CohFTs for GLSMs

joint work w/ Bumsig Kim
& Ciocan-Fontanine — Guéré — Kim — Shoemaker

I GLSMs
II CohFTs
III History
IV Sketch of the Construction

I GLSMs

Data:
- G red. gp
- V G representation
- $\theta : G \longrightarrow \mathbb{C}^\times$
- $\omega \in \mathrm{Sym}\, V^\vee$

$\rightsquigarrow$ $V /\!\!/_\theta G \xrightarrow{\omega} \mathbb{C}$

Assume (Def'n)

① ∂w is proper

② $V^{ss}(\theta) = V^{s}(\theta)$

$\left(\Rightarrow V /\!\!/_{\theta} G \text{ is a smooth DM stack} \right)$

Ex $\mathbb{C}^{\times} \curvearrowright \mathbb{C}^{n+2}$ w/ weights $(\underbrace{1 \ldots 1}_{x_0 \ldots x_n}\ \underbrace{-d}_{p})$

a) $\theta_{+} : \mathbb{C}^{\times} \xrightarrow{\mathrm{Id}} \mathbb{C}^{\times}$

b) $\theta_{-} : \mathbb{C}^{\times} \xrightarrow{i} \mathbb{C}^{\times}$

$\quad t \longmapsto t^{-1}$

- $f(x)$ is a homo gen poly of deg d
- $w := p f(x)$

a) $\mathbb{C}^{n+2} /\!\!/_{\theta_{+}} \mathbb{C}^{\times} = \mathrm{Tot}\ \mathcal{O}(-d)_{\mathbb{P}^n}$, $\quad w$ restricts to $\langle -, f \rangle$

enumerative theory for this GLSM $\rightsquigarrow$ GW theory of $Z(f) \subseteq \mathbb{P}^n$.

b) $\mathbb{C}^{n+2} /\!\!/ \quad = \quad \mathbb{C}^{n+1} \qquad w$ restricts

b) $\mathbb{C}^n /\!/_{\theta^-} \mathbb{C}^\times = \mathbb{C}^n / \mathbb{Z}_d$, ... to f:

$\rightsquigarrow$ FJRW theory of f.

e.g. $n=4$, $d=5$

a) GW theory of quintic
b) FJRW quintic

$\Big\}$ analytic continuation Chiodo-Iritani-Ruan

e.g. $n=5$, $d=4$

a) GW theory of the cubic
b) "FJRW theory of cubic"

In gen you can get GW theory of complete intersections in toric varieties, Grassmannians, etc...

II CohFTs

Def'n (Kontsevich-Manin)

- $\mathcal{H}$ graded vector space
- $\langle -, - \rangle$ super comm. pairing
- $\mathbb{1} \in \mathcal{H}$ (unit)

$\forall g, r \in \mathbb{N}$

- $\Omega_{g,r} : \mathcal{H}^{\otimes r}[[q]] \longrightarrow H^*(\overline{M}_{g,r})[[q]]$
- satisfying natural axioms

  (– permutation invariance
  – tree
  – loop
  – forgetting tails
  – metric)

Ex ① Z smooth variety

$\mathcal{H} = H^*(Z)$

moduli space of maps to Z

$$\underbrace{\begin{tikzcd} & H^*(M(Z)) \arrow[r, "\cap M_{vir}"] & H^*(M(Z)) \arrow[d, "for_*"] \\ H^*(Z)^{\otimes r} = H^*(Z^r) \arrow[ur, "ev^*"] \arrow[rr, dashed, "\Omega_{g,r}"] & & H^*(\overline{M}_{g,r}) \end{tikzcd}}_{\text{GW theory}}$$

② G ...

G is a finite abelian gp

$$\mathcal{H} = \bigoplus_{g\in G} \mathrm{Jac}\left(W|_{V^g}\right)^G$$

$$\underbrace{\mathcal{H}^{\otimes r} \xrightarrow{\ \Omega_{g,r}\ } H^*(M_{g,r})}_{\text{FJRW theory}}$$

③ General Case (for GLSMs)

$$I[V/\!/_\theta G] := \coprod_{g\in G/G} \left[V^g/\!/_\theta C(g)\right]$$

$$\mathcal{H} = \underbrace{H^*\left(IV/\!/_\theta G,\ \left(\Omega^\bullet_{IV/\!/_\theta G},\ dw\right)\right)}_{H^*(IV/\!/_\theta G)_w}$$

Idea:

want to construct $\longrightarrow$ $M_{vir} \in H^*_{M(\partial w)}\left(M(IV/\!/_\theta G)\right)_{-w}$

$$\xrightarrow{\ ev^*\ } H^*\left(M(IV/\!/_\theta G)\right)_w \longrightarrow H^*_{M(\partial w)}\left(M(IV/\!/_\theta G)\right)$$

$$H^*(IV/\!/_\theta G)^{\otimes r}_w \qquad\qquad \downarrow \text{for}_* \qquad H^*(\overline{M}_{g,r})$$

Rmk If $V/\!/_\theta G = \mathrm{tot}_X \mathcal{E}$

$w = \langle -, s\rangle \qquad s \in H^0(X, \mathcal{E}^\vee)$

assume s is regular

then

$$(\Omega^\bullet_{\mathrm{tot}_X \mathcal{E}},\ \wedge d\langle -, s\rangle) \cong (\Omega^\bullet_{Z(s)}, 0)[\mathrm{rk}\,\mathcal{E}+1]$$

$$\Rightarrow H^*(\mathrm{tot}_X \mathcal{E})_{\langle -, s\rangle} \cong H^{*+\mathrm{rk}\mathcal{E}+1}(Z(s))$$

(called a geometric phase of the GLSM)

III History of enumerative theories of GLSMs (V, G, θ, w)

① G is a finite abelian gp. [FJRW-theory]

- Fan-Jarvis-Ruan (2013)
- Polishchuk-Vaintrob (2016) (algebraic vers)

- Kiem-Li (2018) (mostly top. vers., prove it forms a CohFT)

② General GLSMs

- Fan-Jarvis-Ruan (2018) (narrow sector only i.e missing primitive cohomology insertions)

- Ciocan-Fontaine-F-Guéré-Kim-Shoemaker (2018) (Convex hybrid model case but includes primitive cohomology [broad sections])

- F-Kim (2020) (fully general)

Thm (CF-F-G-K-S) (2018)

- When G is finite this recovers FJRW theory.

