Recent Developments in Hodge Theory

Virtual Conference
IMSA Miami, 29 March – 2 April 2021

<u>Program</u>

Monday 29 March 2021

9:00 am – 10:00 am	**June Huh**
10:30 am – 11:30 am	**Kang Zuo**
1:00 pm – 2:00 pm	**Patrick Brosnan**

Tuesday 30 March 2021

9:00 am – 10:00 am	**June Huh**
10:30 am – 11:30 am	**Kang Zuo**
1:00 pm – 2:00 pm	**Philip Engel**

Wednesday 31 March 2021

9:00 am – 10:00 am	**Nicholas Shepherd-Barron**
10:30 am – 11:30 am	**Francis Brown**
1:00 pm – 2:00 pm	**Colleen Robles**

Thursday 1 April 2021

9:00 am – 10:00 am	**Nicholas Shepherd-Barron**
10:30 am – 11:30 am	**Bertrand Toën**
1:00 pm – 2:00 pm	**Colleen Robles**

Friday 2 February 2021

9:00 am – 10:00 am	**Philip Engel**
10:30 am – 11:30 am	**Richard Hain**
1:00 pm – 2:00 pm	**Ron Donagi**

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Abstracts

Periods and Moduli of Elliptic Surfaces I & II

Nicholas Shepherd-Barron

We describe the derivative of the period map associated to the weight 2 Hodge structure of a Kaehler elliptic surface in terms of the derivative of the j-invariant and show that, conversely, this derivative suffices to recover equations for the base curve of the elliptic fibration (provided that the geometric genus of the surface is not too small compared to its irregularity). We go on to prove a generic Torelli theorem for simple elliptic surfaces (those with no multiple fibres).

The key is a plumbing construction for curves and their morphisms that goes back to Fay, and the formulae to which this construction leads. These permit an understanding of the derivatives of period maps in terms of tensors of rank one, from which everything else follows.

Volumes of Definable Sets in O-Minimal Expansions and Affine GAGA Theorems

Patrick Brosnan

The affine GAGA theorem of Peterzil and Starchenko says that a closed analytic subset of complex n-space which is definable in an o-minimal expansion of the ordered field R is actually algebraic. It is a one of the main tools in recent work on Hodge theory by Bakker, Brunebarbe and Tsimerman. In my talk I'll show how to give a very short proof of Peterzil--Starchenko using a much older affine GAGA theorem of Stoll along with a volume estimate for definable sets in an o-minimal structure (due to Kurdyka and Raby in a slightly different form and to Nguyen and Valette in the o-minimal setting).

The main point of my talk is really to advertise the o-minimal methods. So I'll give the (relatively easy) proof of the volume estimate. But I'll also try to say something about the way Peterzil--Starchenko has been used in Hodge theory.

Extensions of Hodge Structure and Completions of Fundamental Groups

Francis Brown

An important problem in the theory of L-functions and motives is to understand how extensions of mixed Hodge structure arise geometrically. One way to approach this problem is first to consider extensions of variations of mixed Hodge structure over a base scheme, and then specialise to a point. The variational problem can be understood, via Tannakian theory, by different notions of fundamental group. To this end, I will explain some of Deligne's work on the unipotent completion of the fundamental group, which classifies unipotent variations, and a generalisation due to Hain, called relative completion, which covers the general case. If time permits, I will outline a geometric interpretation of relative completion, generalising a result of Beilinson for the unipotent fundamental group.

Products of Matrices, Non-Abelian Hodge Theory, Integrable Systems, and Theories of Class S

Ron Donagi

We will describe some unexpected connections between the Deligne-Simpson problem and Hitchin systems, motivated by work on SCF Theories of Class S and 3D mirror symmetry.

