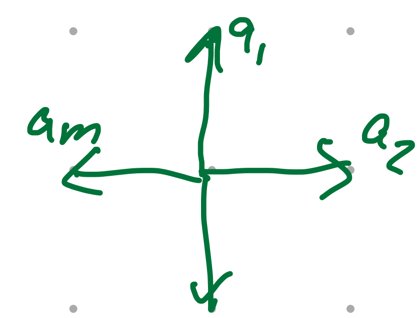
Basic cohomology of moment-angle manifolds

$(\Sigma; a_1, \dots, a_m)$ – complete simpl. fan in $\mathbb{R}^n$ with m gen. s.t.
$(m-n) \,\vdots\, 2$

Associate calibration q:



$$0 \to \ker q \to \mathbb{R}^m \xrightarrow{q} \mathbb{R}^n \to 0$$

$$e_i \mapsto a_i$$

complexification

$$0 \to \ker q^{\mathbb{C}} \to \mathbb{C}^m \xrightarrow[q^{\mathbb{C}}]{} \mathbb{C}^n \to 0$$

$$0 \to \underset{\langle e_1+e_3,\, e_2+e_4 \rangle}{\underset{\shortparallel}{\ker q}} \to \mathbb{R}^4 \to \mathbb{R}^2 \to 0$$

Complex moment-angle manifolds

K_Σ – underlying simplicial complex of Σ

$$\mathcal{U}_{K_\Sigma} = \bigcup_{I \in K_\Sigma} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^* \Big) \subset \mathbb{C}^m$$

Choose $\mathfrak{h} \hookrightarrow \ker q^{\mathbb{C}}$ s.t. $\mathfrak{h} \oplus \ker q = \ker q^{\mathbb{C}}$

$H := \exp(\mathfrak{h}) \subset (\mathbb{C}^*)^m$ $\quad \dim_{\mathbb{C}} \mathfrak{h} = \dfrac{m-n}{2}$

Claim $H \curvearrowright \mathcal{U}_\Sigma$ freely, properly, holomorphically +

$$\mathcal{U}_\Sigma / H \underset{T^m\text{-equiv. diff.}}{\cong} Z_\Sigma = \bigcup_{I \in K_\Sigma} \Big(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1 \Big) \subset \mathbb{C}^m.$$

Canonical foliation:

$$R := \exp(\mathfrak{r}) \subset T^m$$

$$\mathbb{R}^{\mathbb{C}} := \exp(\mathfrak{r}^{\mathbb{C}}) \subset (\mathbb{C}^*)^m$$

$$\begin{array}{c} \mathfrak{r} \\ \shortparallel \\ 0 \to \ker q \to \mathbb{R}^m \to \mathbb{R}^n \to 0 \end{array}$$

$$0 \to \ker q^{\mathbb{C}} \to \mathbb{C}^m \to \mathbb{C}^n \to 0$$

$\mathbb{R}^{\mathbb{C}} \curvearrowright \mathcal{U}_\Sigma$ locally free $\rightsquigarrow$ foliation $\mathcal{F}_\Sigma$ on $\mathcal{U}_\Sigma$

$$(\mathcal{U}_\Sigma, \mathcal{F}_\Sigma) \xrightarrow{pr} (Z_\Sigma, \mathcal{F}_\mathfrak{h})$$

$\mathbb{R}^{\mathbb{C}}/H \cong R$ induces $\mathcal{F}_\mathfrak{h}$

$$\begin{array}{ccc} & & Z_\Sigma \\ & & \downarrow \\ \mathcal{U}_\Sigma/\mathbb{R}^{\mathbb{C}} & \cong & V_\Sigma \end{array}$$

Remark. This is a generalization of Cox construction.

Basic cohomology

Basic de Rham:

$(M, \mathcal{F})$ - foliated mfld

$$(\Omega^*_{\mathcal{F}}(M), d) := \{ \omega \in \Omega^*(M) : i_X \omega = L_X \omega = 0, \ \forall X \in T\mathcal{F} \}$$

$H^*_{\mathcal{F}}(M)$ - cohomology of this DGA

Basic Dolbeault: analogously define

$$(\Omega^{*,*}_{\mathcal{F}}(M; \mathbb{C}), \bar{\partial})$$

$H^{*,*}_{\mathcal{F}}(M)$ - its cohomology

Basic de Rham cohomology

Thm. (Ishida, K., Panov; '18)

$$H^*_{\mathfrak{F}_h}(Z_\Sigma) \cong \mathbb{R}[v_1,\dots,v_m]/(I_{K_\Sigma}+J) \qquad \deg v_i = 2$$

Stanley-Reisner ideal:

I_{K_Σ} gen. by $v_{i_1}\cdots v_{i_k}$ for $\{i_1,\dots,i_k\} \notin K_\Sigma$

J gen. by $\sum \langle u, a_i\rangle v_i$, $u \in \mathfrak{g}' = (\mathfrak{t}/\mathfrak{r})^* \cong (\mathbb{R}^m/\ker q)^*$

Overview:

- (Battaglia, Zaffran; '11) Computed Betti numbers in polytopal case, conjectured ring structure
- (Ishida; '17) Proved conjecture in polytopal case

$$H^*([Z_\Sigma/R];\mathbb{R}) \cong H^*_{\mathfrak{F}_\Sigma}(Z_\Sigma)$$

Basic Dolbeault cohomology

Theorem (Ishida, '17)

