P=W and HMS, II

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joint work with
- L. Katzarkov + V. Przyjalkowski (partially unpublished).
- L. Katzarkov + T. Pantev (in progress).

I Statement of Conjecture

Setup:
- X smooth, projective variety, Y snc anticanonical divisor.
- $X \setminus Y = U$ is a noncompact Calabi-Yau variety.

$$U \leftrightsquigarrow \check{U} \quad \text{mirror of } U$$

Our goal is to describe how filtrations on $H^*(U)$ and $H^*(\check{U})$ are related to one another.

- Hodge filtration
- degree filtration
- weight filtration
- Perverse Leray filtration. (?)

We will also assume for simplicity that $\check{U}$ is quasiprojective.

Basic expectation

$$HH_j(U) = \bigoplus_{q-p=j} H^q(X, \Omega_X^p(\log Y)) = H^{d-j}(\check{U})$$

This is analogous to the standard expectation for compact CY varieties.

Deligne's weight filtration:

- $Y_i = \{\text{points in } Y \text{ of multiplicity} \geq i\}$,
- $Y_i^\circ = Y_i \setminus Y_{i+1}$
- $\overline{Y}_i$ = normalization of Y_i
- $\partial \overline{Y}_i = \left\{ \begin{array}{l} \text{points in } \overline{Y}_i \\ \text{corresponding} \\ \text{to } Y_{i+1} \end{array} \right\}$

$\mathrm{Res}_j \quad \Omega^\bullet_X(\log Y) \longrightarrow \Omega^{\bullet-i}_{\overline{Y}_i}(\log \partial \overline{Y}_i)$

Def: $\quad W_i\Omega^\bullet_X(\log Y) = \ker(\mathrm{Res}_i)$

$$W_i\Omega^p_X(\log Y) = \ker(\mathrm{Res}_i : \Omega^p_X(\log Y) \longrightarrow \Omega^{p-i}_{\overline{Y}_i}(\log \partial\overline{Y}_i))$$

$$\widetilde{W}_i H^q(X, \Omega^p_X(\log Y)) = \mathrm{im}(H^q(X, W_i\Omega^p_X(\log Y)) \longrightarrow H^q(X, \Omega^p_X(\log Y)))$$

$$\widetilde{W}_i HH_j(U) = \bigoplus_{q-p=j} \widetilde{W}_i H^q(X, \Omega^p_X(\log Y))$$

Perverse Leray (flag) filtration.

For any variety, $\exists$ $f: V \longrightarrow W$ (morphism of varieties).

Perverse Leray filtration: $\quad {}^p\tau_{\leq i} Rf_* \mathbb{Q} \in D^b_{\mathrm{constr}}(W, \mathbb{Q})$.

$\rightsquigarrow \quad P_i H^*(Rf_*\mathbb{Q}) \rightsquigarrow P_i H^*(V, \mathbb{Q})$.

de Cataldo-Migliorini: if W is affine, Perverse Leray = Flag

$$\emptyset = \Lambda_{n+1} \subseteq \Lambda_n \subseteq \Lambda_{n-1} \subseteq \dots \subseteq \Lambda_0 = \mathbb{A}^n \supseteq W$$

$$\widetilde{P}_i H^*(V) = \ker(H^*(V) \longrightarrow H^*(f^{-1}(\Lambda_i)))$$

Let V be any variety. There's a map: $f: V \longrightarrow \mathrm{Spec}\,\mathbb{C}[V]$

From now on, $\widetilde{P}_i$ will refer to the perverse Leray filtration of the affinization map

Mirror P = W conjecture If U, $\check{U}$ are mirror dual log Calabi-Yau varieties then

$$\widetilde{P}_i H^{d+j}(\check{U}) \cong \widetilde{W}_i HH_j(U) \quad \text{for all } i, j$$

Remark 1. This conjecture was originally stated on a numerical level. The formulation above literally identifies filtrations.

2. The filtrations on both sides are canonical (given algebraic structures on U, $\check{U}$).

3. If $\check{U}$ is affine, $Gr_i^{\tilde{P}} H^j(\check{U}) \neq 0$ iff $i=j$

$\Longrightarrow$ Delignes MHS on $H^*(U)$ is Hodge-Tate

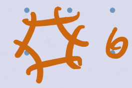
4. The mirror $P=W$ conjecture generalizes and clarifies conjectures of KKP.