Comparing Geometric and Hodge-Theoretic Compactifications I & II

Philip Engel

The The "classical period domains" D of polarized Hodge structures are those for which Griffiths' transversality condition is automatic. Then D is a Hermitian symmetric domain parameterizing either weight 1 Hodge structures, or weight 2 Hodge structures with $h^{2,0}=1$. Sometimes, the period mapping identifies a moduli space of varieties with an arithmetic quotient D/Gamma. For instance, in weight 1, we have moduli spaces A_g of principally polarized abelian varieties, and in weight 2, we have moduli spaces F_g of polarized K3 surfaces, or hyperkahler varieties.

Algebro-geometric compactifications of D/Gamma were first studied by Baily-Borel, and later by Ash-Mumford-Rapaport-Tai, who introduced toroidal compactifications. In the weight 2 case, Looijenga simultaneously generalized these to the semitoroidal compactifications. There are infinitely many such, in bijection with rational polyhedral decompositions of cones in certain lattices, called (semi)fans. In weight 1, the cone is the space of positive-definite quadratic forms on $\mathbb{R}^g$ and in weight 2, it is the positive cone of a hyperbolic lattice.

The motivating question of the talks is whether some distinguished (semi)toroidal compactification is a moduli space of generalized geometric objects: "stable abelian varieties" or "stable K3 surfaces."

I will first review the works of Mumford, Namikawa, and Alexeev who studied this question for abelian varieties. There, the so-called "second Voronoi fan" gives a distinguished toroidal compactification. The cones of the fan are the loci of quadratic forms which have fixed combinatorics of Delaunay-Voronoi decomposition. The combinatorics of the "stable abelian varieties" fibering over the corresponding boundary stratum are encoded by this Delaunay-Voronoi decomposition.

I will then discuss recent joint work with Alexeev on how to extend this to "stable K3 surfaces." These moduli spaces were discovered by Kollar-Sheperd-Barron and Alexeev, and parameterize log general type pairs. Since the moduli spaces in question parameterize Calabi-Yau varieties (which are not general type), one must choose canonically a "polarizing divisor" on the objects of the moduli space.

For abelian varieties, this choice was the theta divisor, and for K3 surfaces, we introduce the notion of a "recognizable divisor". Associated to a recognizable divisor is a semifan whose corresponding semitoroidal compactification is the normalization of the KSBA compactification of K3 pairs. For instance, the "rational curve divisor" is the sum of the rational curves in the polarization class, and gives a recognizable divisor for any F_g. The K3 analogue of the Delaunay-Voronoi decompositions are loci where certain polyhedral decompositions of singular integral-affine spheres have fixed combinatorics.

A Guide to Variations of Mixed Hodge Structure: Generalities and Examples

Richard Hain

In this talk I will recall some basic facts about (admissible) variations of mixed Hodge structure over a smooth variety and then explain how, under favorable circumstances, one can "compute" the MHS on the fibers of the variation from the MHS on one fiber, the global monodromy representation and certain iterated integrals. The basic conduction uses the mixed Hodge structure on the completion of the path torsor of the base manifold relative to the associated weight graded variation (a sum of polarized variation of MHSs). The mixed Hodge structure on the completed torsor of paths from x to y in the base can be used to determine the MHS on the fiber V_y over y from the MHS on the fiber on V_x, the fiber over x. In cases where one knows V_x for some possibly tangential x, one can "reconstruct" the entire variation. I will illustrate how this works with the elliptic KZB variation of MHS over the universal elliptic curve (over the modular curve) with its identity section removed.

