(1)

IMSA Talk. Sottile 19 Oct 2020

My goal is to sketch how the objects of this meeting (Toric Varieties from irrational fans) arose naturally out of questions from Applications

html Describe picture & patches
show movie

(Asi

c. 2000 Krasauskas : Toric Patches

Let $A \subset \mathbb{Z}^n$ be finite

$$\varphi_A : (\mathbb{R}_>)^n \longrightarrow \mathbb{RP}^{|A|-1} =: \triangle^A \quad \text{St. Simplex}$$

$$x \longmapsto [x^a \mid a \in A]$$

Closure of image is Y_A positive Toric Variety

$\omega \in \mathbb{R}^A_>$

$$\triangle^A \xrightarrow{\cdot\omega} \triangle^A$$

$$z \longmapsto [w_a \cdot z_a \mid a \in A]$$

$B = \{b_a \mid a \in A\} \subset \mathbb{R}^d$ $\quad \pi_B : \triangle^A \to \mathbb{R}^d$

$$z \longmapsto \sum b_a \cdot z_a$$

The (image of) $\omega \cdot Y_A$ under π_B is a Toric Bézier patch.

Q: (de Boor) What is the meaning of the cage?
What happens as ω changes.

html $A \subseteq \mathbb{R}^n$ is O.K. | Talk about Birch 1963

When $A \subset \mathbb{Z}^n$ & we replace $\mathbb{R}$ by $\mathbb{C}$
KSZ studied limits of torus translates.

<u>Theorem</u> (Up to normalization)
The space of torus translates of X_A is identified with the toric variety associated to the <u>secondary fan of A</u>
The limiting points are toric degenerations:
$\lambda : A \to \mathbb{Z}$ $\quad t^\lambda$ is a weight
$$\lim_{t \to 0} t^\lambda . w X_A \quad \text{(scheme-theoretic)}$$

Explain secondary fan / Polytope GSZ / $\mathbb{R}$

Postinghel - S - Villamizar: $A \subset \mathbb{R}^n$, $w \in \mathbb{R}^A_>$

$w . Y_A \subset \Delta^A$ compact subset of Δ^A
We studied limits of sequences of torus translates $w . Y_{A}$ $_n$

1) They are all real toric degenerations
$$\lim_{t \to 0} t^\lambda . w X_A$$

(& identifies w, λ)
2) Showed how this is controlled by the secondary fan

IMSA Talk

This work failed in its goal, which was to prove a real version of KSZ.

The problem There was no way to associate a 'Toric Variety' to an irrational fan.

W/ Pir
$N = \mathbb{R}^n \quad M = N^*$
Let $\sigma \subset N$ be a polyhedral cone
$\sigma^\vee := \{ m \in M \mid m(u) \geq 0 \ \forall u \in \sigma \}$

$V_\sigma := \mathrm{Hom}_c(\sigma^\vee, \mathbb{R}_\geq) =$
$\{$ semigroup homomorphisms $\varphi : (\sigma^\vee, +) \to (\mathbb{R}_\geq, \cdot)$
w/ $\varphi(0) = 1$ & φ is continuous on $\varphi^{-1}(\mathbb{R}_{>0})$ $\}$.

- $\varphi^{-1}(\mathbb{R}_{>0})$ is a face of $\sigma^\vee$
- V_σ has the weak topology of pointwise convergence (homeomorphic to σ)

$\tau \subset \sigma \Rightarrow V_\tau \subset V_\sigma$
$\mathbb{T}_> := \mathrm{Hom}(M, \mathbb{R}_{>0}) \simeq (N, +)$
The torus, which acts on everything.

For a fan Σ $\quad Y_\Sigma := \bigcup_{\sigma \in \Sigma} V_\sigma$
w/ usual identifications

__Theorem__ For any fan $\Sigma \subset N$

Y_Σ is a $\mathbb{T}_{>0}$-equivariant cell complex dual to Σ.
it has one orbit $\mathcal{O}_\sigma$ for each cone $\sigma \in \Sigma$
& $\mathcal{O}_\sigma \simeq N/\langle\sigma\rangle$
w/ distinguished point
φ_σ (1 on lineality space $\sigma^\perp \subset \sigma^\vee$
0 elsewhere)

- Closure of $\mathcal{O}_\sigma$ is $Y_{\Sigma/\sigma}$ (link of σ in Σ)

- $\Sigma \to Y_\Sigma$ is functorial under maps of fans
(No equivalence of categories) discuss

- Y_Σ compact $\Leftrightarrow$ Σ is compact
(uses PSV)

- If Σ is the normal fan of a polytope P
$Y_\Sigma \simeq P$ (from Birch's Theorem)

__Theorem__ The Hausdorff space of Toric translates of $Y_A < \Delta^A$ is identified with the secondary polytope of A.