Structural aspects of Fukaya categories of Landau-Ginzburg models

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Plan:
1. Fukaya categories of LG-models
2. Functors
3. Monodromy and decompositions

(based on ideas of Seidel, Abouzaid, Ganatra, Sylvan, Hanlon, Jeffs, ...).

essentially the same picture, in a different language: Nadler, Ganatra-Pardon-Shende, Gammage, ...
and also: Kontsevich, Soibelman, Kapranov, ...

⚠ many of the statements below are only proved in settings more specific than stated.

- **Symplectic Landau-Ginzburg model** := $W : Y \to \mathbb{C}$ with $\begin{cases} (Y,\omega) \text{ symplectic, convex at infinity} \\ F_t = W^{-1}(t) \text{ are symplectic submanifolds} \\ \quad \text{(outside of critical locus)} \end{cases}$

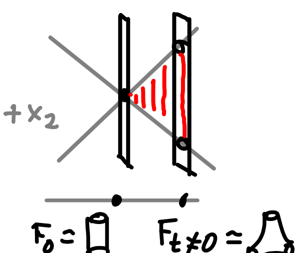
for example Y Kähler (complete) (quasiprojective), W holomorphic (regular function)

- Symplectic parallel transport along horizontal distribution $\mathcal{H} = (\ker dW)^{\perp\omega} = \operatorname{span}(X_{\operatorname{Re} W}, X_{\operatorname{Im} W})$ over a path $\gamma : [0,1] \to \mathbb{C}$ avoiding critval(W) gives symplectomorphisms $F_{\gamma(0)} \xrightarrow{\sim} F_{\gamma(1)}$
- transporting a Lagrangian $\ell \subset F_{\gamma(0)}$ gives a <u>fibered Lagrangian</u> $L \subset Y$ over γ.
- $\ell \subset F_{t_0}$ is a Lagrangian <u>vanishing cycle</u> for path $\gamma : t_0 \rightsquigarrow$ critval if parallel transport collapses ℓ entirely into crit(W); the fibered Lagrangian is then called a <u>thimble</u>.

<u>Ex</u>: <u>Lefschetz fibrations</u> : crit(W) isolated nondegenerate (local model: $\mathbb{C}^n \xrightarrow{\sum z_i^2} \mathbb{C}$)
sing. fibers have ordinary double points; vanishing cycle = Lagrangian $S^{n-1} \subset F$.

<u>Remark</u>: • we include "critical points at ∞" in critval(W), e.g. $(\mathbb{C}^*)^2$
$\downarrow W = x_1 + x_2$
$\mathbb{C}$

• assume critval(W) finite
but crit(W) not assumed isolated, or even proper.



The objects of the Fukaya category $\mathcal{F}(Y,W)$ are admissible Lagrangians

i.e. properly embedded (or immersed) $L \hookrightarrow Y$ disjoint from the stop $Y_{-\infty} = \{\mathrm{Re}\, W \ll 0\}$

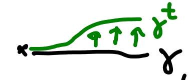
(or other top. equivt. subset near $W^{-1}(-\infty)$).

- with good control over holomorphic curves (maximum principle at ∂), e.g. one of:
    - conical for an exact Liouville structure, or
    - fibered & fiberwise proper outside a compact subset, or
    - monomially admissible, i.e. $Y \setminus \text{compact} = \bigcup_{\text{finite}} U_\nu$, $\arg(z^\nu|_{L \cap U_\nu}) = \text{constant}$)
- unobstructed (no holom. discs with ∂ in L, or cancel by bounding cochain)
- equipped with spin structure, local system, grading, ...

Morphisms: $\hom(L, L') = \varinjlim CF(L^t, L')$, L^t = push L by a positive Hamiltonian isotopy: $\left\{ \begin{array}{l} \to \text{increase } \arg(W) \text{ on ends of } L, \text{ \underline{without} crossing the stop.} \\ \to \text{if } L \text{ isn't fiberwise proper, fiberwise wrapping.} \end{array} \right.$

There are many competing definitions.

Technical aspects of the definition of $\mathcal{F}(Y, W)$



- easiest if L fiberwise proper (eg. thimbles in Lefschetz fibrations: L^t fibers over γ^t) (Seidel) and fibered

  (This suffices if the fibration is loc. trivial at infinity, then fiberwise proper objects generate and fiberwise wrapping isn't needed).