Compa Kazhdan-Lusztig Theory and Singular Hodge Theory for Matroids I & II

June Huh

There is a remarkable parallel between the theory of Coxeter groups (for example the symmetric group or the dihedral group) and matroids (think of your favorite graph or point configuration) from the perspective of combinatorial cohomology theories. I will give an overview of the similarity and report on recent my joint work with Tom Braden, Jacob Matherne, Nick Proudfoot, and Botong Wang on singular Hodge theory for combinatorial geometries: https://arxiv.org/abs/2010.06088; https://arxiv.org/abs/2002.03341

Period Mappings at Infinity: Recent Developments I & II

Colleen Robles

Hodge theory provides a link between:

- Algebraic varieties and their moduli, and
- Algebraic groups, and associated representations and homogenous spaces (period domains)

These talks will focus on the latter as a framework to understand the asymptotic properties of period maps. The prototypical examples here are the classical nilpotent and sl2 orbit theorems. These theorems assign Hodge theoretic invariants to degenerations of smooth projective varieties. As such they, and related results, can be used to construct, describe and study compactifications of moduli spaces.
Historically these applications to moduli have been largely restricted to the setting of ppav and K3s. This is for two closely related reasons:

- The period domain is Hermitian
- The infinitesimal period relation is trivial

I will summarize work that has been done in the last few years to develop Hodge theory with an eye towards applications beyond the classical cases. This overview will cover joint work with Mark Green, Phillip Griffiths, Radu Laza; and Matt Kerr, Greg Pearlstein.

Characteristic Classes of Foliations in Positive Characteristic

Bertrand Toën

I will present a notion of "derived foliations" on arbitrary schemes. I will explain how it can be used in order to extend vanishing results for characteristic classes (Bott and Baum-Bott) to the positive characteristic setting. A key ingredient will be the construction of a Hodge filtration on algebraic de Rham cohomology induced by a derived foliation.

Hodge Theory, Higgs Bundles on Moduli Spaces of Manifolds and Hyperbolicity I & II

Kang Zuo

Given a complex quasi-projective manifold U and a smooth compactification $Y=U+S$, we are particularly interested in the geometry of the log space (Y, S) when U appears as the base of a family $f: V\to U$ parameterizing projective manifolds of non-negative Kodaira-dimension. The powerful theory on variation of Hodge structures (VHS) developed by P. Griffiths shows us a principle of how the variation of the complex structures on the fibers can be mirrored in the global properties of $(Y, S).$

In the first lecture I start to recall two types of graded Higgs bundles. The first one is the so-called system of Hodge bundles (E,θ) arising from a VHS introduced by C. Simpson in his work on the non-abelian Hodge theory.

The second type of Higgs bundle has been studied in my joint work with E. Viehweg, the so-called deformation Higgs bundle (F,τ) appearing as the natural extension of the classical Kodaira-Spencer map of f on the cohomologies of the relative tangent sheaf of higher degree.

In the second lecture I shall discuss on the applications of the existence of the big sub sheaf in $S^\ell\Omega^1_Y(\log S)$ in study of the various hyperbolicities of (Y,S), for example, algebraic hyperbolicity, Viehweg hyperbolicity, Brody and Kobayashi hyperbolicity on moduli stacks due to many people.

In particular, I shall briefly discuss on the recent work by Deng-Lu-Sun-Zuo on Picard's extension theorem. We show an analytic version of a big sub sheaf in $S^\ell\Omega^1_Y(\log S)$ by constructing a Finsler pseudo metric on $\Omega^1_Y(\log S)$ satisfying a strongly negative curvature inequality. As a generalization of the origin Lu's extension theorem, we show Picard's extension theorem holds true for any log space endowed with such a Finsler pseudo metric. The proof is highly inspired by the higher dimensional generalization of value distribution theory, also known as Nevanlinna theory developed by Griffiths and King.

I shall also mention on a sharp form of Arakelov inequality for families of n-folds of general type in a recent joint work with Lv and Yang. The proof relies on the Higgs map ρ and Simpson's work on the equivalence between the category of integrable connections and the category of semi-stable Higgs bundles with trivial Chern classes. I shall briefly sketch the recent progress on the rigidity problem with Javanpekar, Lu and Sun.

If time permits, I'd like to raise a question on the topological hyperbolictiy of moduli stacks. Namely, does the fundamental group of U grow exponentially? So far, there are some evidence to support this question observed by Lu, Sun and myself recently.