Let Σ be a polytopal fan. Then

$$H^r_{\mathcal{F}_h}(Z_\Sigma, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\mathcal{F}_h}(Z_\Sigma)$$

Moreover,

$$H^{*,*}_{\mathcal{F}_h}(Z_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m] / (I_\Sigma + J)$$

$$v_i \in H^{1,1}_{\mathcal{F}_h}(Z_\Sigma)$$

Remark $\mathcal{F}_h$ is transverse kähler $\Leftrightarrow$ Σ - polytopal

$\exists\, \omega \in \Omega^2_{\mathcal{F}}(M)$ $\ker \omega = T\mathcal{F}$

$\omega|_{TM/T\mathcal{F}}$ - kähler

Thm (K., Panov; '19)

Same holds for general Σ.

Idea: Reduction to transverse kähler case.

Fujiki foliations

Given $(M, \mathcal{F})$ - holomorphic foliation

Def. $(M, \mathcal{F})$ — Fujiki

$$f : (M', \mathcal{F}') \twoheadrightarrow (M, \mathcal{F})$$

transversely kähler

$$H^{top}_{\mathcal{F}}(M) \neq 0$$

Lemma $f : (M', \mathcal{F}') \to (M, \mathcal{F})$ as above

$$\Rightarrow \quad H^{*,*}_{\mathcal{F}}(M) \hookrightarrow H^{*,*}_{\mathcal{F}'}(M')$$

So, $\mathcal{F}$ admits Hodge decomposition.

Want to find

with Σ' - polytopal

$$f : (Z_{\Sigma'}, \mathcal{F}_{\eta'}) \twoheadrightarrow (Z_{\Sigma}, \mathcal{F}_{\eta})$$

Consider map $L: \Sigma' \to \Sigma$ induced by $L: \mathbb{R}^{m'} \to \mathbb{R}^{m}$ satisfying

$$L(a_i') \in \mathbb{Z}\langle a_{i_1}, \dots, a_{i_k} \rangle$$

where $L(a_i') \in \sigma_{i_1 \dots i_k}$

$\rightsquigarrow$ It induces

$$\begin{tikzcd}
(\mathcal{U}_{\Sigma'}, \mathcal{F}_{\Sigma'}) \arrow[r] \arrow[d] & (\mathcal{U}_{\Sigma}, \mathcal{F}_{\Sigma}) \arrow[d] \\
(Z_{\Sigma'}, \mathcal{F}_{h'}) \arrow[r] & (Z_{\Sigma}, \mathcal{F}_{h})
\end{tikzcd}$$

Now we need to solve combinatorial problem!

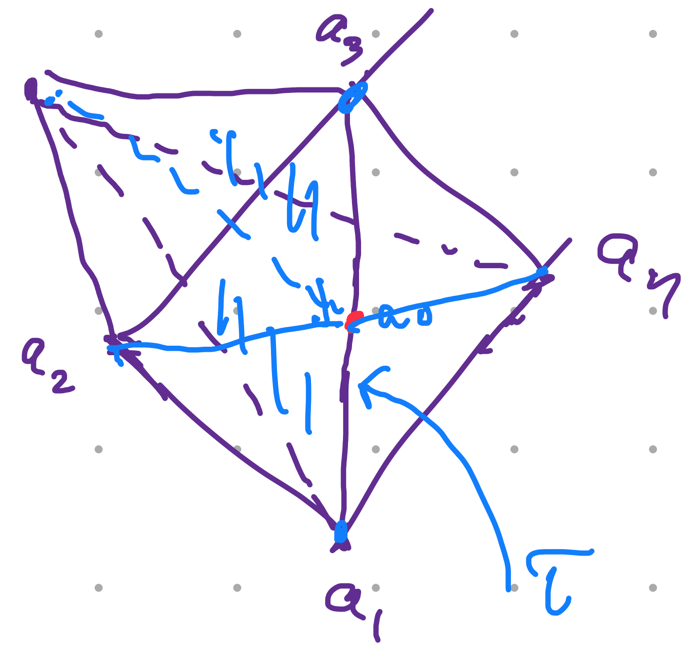
$$\boxed{\text{Given } \Sigma, \text{ find } L: \Sigma' \twoheadrightarrow \Sigma \text{ s.t. } \Sigma' \text{ is polytopal.}}$$

Stellar subdivision

$$\Sigma_\tau \to \underset{\substack{\cup \\ \tau}}{\Sigma}$$

$\tau = \langle a_1 \ldots a_k \rangle$

$a_0 = a_1 + \ldots + a_k$



Thm For any complete simplicial fan Σ ~~there~~ exist a sequence of stellar subdivisions s.t. resulting fan Σ' is polytopal.

Rmk. For rational fans it was proved by De Concini and Procesi

toric blow ups

$$\underbrace{V_{\Sigma'}}_{\text{projective}} \to \ldots \to V_\Sigma$$

Rmk Strong factoriz. conj.

$$\begin{tikzcd} & \Sigma' \arrow[dl] \arrow[dr] & \\ \Sigma_1 & & \Sigma_2 \end{tikzcd}$$

same generators

Does it hold for irrational fans?