- When Y is exact Liouville (eg. affine variety), can use Sylvan's partially wrapped Fukaya category or view $Y \setminus Y_{-\infty}$ as a sector (Ganatra-Pardon-Shende)

  (This is the most versatile, but the non-exact case hasn't been developed yet)

  Note: we don't consider arbitrary sectors – the stops of LG models are swappable)

- When Y is toric, can use monomial admissibility (Hanlon, Abouzaid-A.)

★ In all cases, the picture becomes richest if we allow ourselves to split W into a sum of two terms – a "main" term W_0, and an "auxiliary" term W_F defining stops on the fibers of W_0. Ie., we decompose the stop of (Y, W) into two components, and explore the geometry of Y relative to the part of the stop which corresponds to W_0.

We change notation and consider F = fiber of W_0, rather than the whole fiber/stop.

3)

The fiberwise wrapped Fukaya category of $(Y, w_0 + w_F)$ (Y toric, w_0 monomial) (Abouzaid-A.)

<u>Objects</u>: properly embedded Lagrangians $L \subset Y$ (+ extra data: spin str, grading, ...) which are unobstructed + monomially admissible: (cf. A. Hanlon's thesis)

1) for $|w_0| \gg 1$, $\arg(w_{0|L})$ is loc. constant $\in (-\frac{\pi}{2}, \frac{\pi}{2})$ (ie. $w_{0|L} \in$ union of radial arcs)
2) inside the fibers F_t of w_0 (again toric!), impose admissibility w.r.t. a collection of toric monomials z^ν (incl. all terms in w_F), ie. $\arg(z^\nu) =$ loc. constant over subsets U_ν.

<u>Note</u>:
- monomial admissibility gives control over discs in Floer products via maximum principle
- use a specific toric Kähler form for which $\{\log w_0, \log z^\nu\}$ Poisson-commute in U_ν.

L admissible $\rightsquigarrow$ flow L^t (Ham. isotopic to L; admissible)

The flow increases the values of $\arg(w_0)$ and $\arg(z^\nu)_{|U_\nu}$ at ∞.

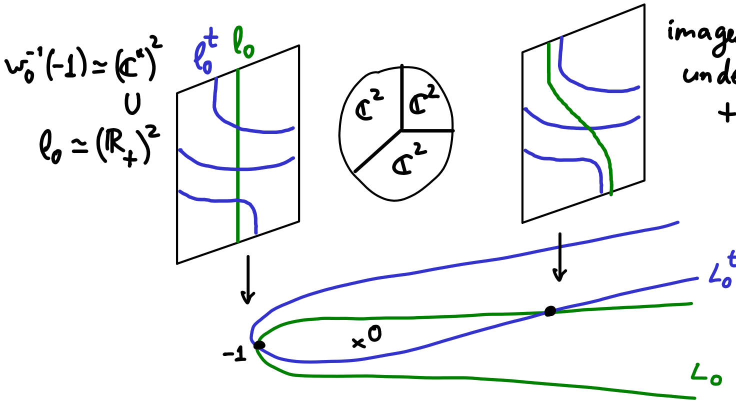
(within $\arg \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for w_0 and terms of w_F: <u>STOP</u> at $-\infty$; else $\arg \uparrow \infty$: <u>WRAP</u>)

<u>Define</u> $\hom(L_0, L_1) := \varinjlim_{t\to\infty} CF^*(L_0^t, L_1)$ under natural continuation maps.

<u>Conj</u>: can make a similar definition outside of toric setting & the resulting "two-stage category" (whose objects are fibered Lagrangians wrt w_0, rather than $w_0 + w_F$), agrees with other versions.

Example: $(\mathbb{C}^3, w_0 = -z_1 z_2 z_3)$ (mirror of [pair of pants] $= \{(x_1,x_2)\in(\mathbb{C}^*)^2 \mid 1+x_1+x_2=0\}$).