II Examples.

Example 1: Del Pezzo surfaces.

$$(X_n \text{ del Pezzo of degree } n,\ Y = \text{smooth E.C.})$$

$$\Updownarrow$$



$$\left(\begin{array}{c} Z_n = \text{rational E.S. s.t. } g: Z_n \longrightarrow \mathbb{P}^1 \text{ has } I_n \text{ type fiber} / \infty \\ D_n = g^{-1}(\infty) \end{array} \right)$$

Affinization: $f: X_n \setminus Y \longrightarrow X_n \setminus Y$ (identity).

$\check{f} = g|_{Z_n \setminus D_n} : Z_n \setminus D_n \longrightarrow \mathbb{A}^1$ (proper).

Flags: $\left.\begin{array}{l} f^{-1}(\Lambda_2) = n \text{ points} \\ f^{-1}(\Lambda_1) = n\text{-punctured E.C} \end{array}\right\}$ flag on $X_n \setminus Y$

$\left.\check{f}^{-1}(\Lambda_1) = \text{smooth fiber of } \check{f}\ \right\}$ flag on $Z_n \setminus D_n$

Therefore the flag filtrations on $H^*(U_n)$, $H^*(\check{U}_n)$ have lengths 2, 1 respectively.

Since $Y \subseteq X_n$ is smooth, $W_\bullet HH_\bullet(X_n \setminus Y)$ has length 1
Since $D_n \subseteq Z_n$ has singular points, $W_\bullet HH_\bullet(Z_n \setminus D_n)$ has length 2

Theorem (H.-Katzarkov-Przyjalkowski) The mirror $P=W$ conjecture holds numerically for del Pezzo / smooth anticanonical pairs and their mirrors.

(This is a relatively easy computation
it can be extended to pairs with singular anticanonical divisors)

Higher dimensions?

Example 2: Toric threefolds.

Let Δ be a reflexive polytope in dimension 3
$\rightsquigarrow$ X_Δ: crepant resolution of the associated toric variety
S_Δ: smooth anticanonical K3 in X_Δ
(X_Δ, S_Δ) log Calabi-Yau pair.

Mirror? Δ° the polar of Δ, X_{Δ°, crepant resolution
Let $D_{\Delta^\circ} \subseteq X_{\Delta^\circ}$, toric boundary, S_{Δ° smooth anticanonical.

$$U_{\Delta^\circ} = \left\{ \begin{array}{l} \text{Sequential blow up of} \\ \text{components of } S_{\Delta^\circ} \cap D_{\Delta^\circ} \end{array} \right\} \setminus \left\{ \begin{array}{c} \text{Proper transform} \\ \text{of } D_{\Delta^\circ} \end{array} \right\}.$$

Remark (X_Δ, D_Δ), $(X_{\Delta^\circ}, D_{\Delta^\circ})$ are mirror log CY pairs.

Smoothing D_Δ $\leftrightsquigarrow$ Blowing up in $D_{\Delta^\circ} \cap S_{\Delta^\circ}$

This construction can be modified:

partial smoothing of D_Δ $\leftrightsquigarrow$ Blowing up in certain components of $D_{\Delta^\circ} \cap S_{\Delta^\circ}$

Example $\Delta = \mathrm{Conv}\{(-1,-1,1), (-1,1,-1), (1,-1,-1), (-1,-1,-1)\}$

X_Δ = Resolution of $V(x^4 - yzwt)$

$\Delta^\circ = \mathrm{Conv}\{(1,0,0), (0,1,0), (0,0,1), (-1,-1,-1)\}$

U_{Δ° = Blow up of $\mathbb{P}^3$ in quartic curves in $V(x_i)$ $\setminus$ Proper transform of $V(x_0x_1x_2x_3)$

Theorem (H.-Katzarkov-Przyjalkowski) (Non-Fano examples!).

For any reflexive polytope in dim = 3, (X_Δ, S_Δ) and U_{Δ° satisfy the mirror P=W conjecture numerically.

Remark: This can be extended to the partial smoothing / partial resolution case as well.

