$P = W$ in genus 2
&
Hodge # of O'Grady10

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$P = W$ & OGIO

w/ J. Shen, D. Maulik — w/ A. Rapagnetta, G. Saccà

both on arXiv

C curve $g \geq 2$.

n the rank, deg the degree $\Big\} (rk, deg) = 1$

M_D moduli Higgs bundles $E \xrightarrow{\Phi} E \otimes \omega_C$ w/ slope stability;

q-proj; irreducible; nonsingular;

M_D^d $d = m = n^2\, 2(g-1)$

~~affine~~ ~~proj~~.

M_B moduli of $\pi_1(C \setminus \{x_0\}) \to GL(n, \mathbb{C})$ 2

$$\circlearrowleft_{x_0} \longmapsto \zeta^{\deg/n}\, \mathrm{Id}_n$$

affine; irreducible; nonsingular;

$\dim M_B = \dim M_D$.

<u>NAHT</u>: (Simpson) $\quad M_D \simeq M_B$ diffeo ~~algebraic~~ ~~biholomorphic~~

$$\Downarrow$$

$$H^\bullet(M_D) \simeq H^\bullet(M_B) \quad (\mathbb{Q}).$$

<u>FACTS</u>

(1) $H^\bullet(M_D)$ pure. (2) $H^\bullet(M_B)$ Hodge-Tate.
mixed

rk $=1$ degree 0

$M_D = T^*C$ pure $\qquad M_B = (\mathbb{C}^*)^{2g}$ HT

$$H M_D \simeq H M_B$$

$$W_D \not\Leftrightarrow W_B$$

$$? \Longleftrightarrow W_B .$$

Hitchin morphism

$$M_D \xrightarrow{h} A = \bigoplus_{i=1}^{n} H^0(C, i\omega_C)$$

$(E,\phi) \longmapsto$ car poly of ϕ

$t^n + a_1 t^{n-1} + \dots + a_n$

eq. of spectral curve inside $\mathrm{Tot}(\omega_C)$

4

h projective, surjective,
flat, connected fibers.

h^{-1}(smooth spectral curve in $\mathbb{A}$) = $\ddot{\mathrm{Jac}}$(said curve).

<u>FACT</u> (Leray) $L^h_D = W_{\supset}(\cancel{\Leftrightarrow} W_B)$.

So $L^h_D \cancel{\Leftrightarrow} W_B$ ...

$$\underset{c_1(h\text{-ample})}{H^2 M_D} \supset \underset{(2,2)}{\alpha \in H^2 M_B} \quad \text{rem: } M_B \text{ affine})$$

CHL $(n=2$ H+ - ; $\forall n \geq 2$ Mellit)
curious hard Lefschetz

$$Gr^{W_B}_{\cdots} \ddot{H} M_B \xrightarrow[\alpha^{\cdots}]{\sim} Gr^{W_B}_{\cdots} \ddot{H} M_B$$

5

RHL (BBDG)

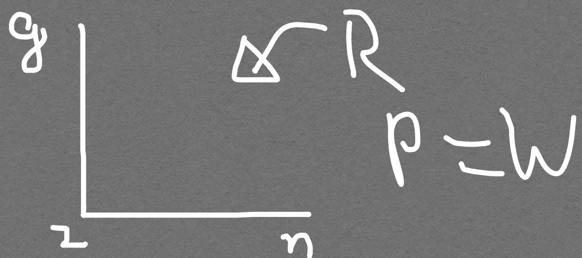
relative hard Lefschetz

$$Gr^{P_h}_{D} H M_D \xrightarrow[\alpha^{\cdots}]{\simeq} Gr^{P_h}_{D} H M_D$$

Conjecture (P=W) (d-Hausel-Migliorini 2010)

P = W after re-numbering)

<u>THM</u> d-H-M, 2010) P=W for: $\forall g \geq 2$; $n=2$



<u>THM</u> (d-Maulik-Junliang Shen, 2019)

- P=W for: $g=2$; $\forall n$
- g, n P=W on $R^{\cdot} \subseteq H^{\cdot} M_D$ or M
  subalgebra of certain taut. classes
- $\forall g, n$ P=W holds on all taut classes.

6

RMK – multiplicativity $\iff$ $P=W$
$P_i \cup P_j \subseteq P_{i+j}$

– $P=W \iff P\subseteq W \iff W \subseteq P$.