L_0 = parallel transport $\ell_0 = (\mathbb{R}_+)^2 \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1)$ along U-shaped arc.



images of ℓ_0 under monodromy + wrapping Ham. $\arg(z_i) = f_i(\log|z|)$

$\arg(z_i) = 0$ wherever $|z_i| \gg \min_j |z_j|$

wrapping t (here $t \to +\infty$)

$$\hom(L_0, L_0) \simeq \underset{\underset{\mathbb{K}[x_1^{\pm1}, x_2^{\pm1}]}{\wr\wr}}{CW^*(\ell_0,\ell_0)} \oplus CW^*(\ell_0,\ell_0)[-1]$$

∂ = multiplication by $1+x_1+x_2$ (Abouzaid-A.)

$$H^*\hom(L_0,L_0) \underset{\text{(ring iso.)}}{\simeq} \mathbb{K}[x_1^{\pm1}, x_2^{\pm1}]/(1+x_1+x_2) \simeq \hom(\mathcal{O},\mathcal{O}) \text{ in } D^b\mathrm{Coh}(\text{[pair of pants]}) \quad \checkmark$$

Example: $(\mathbb{C}^3, w_0 + w_F = -z_1z_2z_3 + q(z_1+z_2+z_3))$: $(F_t, w_F) \simeq ((\mathbb{C}^*)^2, z_1+z_2+\frac{t}{z_1z_2})$

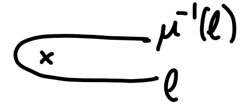
then $\mathcal{F}(F_t, w_F) \simeq D^b(\mathbb{P}^2)$ and $\mathcal{F}(\mathbb{C}^3, w_0+w_F) \simeq D^b(\{(x_0:x_1:x_2) \mid x_0+x_1+x_2=0\})$ (ie. $\mathbb{P}^1 \subset \mathbb{P}^2$).

Abouzaid-A.: this approach leads to HMS for hypersurfaces (& complete intersections) in $(\mathbb{C}^*)^n$ & toric var's.

5 The Fukaya categories of $(Y, w_0 + w_F)$ and $(F = w_0^{-1}(t), w_F)$ are related by functors

monodromy of w_0 around origin $\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0 + w_F) \circlearrowleft \sigma$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

$\cup \ell$ = parallel transport $\ell \subset F$ (admissible) along U-shape



$=: \cup \ell$

$\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$ (actually $\in \mathrm{Tw}\, \mathcal{F}(F, w_F)$ if $w_{0|L}$ has more than one end).

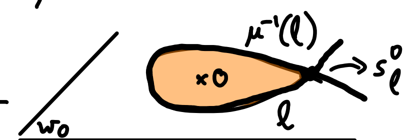
+ exact triangle of functors on $\mathcal{F}(F, w_F)$:

$$\boxed{\begin{tikzcd} \mu^{-1} \arrow[rr, "s"] & & id \arrow[dl] \\ & \cap\cup \arrow[ul] & \end{tikzcd}}$$

(Abouzaid-Ganatra, Sylvan).

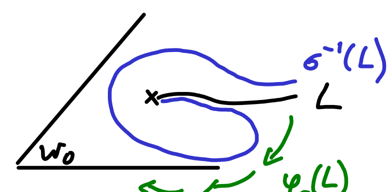
where s = section-counting natural transformation from μ^{-1} to id [Seidel]

$\forall \ell \subset F$, $s^0_\ell \in CF^0(\mu^{-1}(\ell), \ell)$ counts holom. sections of w_0 over



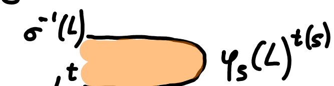
+ exact triangle of functors on $\mathcal{F}(Y, w_0 + w_F)$:

$$\boxed{\begin{tikzcd} id \arrow[rr, "\kappa"] & & \sigma^{-1} \arrow[dl] \\ & \cup\cap \arrow[ul] & \end{tikzcd}}$$



κ = continuation element for non-positive Hamiltonian isotopy $id \longrightarrow \sigma^{-1}$

i.e. κ^0_L counts holom. discs



monodromy of w_0 around origin $\quad \mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

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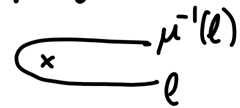


$=: \cup \ell$

$\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$ (actually $\in \mathrm{Tw}\, \mathcal{F}(F, w_F)$ if $w_0|_L$ has more than one end).

