Let S be a nonsingular projective surface with $H^1(\mathcal{O}_S) = 0$ and let $h = c_1(\mathcal{O}_S(1))$. Let

$$\vartheta_S(r, \gamma, m) \in \bigoplus_{i=0}^{2} H^{2i}(S, \mathbb{Q})$$

$\longrightarrow$

Fix a Riemannian 4-mfd S

Fix a cpct Lie group G and a principal G-bdl $P \to S$

let $\mathcal{A}_P := \{ G\text{-connections on } P \}$

$\Omega^i(\mathfrak{g}_P)$ the C^∞ i-forms on P with values in the adjoint of P. Finally let $\Omega^+ \subset \Omega^2$: Self dual 2-forms with respect to Riemannian metric.

The VW equations are then:

$$\mathcal{A}_P \times \Omega^+(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \longrightarrow \Omega^+(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P)$$

$$(d_A, B, \Gamma) \longrightarrow \Big(F_A^+ + \underbrace{[B,B]}_{\text{Lie bracket on } \mathfrak{g}_P} + [B, \Gamma],\ d_A\Gamma + d_A^* B\Big)$$

$T^{*\otimes 2} \otimes T^{*\otimes 2} \longrightarrow T^{*\otimes 2}$ contraction is

If S is kahler surface (S, ω)

then the Vafa Witten equations are

$$\begin{cases} F_A^{0,2} = 0 \\ F_A^{1,1} \wedge \omega + [\phi, \bar{\phi}] = c \cdot id_E \, \omega^2 \\ \bar{\partial}_A \phi = 0 \end{cases}$$

topological number.

where $\varphi \in \Omega^{2,0}(\mathfrak{g}_P \otimes \mathbb{C})$

We are interested in cases where $G = U(r)$
or
$G = SU(r)$

and hence P is a frame bundle of a Hermittian vector bundle E. Then the first equation defines an integrable holomorphic structure on E with respect to which ϕ is hol.

Since the second equation is a moment map for the gauge group action one can expect that solutions (modulo $U(r)$ or $SU(r)$ gauge transformation) are equivalent to holomorphic pairs (E, ϕ) modulo $GL_r(\mathbb{C})$ or $SL_r(\mathbb{C})$ gauge transformations).

Let $M_h(v)$: moduli Space of h-Semistable sheaves on S with Chern char v.

$M(v)$ is a projective Scheme.

We assume that

Slope semistability $\Longrightarrow$ Slope stability

We assume $M_h(v)$ admits a universal family

the Condition $\gcd(r, \gamma \cdot h) = 1$ implies these requirements

Here is a fact on
$$\begin{array}{c} \tilde{\mathbb{F}} \\ \wr \\ S \times M^h(v) \xrightarrow{p} M_h(v) \\ \downarrow \\ S \end{array}$$

$R_{p_*} R\mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\mathcal{F}})_0[1]$ is the virtual tangent bundle of a trace-free perfect obstruction theory on $M_h(v)$

That gives virtual Class $[M^h(v)]^{vir}_0$ next

Now let $\mathcal{L}$ be a line bundle on S Such that $H^0(\mathcal{L} \otimes \omega_S^{-1}) \neq 0$ and let $X : \mathcal{L} \xrightarrow{q} S$

be the total space of this line bundle on S.

X ; noncompact, $\omega_X = q^*(\mathcal{L}^{-1} \otimes \omega_S)$, in particular if $\mathcal{L} = \omega_S$ then $\omega_X \cong \mathcal{O}_X$ and X is Calabi Yau.

Fact. The one dimensional Torus $\mathbb{C}^* \curvearrowright X$ by multiplication on the fibers of q so that

$$q_* \mathcal{O}_X = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \otimes t^{-i}$$

$t \leadsto$ trivial bundle on S with $\mathbb{C}^*$-action weight 1 on the fibers.

cat $\mathrm{Coh}_c(X) \subset \mathrm{Coh}(X) \leadsto$ abelian Cat Comptly Supported Sheaves

We define a slope function on $\mathrm{Coh}_c(X) \setminus 0$

$$\mu_h(\mathcal{E}) = \frac{c_1(q_*\mathcal{E})\cdot h}{\operatorname{rank}(q_*\mathcal{E})} \in \mathbb{Q} \cup \{\infty\}$$

determining a slope fcn. on $\mathrm{Coh}_C(X)$

Let $\mathcal{M}_h^{\mathcal{L}}(\delta)$: moduli space of μ_h-stable sheaves $\mathcal{E} \in \mathrm{Coh}_C(X)$ with $ch(q_*\mathcal{E}) = \delta$

Assume $\mathcal{M}_h^{\mathcal{L}}(\delta)$ admits a universal family denoted by $\bar{\mathcal{E}}$. $\Leftarrow$ $\gcd(r, \gamma \cdot h) = 1$ ensures This!