– former student Zili Zhang:
mult. holds in related contexts esp stemming from surfaces w/ trivial ω

mult. characterized (& can fail) in related contexts.

~ o ~

Sketch of main ideas

$C \hookrightarrow Jac(C) =: A$ abelian surface

D orm tion to normal cone

appropriate moduli spaces of sheaves on curves inside $\exists$

$T^\vee C$ for M_D $\quad$ A for M_A

$|nC|$

Sending sheaf $\mapsto$ Fitting support

$$\begin{tikzcd}
M_D \arrow[d, "h_D"'] & M_A \arrow[d, "h_{LP}"] \\
\mathbb{A} & {|nC|}
\end{tikzcd}$$

$0 \qquad t \qquad \mathbb{A}^1$

$$\begin{tikzcd}
H^\bullet M_D \arrow[r, shift left, "\overset{?}{\cancel{\mathrm{sp}}}"] & H^\bullet M_A \arrow[l, shift left, "\mathrm{cosp}"]
\end{tikzcd}$$

sp : M_D not compact: no sp!

cosp: same arrow always defined,
but target is not what one may expect.
but here, it is!
also cosp & P compatible: needs proof.
ok!

$$(HM_D, P^h) \xleftarrow[\mathbb{Q}\text{-alg}]{\cos} (HM_A, P^h)$$

Markman: $H(M_D / M_A)$ $\mathbb{Q}$-alg
generated by tautological
classes $c(k, \gamma)$

$$\begin{array}{c} \mathcal{E} \\ | \\ C \times M \end{array}$$

$ch(\mathcal{E})$ in $H^\cdot(C \times M)$
integrate ch against $H^\cdot C$
to get these tautological classes c

__If__ : the c's are in the "right" place in P & we have P multiplicative,

__Then__ : cosp gives $P=W$.

9

Deform, again : $A \sim E\times E'$

but take nC $\quad 5+\tilde{n}E'$

different divisibility in H^2

$$\begin{tikzcd} M_A \arrow[r, "\text{def}", "\sim"'] \arrow[d, "h_A"'] & M_{E\times E'} \arrow[d, "h_{E\times E'}"] \\ {|nC|} \arrow[r, "\cancel{\text{def}}"', "\cancel{\sim}"] & B \end{tikzcd}$$

$\circledast$ $M \to A\times\check{A}$ we take the fiber, that's $\circledast$.

How to relate $(H(M_A), P)$ & $(H(M_{E\times E'}), P)$?

J. Shen & Z. Zhang

Markman's monodromy operators are symmetries of $H^\bullet(M\_)$ that respect taut'l classes and their cup products.

11

What about P? We identify classes in $H^\bullet(M)$ that "carry" P and then verify these can be made to correspond under the monodromy operators.

<u>If</u> we have tautological classes & their products in the right place in P for $M_{E\times E'}$

<u>Then</u>, via Markman's operators and our analysis, the same is true for M_A

&

<u>then</u>, via cosp, the same is true for M_D.

<u>THM</u> (Junliang Shen, Zili Zhang 2018): The <u>IF</u> is ok.

$\square$

OG10

11

4 Known types of IHSV

compact Kähler

$\pi_1 = 1$

$h^0(\Omega^2) = 1$
$\langle \sigma \rangle$

$G: T \xrightarrow{\sim} T^\vee$

K3	Abelian	
$K3^{[n]}$	$Ku_{[n]}$	dim 2n
OG10	OG6	sporadic

Hodge #: $K3^{[n]}, Ku_{[n]}$ Göttsche-Sorgel

OG6 Mongardi-Rapagnetta-Saccà

THM (d-R-S) OG10 & when K3 ⟶ OG10

$$PHS_{OG10} = K3^{[5]} + 2(K3^{[2]})^{(2} - (K3^{[2]}) + K3^{[2]}$$

<u>RMK</u> odd Betti $= 0$

Shortly after;

- Green–Kim–Laza–Robles:
  by different methods
  <u>if</u> odd Betti $= 0$ <u>then</u> h^{pq} OG10.
- Floccari–Fu–Zhang: $\Rightarrow$ odd Betti $= 0$

So, by combining: different proof of h^{pq} OG10.