On $\mathcal{F}(F, w_F)$, $\cap\cup \simeq \mathrm{Cone}(\mu^{-1} \xrightarrow{s} id)$ $\qquad$ On $\mathcal{F}(Y, w_0+w_F)$, $\cup\cap \simeq \mathrm{Cone}(id \xrightarrow{\kappa} \sigma^{-1})$

HMS for log CY/Fano (X, D_0), $D_0 + D' = -K_X$

$\rightsquigarrow$ $(Y, W = w_0 + w_F)$ mirror to X

$(F = w_0^{-1}(t), w_F)$ mirror to D_0

$$\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$$

HMS for D_0 $\wr\!\wr$ $\qquad$ $\wr\!\wr$ HMS for X

$$\mathrm{Perf}(D_0) \underset{i^*}{\overset{i_*}{\rightleftarrows}} \mathrm{Perf}(X)$$

<u>Ex</u>: $\mathcal{F}\left((\mathbb{C}^*)^2, z_1 + z_2 + \frac{q}{z_1 z_2}\right) \rightleftarrows \mathcal{F}\Big($

$\Big)$

mirror to $\mathbb{P}^2$ $\qquad$ mirror to

$x_0 x_1 x_2 = 0$

HMS for hypersurface $H \subset V$ [AAK]

$\rightsquigarrow$ $(Y, W = w_0 + w_F)$ mirror to H

$(F = w_0^{-1}(t), w_F)$ mirror to V

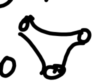
$$\mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$$

HMS for V $\wr\!\wr$ $\qquad$ $\wr\!\wr$ HMS for H

$$\mathrm{Perf}(V) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Perf}(H)$$

<u>Ex</u>. $\mathcal{F}(\mathbb{C}^3, -xyz) \rightleftarrows \mathcal{W}((\mathbb{C}^*)^2)$

mirror to



$1 + x_1 + x_2 = 0$ $\qquad$ mirror to $(\mathbb{C}^*)^2$

- In this language, the above calculation for mirror (Y, w_0+w_F) of hypersurface $H\subset V$ is:

$$\hom_Y(\cup\ell, \cup\ell') \simeq \hom_F(\ell, \cap\cup\ell') \simeq \mathrm{Cone}\big(\hom_F(\ell,\mu^{-1}(\ell')) \xrightarrow{S} \hom_F(\ell,\ell')\big)$$

& homological mirror symmetry is proved by matching this with $D^b\mathrm{Coh}(H) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^b\mathrm{Coh}(V)$

$$\hom_H(i^*\mathcal{L}, i^*\mathcal{L}') \simeq \hom_V(\mathcal{L}, i_*i^*\mathcal{L}') \simeq \mathrm{Cone}\big(\hom_V(\mathcal{L},\mathcal{L}'\otimes\mathcal{O}(-H)) \xrightarrow[f]{} \hom_V(\mathcal{L},\mathcal{L}')\big)$$

[Abouzaid-A.].

- The two functors $\cup$, $\cap$ are defined differently, but play symmetric roles in the spherical functor package. Conj: Fiber and total space can be swapped around by stabilization:

$$(Y,\ W=w_0+w_F) \quad F=w_0^{-1}(t)\ (t\gg0) \rightsquigarrow \big(\tilde Y = Y\times \underset{z}{\mathbb{C}},\ \tilde W = z(1-t^{-1}w_0)+w_F\big)$$

- "A-model Knörrer periodicity" (M. Jeffs) $\Rightarrow \mathcal{F}(Y\times\mathbb{C},\tilde W)\simeq\mathcal{F}(F,w_F)$
- the levels of $\tilde w_0 := z$ are $\simeq (Y, w_0+w_F)$ by considering thimbles for $z(1-t^{-1}w_0)$ (Morse-Bott along $F\times\{0\}$)

- Jeffs' result is for splitting $\tilde W = \underbrace{z(1-t^{-1}w_0)}_{\text{main}} + w_F$, while $\tilde F \simeq (Y, w_0+w_F)$ is for $\tilde W = \underbrace{z}_{\text{main}} + \underbrace{(-t^{-1}zw_0+w_F)}_{\text{fiberwise}}$.
- $\widetilde{(\cap,\cup)}$ not quite $\sim(\cup,\cap)$: left vs. right adjoint, $\circlearrowright$ vs. $\circlearrowleft$