- when $V/\!/_\theta G = \begin{array}{c} \mathrm{tot}\, E \\ \downarrow \\ X \end{array}$ $\quad w = \langle s, - \rangle$, $s \in H^0(X, E^\vee)$

  this recovers GW theory as defined by the Chang-Li virtual cycle.

Thm (Kim-Oh) (2018) The Chang-Li

virtual cycle agrees with the Behrend-Fantechi virtual cycle.

<u>Thm</u> (F-Kim) (2020) $\exists$ a general construction of $\Omega_{g,r}$ $\forall$ GLSM.

Furthermore this collection of maps form a CohFT $\forall$ GLSM.

<u>IV Idea of the Construction</u>

$$
\begin{tikzcd}
M(V/\!/_\theta G) \arrow[rr] \arrow[dd, "ev"'] \arrow[ddrr, "for"] & & A \arrow[ddll, dashed, "\widetilde{ev}"] \arrow[dd, dashed, "\widetilde{for}"] \\
& & \\
V/\!/_\theta G & & \overline{M}_{g,r}
\end{tikzcd}
$$

- find a matrix factorization K on A (supported on $\underbrace{M(\partial w)}_{\text{proper}}$)

- virtual cycle $\simeq$ td(B) Atiyah(K) $\in H^*_{M(\partial w)}(A)_{\widetilde{ev}^*(w)}$

to be explained

How to build A. $\mathcal{P}(V) := P \times_G V$

from ω $\rightsquigarrow$ α

$$\begin{tikzcd} & \mathrm{tot}\, p^*B \arrow[d] \arrow[d, bend left, "\beta = d\,\mathrm{tot}_A"] & & \mathcal{P}(V) \arrow[d] \\ Z(B) \subset & \mathrm{tot}\, A \arrow[dr, "p"] & & \mathcal{C} \arrow[dl, "\pi"] \\ & & \mathrm{Bun}_G & \end{tikzcd}$$

$$\boxed{R\pi_* \mathcal{P}(V) = [A \xrightarrow{d} B]}$$

$$\mathrm{Bun}_G = \{(C, q, P, K)\}$$

- C curve
- $q = (q_1, \dots, q_r)$ - r-marked pts
- P is principal G-bundle (actually a Γ-bundle)
- K: $\omega_C^{\log} \cong \mathbb{C}_\chi$ <- - - - - $\chi : \Gamma \to \mathbb{C}^\times$ s.t. $G = \ker \chi$.

$$Z(B) := \{C, q, P, K, s \in H^0(\mathcal{P}(V))\}$$

(quasi-map theory) $= \mathrm{Maps}(\mathcal{C}, [V/G])$

now consider an open set in $\mathrm{tot}\, A$ defined by stability of curves & forcing maps to factor through $V/\!\!/_\beta G$..

That's our space $A \underset{\text{open}}{\subseteq} \text{tot}\,A$

on it lives a matrix factorization.

$$\mathcal{O}_{\text{tot}\,A} \xrightarrow{\beta} p^*B \xrightarrow{\alpha} \mathcal{O}_{\text{tot}\,A} \qquad (\beta = d \circ \text{taut})$$

$\mathbb{R}\Gamma_{\pi_*} P(\mathcal{V})$

$$= [A \xrightarrow{d} B]$$

$$K = \left[\Lambda^{\text{even}} p^*B^\vee \underset{\partial}{\overset{\partial}{\rightleftarrows}} \Lambda^{\text{odd}} p^*B^\vee \right]$$

$$\partial = \lrcorner\beta + \Lambda\alpha$$

<u>Problem</u> In general, w only gives a map in the derived category

$\rightsquigarrow \alpha : \text{Sym}\, A^\vee[1] \longrightarrow \text{Sym}[B^\vee \to A^\vee]$

exists in the derived

derived category

$$\begin{tikzcd}
 & & \mathrm{Sym}\, A^\vee \arrow[d, "\alpha"] & \\
\cdots \arrow[r] & \Lambda^2 B^\vee \otimes \mathrm{Sym}\, A^\vee \arrow[r] & B^\vee \otimes \mathrm{Sym}\, A^\vee \arrow[r] & \mathrm{Sym}\, A^\vee
\end{tikzcd}$$

Solution: Idea

X - DM stack

A - sheaf of CDGAs over $\mathcal{O}_X$ ← differential d

$\omega \in \Gamma(X, \mathcal{O}_X)$

$\alpha \in \Gamma(X, A_{-1})$ $\quad (d(\alpha) = \omega)$

Then you make a MF.

$$A^{even} \underset{\partial}{\overset{\partial}{\rightleftarrows}} A^{odd}$$

$$\partial = d + \wedge \alpha$$

graded Leibnitz rule $\Rightarrow \partial^2 = \omega$.

To get α is general consider the sheaf of CDGAs $\mathcal{K}(B)$

replace $\mathcal{K}(B)$ by a Γ-acyclic quasi-isomorphic sheaf of CDGAs.

(obtained by taking the (cosimplicial) Godement resolution & then apply the Thom-Sullivan construction)