\frac{1}{u}\mathbf{K}_{X,0}+\mathbf{G}_X$ and $(m-1)$ copies of $\frac{ud}{du}+\frac{1}{u}\mathbf{K}_{Y,0}+\frac{z}{u}+\mathbf{G}_Y$. Gluing to the same connection $\frac{ud}{du}+\frac{1}{u}\mathbf{K}_{\widetilde{X},0}+\mathbf{G}_{\widetilde{X}}$ over $\mathbb{C}[u^{-1}]$ we obtain again a trivial bundle over $\mathbb{C}P^1$.

One can read flat coordinates in a canonical way, and obtain a non-linear map

$$\mathcal{M}_{\widetilde{X}}^{alg}\to\mathcal{M}_X^{alg}\times(\mathcal{M}_Y^{alg})^{m-1}$$

Conjecture: the pullback of the flat connection on $\mathbb{P}^1_u \times \mathcal{M}^{alg}_X \times (\mathcal{M}^{alg}_Y)^{m-1}$ to $\mathbb{P}^1_u \times \mathcal{M}^{alg}_{\widetilde{X}}$ coincides with those given by GW-invariants of $\widetilde{X}$.

This is a bit non-explicit description of the quantum product of $\widetilde{X}$ in terms of those for X and Y, and some data from the classical topology (restriction morphisms, cup-products on cohomology, and characteristic classes of normal/tangent bundles).

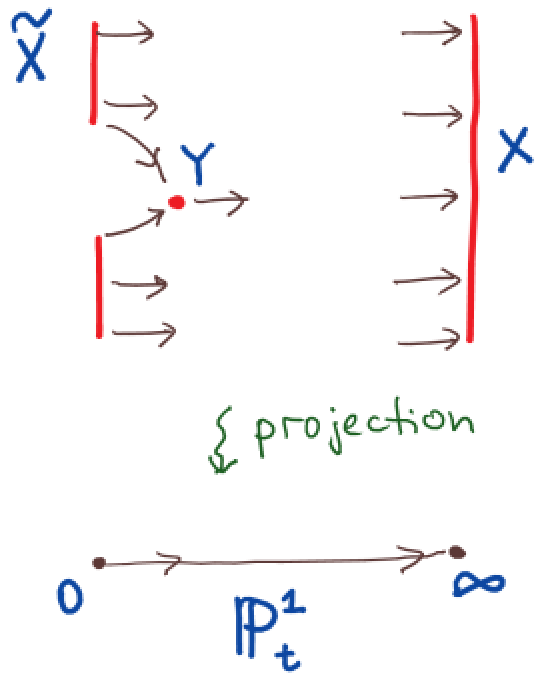
The Blow-up conjecture is still not proven (and not refuted).

I will finish this talk with the description of the strategy, which (I hope) can work. The main statement which I will try to prove is that the genus zero GW-invariants of $\widetilde{X}$ are *canonically* determined in terms of those for X and Y and the classical data. Then the Blow-up conjecture will be reduced to certain formal identity.

Main idea: introduce a new manifold

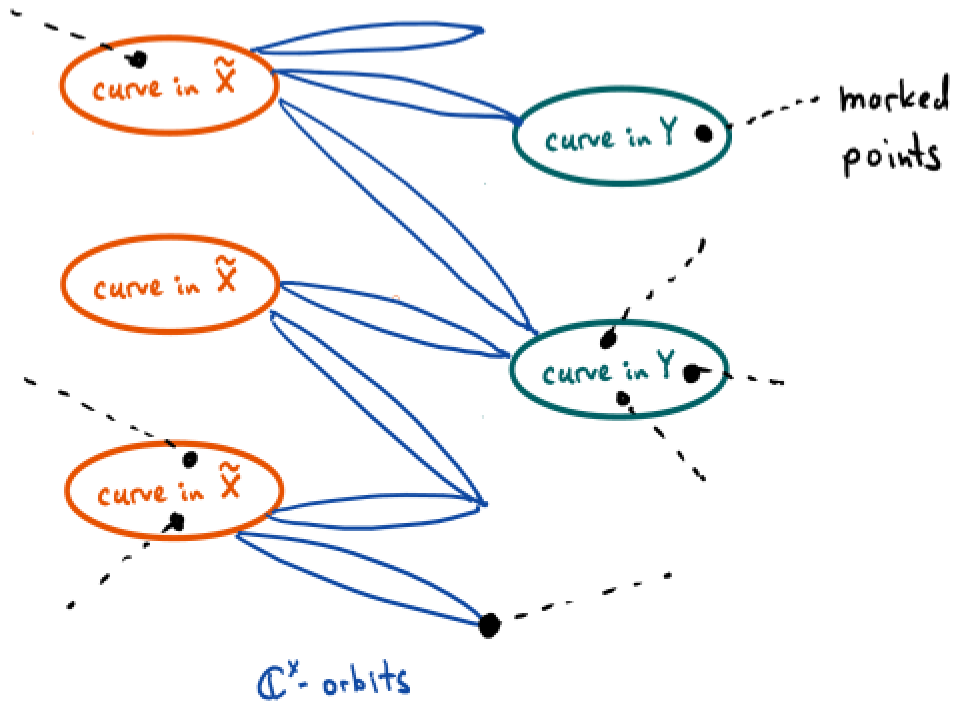
$$\widehat{X} := Bl_{Y\times\{0\}}(X\times\mathbb{P}^1)$$

It carries $\mathbb{C}^\times$-action by rescaling the canonical coordinate t on $\mathbb{P}^1$. The locus of fixed points consists of 3 components $\widetilde{X}\times\{0\}$, $Y\times\{0\}$ and $X\times\{\infty\}$.



Now consider moduli spaces of genus zero curves on $\widehat{X}$ of all possible degrees $\widehat{\beta} \in H_2(\widehat{X}, 0)$ such that the image in $H_2(\mathbb{P}^1, \mathbb{Z})$ vanishes ("vertical curves").

The locus of fixed points in $\overline{\mathcal{M}}_{0,n}(\widehat{X}, \widehat{\beta})$ consists either of curves in $X \times \{\infty\}$, or of trees of curves in $\widetilde{X} \times \{0\}$ and $Y \times \{0\}$ joined by cyclic covers of orbits of $\mathbb{C}^\times$-action connecting points of $\widetilde{X} \times \{0\}$ and $Y \times \{0\}$.



The sum of contributions of the fixed loci (by Bott formula) should have *vanishing* coefficients for *strictly negative* powers of the equivariant parameter. This gives an infinite bunch of identities, and there are good signs that these identities determine genus zero GW invariants of $\widetilde{X}$ **uniquely**.