Localizations / quotients

monodromy of w_0 around origin $\quad \mu \circlearrowright \mathcal{F}(F, w_F) \underset{\cap}{\overset{\cup}{\rightleftarrows}} \mathcal{F}(Y, w_0+w_F) \circlearrowleft \sigma$ wrap once past the stop $w_0 \to -\infty$. (spherical functor)

$\cup \ell$ = parallel transport $\ell \subset F$ (admissible) along U-shape, $\cap L$ = ends of $L \subset Y$ at $w_0 \to \infty$

On $\mathcal{F}(F, w_F)$, $\cap\cup \simeq \mathrm{Cone}(\mu^{-1} \xrightarrow{S} id)$ On $\mathcal{F}(Y, w_0+w_F)$, $\cup\cap \simeq \mathrm{Cone}(id \xrightarrow{K} \sigma^{-1})$

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① Localizing $\mathcal{F}(Y, w_0+w_F)$ wrt $id \xrightarrow{K} \sigma^{-1}$ $\iff$ quotient by $\mathrm{Im}(\cup)$

(Abouzaid-Seidel, Sylvan, GPS) $\iff$ stop removal at w_0

yields $\mathcal{F}(Y, w_F)$ (if no fiberwise stop, $\mathcal{W}(Y)$ – the fully wrapped Fukaya category)

In HMS for log CY / Fano, with fiber mirror to $D_0 \subset X$, $D_0 + D' = -K_X$, σ^{-1} corresponds to $-\otimes\mathcal{O}(D_0)$ and $\cup$ to i_*; localization gives mirror to $X - D_0$

② Localizing $\mathcal{F}(F, w_F)$ wrt $\mu^{-1} \xrightarrow{S} id$ $\iff$ quotient by $\mathrm{Im}(\cap)$ (vanishing cycles)

when $\mathrm{critval}(w_0) = \{0\}$, can be interpreted as the Fukaya cat. of the singular fiber F_0.

Eg. $\mathcal{W}(\{xy=0\} \subset \mathbb{C}^2) = \mathcal{W}(T^*S^1)/\langle S^1 \rangle \simeq \mathrm{Perf}($  $)$, $\mathcal{W}(\{xyz=0\}) \simeq \mathrm{Perf}((\mathbb{C}^*)^2 - \{1+x_1+x_2=0\})$. 2d pants

Thm (Jeffs): (Knörrer periodicity) $\boxed{\mathcal{W}(F_0) \simeq \mathcal{F}(Y \times \mathbb{C}, z w_0)}$ (expect $\mathcal{F}(F_0, w_F) \simeq \mathcal{F}(Y \times \mathbb{C}, z w_0 + w_F)$)

Families of LG models and monodromy

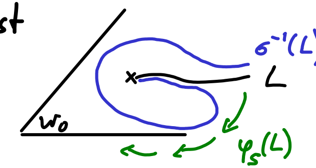
(Y, W_t) s.t. topology of the stop doesn't change with $t \Rightarrow \mathcal{F}(Y, W_t)$ indep^t of t.

For a loop $(W_t)_{t\in S^1}$, get a monodromy functor $\phi \in \mathrm{Auteq}\, \mathcal{F}(Y,W)$.

+ if loop is "positive", Floer continuation maps give a natural transformation $\mathrm{id} \to \phi$.

Main example: $(Y, W = e^{i\theta} w_0 + w_F) \Rightarrow$ the monodromy functor is σ^{-1}, with $\mathrm{id} \xrightarrow{K} \sigma^{-1}$.

(wrap once clockwise past the stop of w_0)



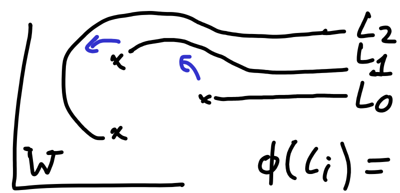
Ex (Hanlon) V toric var. $\longleftrightarrow$ $((\mathbb{C}^*)^n, W = \sum q_\nu z^\nu)$

toric divisors D_ν

rays of fan = $\nu \in \mathbb{Z}^n$

monodromy for $e^{i\theta} q_\nu$ is $\longleftrightarrow$ $-\otimes \mathcal{O}(D_\nu)$.