$\exists$ a natural perfect obstruction theory

$$E^\circ \longrightarrow \mathbb{L}^\circ_{\mathcal{M}_h^{\mathcal{L}}(\delta)} \qquad \begin{array}{c}\bar{\mathcal{E}} \\ \wr \\ X \times \mathcal{M}_h^{\mathcal{L}}(\delta)\end{array} \xrightarrow{\bar{p}} \mathcal{M}_h^{\mathcal{L}}(\delta)$$

where $E^{\circ\vee} = \tau^{[1,2]}\left(R\mathcal{H}om_{\bar{p}}(\bar{\mathcal{E}}, \bar{\mathcal{E}})\right)[1]$

if $\mathcal{E} \in \mathcal{M}_h^{\mathcal{L}}(\delta)$ is a closed point

Then $\operatorname{Rank}(E^\circ) = \operatorname{ext}^1(\mathcal{E},\mathcal{E}) - \operatorname{Ext}^2(\mathcal{E},\mathcal{E})$

$$= 1 - k - \sum_{i=0}^{3} (-1)^i \operatorname{ext}^i(\mathcal{E},\mathcal{E})$$

$k=1$ if $L = \omega_S$

$k=0$ otherwise

$$\mathrm{Rank}(E^{\circ}) = \begin{cases} 0 & L = \omega_S \\ r_{C_1}^2(L) \cdot (c_1(L) - \omega_S)/2 + 1 & L \neq \omega_S \end{cases}$$

if L is ω_S then E° is known to be Symmetric

Graber–Pandharipande we obtain $\mathbb{C}^*$-fixed obs. Th.

$$E^{\circ, fix} = \left(\left(\tau^{[1,2]} R\mathcal{H}om_{\bar{p}}(\bar{\mathcal{E}}, \tilde{\mathcal{E}})[1] \right)^{\vee} \right)^{\mathbb{C}^*} \longrightarrow \mathbb{L}^{\circ}_{\mathcal{M}^{L}_{h}(0)^{\mathbb{C}^*}}$$

$$\widehat{DT}_h(0;\alpha) = \int_{[\mathcal{M}^{L}_{h}(0)^{\mathbb{C}^*}]^{vir}} \frac{\alpha}{e\left((E^{\circ, mov})^{\vee} \right)}$$

$\alpha \in H^{*}_{\mathbb{C}^*}(\mathcal{M}^{L}_{h}(0), \mathbb{Q})_S \hookrightarrow$ localized at S: equivariant Parameter

Remark If $\mathcal{M}^{\mathcal{L}}_h(0)$ is compact then $\widehat{DT}^{\mathcal{L}}_h(0)$ is equal to $\int_{[\mathcal{M}^{\mathcal{L}}_h(0)]^{vir}} \alpha$ via virtual localization formula. This is the case when $c_1(\mathcal{L})\cdot h<0$ all sheaves are scheme theoretically supported on $\mathcal{S}$.

Remark If $\mathcal{L}=\omega_S$ (X $CY3$) one can define invariants by taking weighted Euler char of the moduli spaces $\int_{\mathcal{M}^{\omega_S}_h(0)} \nu_{\mathcal{M}^{\omega_S}}\, d\chi$

where $\nu_{\mathcal{M}^{\omega_S}}$ is Behrend's constructible function on $\mathcal{M}^{\omega_S}_h(0)$. By localization this coincides with integration of $\nu_{\mathcal{M}^{\omega_S}}$ on $\mathcal{M}^{\omega_S}_h(0)^{\mathbb{C}^*}$ $\longrightarrow$ Tanaka Thomas

Remark When $P_g(S)>0$ The fixed part of E°

contains a trivial factor; then we reduce

NO!

The obstruction theory

$$\dot{C}^\circ = \mathrm{Cone}\left[q_* R\mathcal{H}om_{X\times \mathcal{M}^{\mathcal{L}}_h(0)}(\tilde{\mathcal{E}},\tilde{\mathcal{E}}) \xrightarrow{q_*} R\mathcal{H}om_{S\times \mathcal{M}^{\mathcal{L}}_h(0)}(q_*\tilde{\mathcal{E}}, q_*\tilde{\mathcal{E}}) \xrightarrow{tr} \mathcal{O}_{S\times \mathcal{M}^{\mathcal{L}}_h(0)} \right]$$

and let $C^\circ = R_{p_*} R\mathcal{H}om\ \dot{C}$

$$S\times \mathcal{M}^{\mathcal{L}}_h(0) \xrightarrow{P} \mathcal{M}^{\mathcal{L}}_h(0)$$

(q is affine $\Rightarrow R^i q_* = 0 \quad i>0$)

$$E^\circ_{red} := \left(\tau^{\leq 1} C^\circ\right)^\vee \rightsquigarrow \text{perfect of } [0,1]$$

and $h^0\left(\tau^{\leq 1}(C^\bullet)\right) \cong \mathcal{E}xt^1_p(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$ Next!