—— o ——

$$\begin{tikzcd}
\mathrm{OG10}-\widetilde{M}^{10} \arrow[d, "b"'] \arrow[ddr, "\widetilde{m}"] & P \circlearrowright & N^{10} \arrow[ddl, "n"] \\
M^{10} \arrow[dr, "m"'] & & \\
& B = |2C| \simeq \mathbb{P}^5 &
\end{tikzcd}$$

13

$(K3, C)$
$g=2$
general
then can
remove that

M sing
moduli of sheaves on $K3$ — $(0, 2C, \text{even})$ / $(0, 2C, \text{odd})$
N smooth

$\widetilde{M} \to M$ blow up of strictly semistable locus

m, n : sheaf to Fitting support, a curve in $|2C|$.

$P = \mathrm{Pic}^0(\mathscr{C}/B)$ $\quad \mathscr{C}$ univ. curve $\subseteq |2C| \times K3$.

$P \circlearrowright_{/B} M, N$ via $L \cdot \mathcal{F} := L \otimes \mathcal{F}$

$P \circlearrowright \widetilde{M}$ also after blow up.

MAIN IDEA: compare $\widetilde{m}_*\mathbb{Q}$ & $h_*\mathbb{Q}$ 14

Decomposition Thm (BBDG)

$f: X \to Y$

$$f_* IC_X \simeq \bigoplus_{S \in \Sigma} K_S$$

Σ finite set of supports, $S \hookrightarrow Y$

$$K_S \simeq \bigoplus IC_S(\mathcal{L}_{S,i})[-i]$$

$S = ?$ $\quad \mathcal{L}_{S,i} = ?$

— o —

P/B $\quad 0 \to R_b \to P_b \to A_b \to 1$

Chevalley devissage at $b \in B$

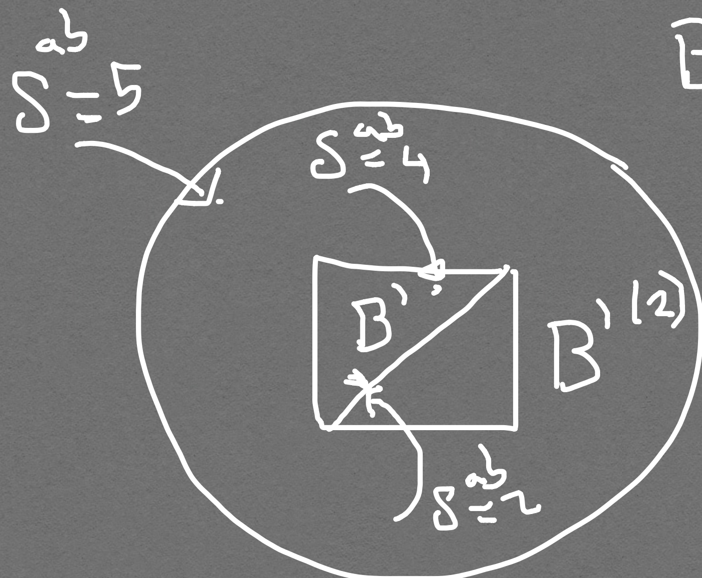
$R_b \simeq \mathbb{G}_a^{\cdots} \times \mathbb{G}_m^{\cdots}$ some dim δ_b^{aff} $\quad$ A_b ab. var some dim δ_b^{ab}

$$B = |2C| \simeq \mathbb{P}^5 = \bigsqcup B_\delta$$

$\uparrow$ where $\delta^{ab} = \delta$

<u>PROP</u> δ among the B_δ.

$B' = |C| \simeq \mathbb{P}^2$



$B^5 \simeq |2C|$

$B \quad B'^{(2)} \quad B'$

These turn out to be <u>the</u> supports

Ngô Support Theorem:

If S is a support,

then S contributes to $R^{10}\tilde{m}_*\mathbb{Q}$:

$\exists$ L_S loc. const on $S^0 \overset{j}{\hookrightarrow} S$

s.t. $j_! L_S \overset{\oplus}{\subseteq} R^{10}\tilde{m}_*\mathbb{Q}$.