Ex: for $\mathbb{P}^n$, $W = z_1 + \dots + z_n + \dfrac{e^{i\theta} q}{z_1 \cdots z_n} \Rightarrow$



$\phi(L_i) = L_{i+1}$.

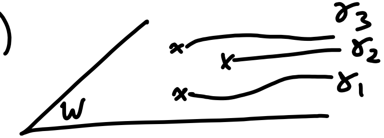
This suggests:

- every line bundle on a toric var. can be represented by a Lefschetz thimble. [Hanlon]
- certain mutations of exceptional collections (or semi$\perp$ decomp^s) are related by Auteq's.

Critical values and semi-orthogonal decompositions

- When W has multiple critical values, a choice of basis of disjoint paths γ_i critval$(W) \to +\infty$ determines a semiorthogonal decomposition (sod) of $\mathcal{F}(Y,W)$

$\mathcal{F}(Y,W) = \langle \mathcal{F}_{\gamma_1}, \dots, \mathcal{F}_{\gamma_r} \rangle$ $\quad (L_i \in \mathcal{F}_{\gamma_i}, L_j \in \mathcal{F}_{\gamma_j}, \ i>j \Rightarrow \hom(L_i, L_j) = 0)$



Roughly, $\mathcal{F}(Y,W)_{\gamma_i}$ = Lagrangians supported in a nbd. of $W^{-1}(\gamma_i)$.

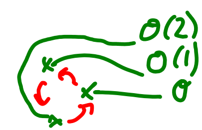
(For Lefschetz fibrations, we get an exceptional collection: $\mathcal{F}_{\gamma_i} = \langle$Lefschetz thimble$\rangle$).

- Morsification of an isolated singularity decomposes its Fukaya category ($\rightsquigarrow$ exc. collection) but this can't be done for general (non-isolated) singularities (if $\mathcal{F}$ doesn't have s.o.d.!) (eg. Morse–Bott singularities: $\mathcal{F}_{\gamma_i}$ is CY category, usually can't decompose).

- Change of paths $\Rightarrow$ braid group action (by mutation) on s.o.d.'s of $\mathcal{F}(Y,W)$

When the change of basis is induced by a loop of LG models (Y, W_t), the mutated sod is also the image of the original one under monodromy autoequiv^{ce}.

Ex: $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle = \langle \mathcal{O}(1), \dots, \mathcal{O}(n+1) \rangle$ vs. $\left((\mathbb{C}^*)^n, W = z_1 + \dots + z_n + \dfrac{e^{i\theta} q}{z_1 \cdots z_n}\right)$



mutation $\Longleftrightarrow$ autoeq. $-\otimes \mathcal{O}(1)$

A question for experts: Semiorthogonal decompositions from stability conditions?

Stability conditions on Fukaya categories of LG models are very poorly understood, outside of a few cases (open Riemann surfaces: Haiden-Katzarkov-Kontsevich) or via HMS.

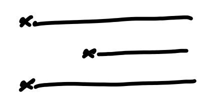
Analogue of Joyce's conjecture: (Y, W) LG model, Ω holom. volume form on Y
Y quasiproj. CY (meromorphic on $\overline{Y}$)

Expect: W, Ω determine a stability condition on $\mathcal{F}(Y,W)$ st. L admissible Lagrangian

$$(\star)\quad \arg\left(\exp(-W/\hbar)\,\Omega_{|L}\right) \text{ constant} \Rightarrow L \text{ is stable with } Z(L) = \int_L \exp\left(-\frac{W}{\hbar}\right)\Omega.$$

For $\hbar$ small, the only way for a thimble to satisfy $(\star)$ is to have $\mathrm{Im}(W) \simeq$ const. i.e. this forces stable objects to stay near horizontal base paths; for a Lefschetz fibration we get a stab. cond^{n} whose only stable objects are an exceptional collection of thimbles.

(...? some numerical evidence in Z. Chen's undergrad thesis)



Q: → does this actually work in more general settings?
→ is there a mirror counterpart - a way to get a stability condition on $D^b(X)$ that is governed by $(c_1(X) *_q -)$ and induce SOD's in this way?