Variation: Victor has constructed prospective mirror duals to Fano 3-fold + smooth anticanonical pairs. (Construction is similar)

Theorem (H. Katzarkov-Przyjalkowski)

For any Fano 3-fold smooth anticanonical pair + Przyjalkowsk mirror, the mirror P=W conjecture holds numerically.

Remark The previous two results are easy consequences of proofs of cohomological conjectures of KKP

Fano $\leftrightsquigarrow$ LG pairs.

There's a close relation between the mirror P=W and KKP conjectures.

New directions? Everything I've said was known in 2018

Results are obtained via direct computation.

We would like to understand how this interacts with mirror symmetry and build tools to prove it more generally

III Interaction with torus fibrations

Assume that U, $\check{U}$ admit dual torus fibrations.

$$\begin{tikzcd} U \arrow[dr, "\pi"'] & & \check{U} \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

Goal "Geometrize" the Mirror P=W conjecture Using these fibrations.

Assumptions (Apologetically, the rest is notation-heavy).

- B_0 = smooth locus of π, $j : B_0 \hookrightarrow B$.
- $\pi_0 = \pi|_{\pi^{-1}(B_0)}$, $\check{\pi}_0 = \check{\pi}|_{\check{\pi}^{-1}(B_0)}$.
- π and π_0 are $\mathbb{Q}$-simple, $j_* R^i \pi_{0*}\mathbb{Q} \cong R^i\pi_*\mathbb{Q}$
  $j_* R^i \check{\pi}_{0*}\mathbb{Q} \cong R^i\check{\pi}_*\mathbb{Q}$

This condition basically means that we will ignore singular fibres from here on.

Duality: $R^i\pi_{0*}\mathbb{Z} \cong R^{d-i}\check{\pi}_{0*}\mathbb{Z}$ $\forall i$, $d = \dim U$

$$R^{d-1}\pi_{0*}\mathbb{R} / R^{d-1}\pi_{0*}\mathbb{Z} = \pi^{-1}(B_0).$$

$$R^{d-1}\check{\pi}_{0*}\mathbb{R} / R^{d-1}\check{\pi}_{0*}\mathbb{Z} = \check{\pi}^{-1}(B_0).$$

Hodge numbers

$$H^p(X, \Omega^q_X(\log Y)) \overset{?}{=} H^p(B, R^q\pi_*\mathbb{C})$$

Goal: Build filtrations on $H^p(B, R^q\pi_*\mathbb{C})$, imitating the weight and Perverse Leray filtrations.

Assumption • $\pi : X \setminus Y \longrightarrow B$ extends to $\bar{\pi} : X \longrightarrow \bar{B}$

Where $\bar{B}$ is a manifold with corners.

$$\overline{\partial B}_n \subseteq \overline{\partial B}_{n-1} \subseteq \ldots \subseteq \bar{B} \quad , \quad \partial B_i = \overline{\partial B}_i \setminus \overline{\partial B}_{i+1}$$

$$u_i : \partial B_i \longrightarrow \bar{B}$$

• $\pi|_{Y_i^\circ} : Y_i^\circ \longrightarrow \partial B_i$ is a $\mathbb{Z}$-simple torus fibration. $\forall i$.

Weight filtration A is any coefficient ring

$$\mathrm{Res}_i : (U \hookrightarrow X)_* A_U \longrightarrow (Y_i^\circ \hookrightarrow X)_* A_{Y_i^\circ}[i]$$

$$\rightsquigarrow \quad u_* R\pi_* A_U \longrightarrow u_{i*} R\pi_{i*} A_{Y_i^\circ}[i].$$

Proposition the induced map respects the Leray (canonical) filtration

$\Rightarrow$ There are maps $\mathrm{Res}_{i,q} : u_* R^q \pi_* A_U \longrightarrow u_{i*} R^{q-i} \pi_{i*} A_{Y_i^\circ}$

Definition

$$W_i u_* R^q \pi_* \mathbb{Q}_U = \ker(\mathrm{Res}_{i,q})$$

$$\widetilde{W}_i H^p(u_* R^q \pi_* \mathbb{Q}_U) = \mathrm{im}\left(H^p(W_i u_* R^q \pi_* \mathbb{Q}_U) \longrightarrow H^p(u_* R^q \pi_* \mathbb{Q}_U)\right)$$