Ngô strings:

Λ^i_S loc. const on S^0 with fiber $= \Lambda^i H^1(A_s)$

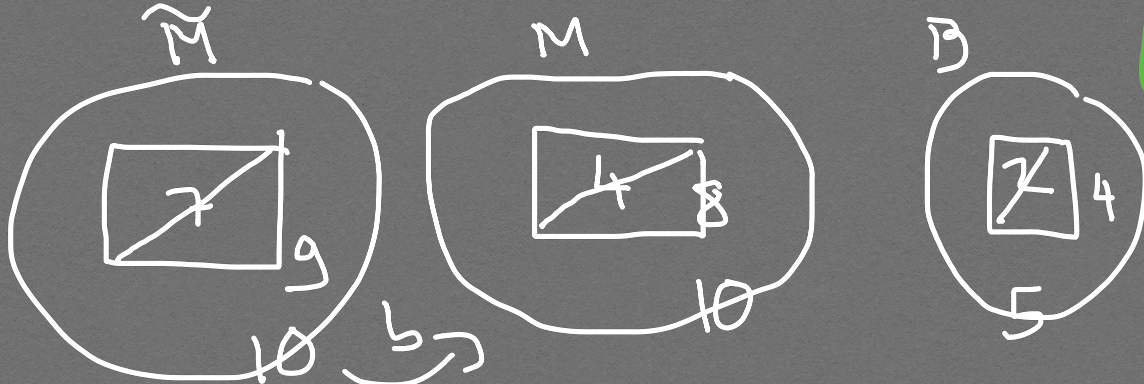
$= H^i(A_s)$

$$K_S = \bigoplus_{i=0}^{2\delta_S} IC_S(\Lambda^i_S \otimes L_S)[\delta_S - i]$$

PROP ${}^5B = IC1$ are supports

${}^4B^{1(2)}$

${}^2B^1 = IC$



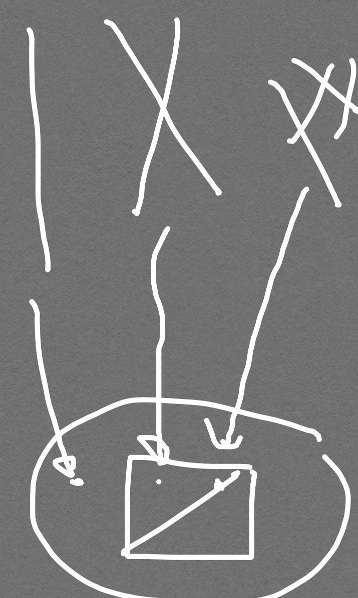
4, 8, 10 are supports of b_*

so 2, 4, 5 are supports via $\widetilde{m}_* = m_* b_*$

$\square$

What about $L_B, L_{B'(2)}, L_{B'}$?

Exhamine fibers of $\widetilde{m}$



18

$L_B = \mathbb{Q}_B$

no monodromy in X: $L_{B'(2)} = \mathbb{Q}_{B'(2)}$

L_B & $L_{B'(2)}$ contribute $\mathbb{Q}_b^2$ to $\left(R^{10} m_* \mathbb{Q}\right)_b$.

So $L_{B'} = \mathbb{Q}_{B'}^{1+\varepsilon}$

$\varepsilon = 0,1 \iff \mathcal{H}^8(\partial\mathcal{C}(\Lambda_B^2)) = 1,0$.

So:

$$\tilde{m}_* \mathbb{Q} \simeq \partial\mathcal{C}_B\left(\bigoplus_0^{16} \Lambda_B^i\right)[-i]$$

$$\oplus\, \partial\mathcal{C}_{B'(2)}\left(\bigoplus_0^{8} \Lambda_{B'(2)}^i\right)[-i][-2]$$

$$\oplus \left(\partial\mathcal{C}_{B'}\left(\bigoplus_0^{4} \Lambda_{B'}^i\right)[-i][-2]\right)^{\oplus 1+\varepsilon}$$

Repeat the analysis for $N \Rightarrow B$ 19

$h_* \mathbb{Q} \simeq$ same

same but Λ^i twisted by $L_{\cdot 1}$

same but $\oplus\, \varepsilon$

the same ε! comes from P not $\tilde{M}$, not N!

Take cohomology (Betti, PHS):

$$\tilde{M} = B + \mathrm{Sym}^2 K3^{[2]} + \left(K3^{[2]}\right)^{1+\varepsilon}_{\varepsilon}$$

$$N = B + \mathrm{Anti}\,\mathrm{sym}^2 K3^{[2]} + K3^{[2]}$$

$$\begin{array}{l} \| \\ K3^{[5]} \end{array}$$

$$\tilde{M} = 2\,\mathrm{Sym}^2 K3^{[2]} - \left(K3^{[2]}\right)^2 + K3^{[2]}.$$

$\square$

Ben Wu: OG6

- P disconnected fibers
- fibers of $\tilde{M} \to B$
  many irred comp