$\Rightarrow$ There's an induced weight filtration on the affine Hodge groups

Is there a corresponding geometric filtration on $H^p(R^q \check{\pi}_* \mathbb{Q}_{\check{U}})$?

$$\begin{array}{cccc} \mathrm{Res}_{i,d-1} : & u_* R^{d-1} \pi_* A_U & \longrightarrow & u_{i*} R^{d-i-1} \pi_{i*} A_{Y_i^\circ} \\ & \wr\| & & \wr\| \\ & u_* R^1 \check{\pi}_* A_{\check{U}} & \longrightarrow & u_{i*} R^1 \check{\pi}_{i*} A_{\check{Y}_i^\circ} \end{array}$$

$\rightsquigarrow$ dual torus fibration

$$\mathrm{Res}^*_{i,d-1} : u_{i*} H_1 \check{\pi}_{i*} A_{\check{Y}_i^\circ} \longrightarrow u_* H_1 \check{\pi}_* A_{\check{U}}$$

Sheaf of homology groups

$\Rightarrow$ There's an embedding of torus bundles on $\partial\overline{B}_i \subseteq \overline{B}$ so that the induced maps in cohomology sheaves agrees with the residue map

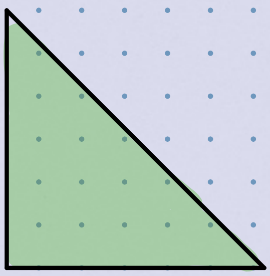
Let $\check{Y}_i$ be the union of torus bundles on $\partial\overline{B}_i$ i.e.

$$W_i H^p(u_* R^q \pi_* \mathbb{Q})$$
$$= \ker\left(H^p(u_* R^{d-q} \check{\pi}_* \mathbb{Q}_{\check{U}}) \longrightarrow H^p(u_{i*} R^{d-q} \check{\pi}_i \mathbb{Q}_{\check{Y}_i}) \right)$$

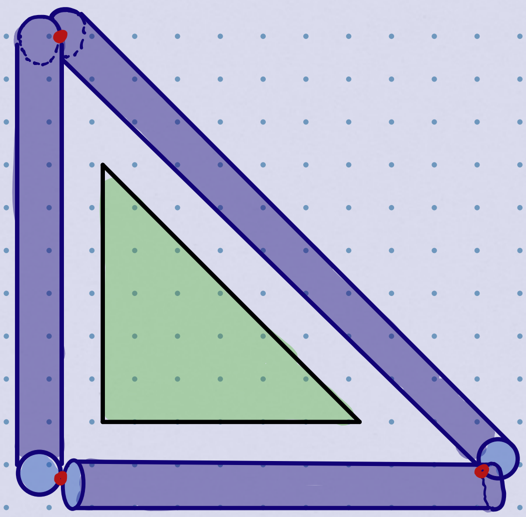
Conjecture $\check{Y}_i$ is homotopic to Λ_i.

Example $(\mathbb{CP}^2 \setminus V(xyz))$.

$\pi : \mathbb{CP}^2 \longrightarrow \Delta$



duality process
$\longrightarrow$
outlined above.



$\big\downarrow$ Mirror dual.

Union of tubes $= \check{Y}_1$
Union of points $= \check{Y}_0$

$\mathbb{C}^{*2} \xrightarrow{(x,y,\frac{1}{xy})} \mathbb{C}^{*3} \supseteq \mathbb{P}^3$ Flag: $\left\{ \begin{matrix} \text{thrice punctured} \\ \text{E.C} \end{matrix} \right\} \supseteq \left\{ 3 \text{ points} \right\}$

$\text{thrice punctured E.C} \cong$ Union of tubes

$3 \text{ points} \cong$ Union of points.

Outlook

- Proving the mirror $P=W$ conjecture for toric complete intersection

  Combining the SYZ inspired approach above, along with tropical Hodge theory should allow us to prove the mirror $P=W$ conjecture numerically for many toric complete intersections.

- Understanding the relationship between mirror $P=W$ and HMS when $Y \subseteq X$ is not smooth

  $\rightsquigarrow$ Multipotentials?

  $\rightsquigarrow$ Topological decomposition of anticanonical hypersurfaces?

- A-side VMHS for log Calabi-Yau varieties?