Thm (Gholampour, —, Yau 2018)

$(E^\bullet_{red})^\vee = \tau^{\leq 1}(C^\bullet)$ is the virtual tangent bundle of obstruction theory over $\mathcal{M}^R_h(\sigma)$

Remark $\operatorname{Rank}(E^\bullet_{red}) = \operatorname{rank}(E^\bullet) + p_g(S)$ in particular when $p_g(S) = 0 \Rightarrow E^\bullet = E^\bullet_{red}$; The reduction only affects the fixed part of the virtual tangent bundles; that is $E^{\bullet, mov} = E^{\bullet, mov}_{red}$.

Cor. $E^{\bullet, fix}_{red}$ gives a perfect obs Thy on $\mathcal{M}^{\mathbb{C}^*}_h(\sigma)$ and hence a virtual fundamental class

$$[\mathcal{M}^{\mathcal{L}}_{h}(\theta)^{\mathbb{C}^*}]^{vir}_{red} \in A_*(\mathcal{M}^{\mathcal{L}}_{h}(\theta)^{\mathbb{C}^*})$$

$$DT^{\mathcal{L}}_{h}(\theta,\alpha) = \int_{[\mathcal{M}^{\mathcal{L}}_{h}(\theta)^{\mathbb{C}^*}]^{vir}_{red}} \frac{\alpha}{e\left((E^{\bullet,mov})^{\vee}\right)} \in \mathbb{Q}[s,s^{-1}]$$

$$\alpha \in H^*_{\mathbb{C}^*}(\mathcal{M}^{\mathcal{L}}_{h}(\theta),\mathbb{Q})[s]$$

Vafa-Witten invariants: Tanaka-Thomas define Vafa-Witten invariants by constructing a symmetric perfect obs. Theory on moduli space of Higgs pairs (G,ϕ) on S such that $tr(\phi)=0$

$$G \xrightarrow{\phi} G \otimes \omega_S$$

① Infact they show that moduli space of Higgs pairs is isomorphic to $\mathcal{M}^{\omega_S}_{h}(\theta)$.

② The moduli space of Higgs pairs is equipped with

② $\mathbb{C}^*$ - action on Higgs pair moduli space is the same as $\mathbb{C}^*$ - action on $\mathcal{M}_h^{\omega_S}(\theta)$

③ Over the fixed locus of moduli space of Higgs pairs $tr(\phi)=0$ automatically. As a result the fixed locus of $\mathcal{T}$ is identified with $\mathcal{M}_h^{\omega_S}(\theta)^{\mathbb{C}^*}$

④ The fixed part of $\mathcal{T}$ obs Th is equivalent in K-Theory to $(E_{red}^{\bullet,fix})^\vee$ and the moving parts differ (in K-Theory) by trivial bundle of rank $p_g(S)$. In fact in K-Theory.

$$\left.\begin{aligned} h^0(E^a_{red}) &= h^0(E^{\circ}_{\mathcal{T}})^\vee + p_* \omega_S \\ h^1(E^{\circ}_{red}) &= h^1(E^{\circ}_{\mathcal{T}})^\vee \end{aligned}\right\} \begin{aligned} &[\mathcal{M}_{\mathcal{T}}^{\mathbb{C}^*}]^{vir}_a \\ &= [\mathcal{M}_h^{\omega_S}(\theta)^{\mathbb{C}^*}]^{vir}_{red} \end{aligned}$$

when $\alpha = 1$

$$\Rightarrow \quad DT^{ws}_{h}(\vartheta\,; 1) = \delta^{-p_g} VW_{h}(\vartheta)$$

Fixed loci: Suppose $\mathcal{E} \in \mathcal{M}^{\mathcal{L}\ \mathbb{C}^*}_{h}(0)$ is a closed point. Since $\mathcal{E}$ is a pure and $\mathbb{C}^*$-equiv sheaf we can assume that

$$q_*\mathcal{E} = \bigoplus_{i \geq 0}^{\ell(\lambda)-1} E_{-i} \otimes t^{-i}$$

for $\lambda \vdash r$ a partition of rank $\mathcal{E}$ $\lambda : (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\ell(\lambda)})$

where E_{-i} is rank λ_{i+1}; torsion-free on S

and the $\mathcal{O}_X$-module structure on $\mathcal{E}$ is given by a collection of injective maps.

of $\mathcal{O}_S$-modules $\psi_i : E_{-i} \longrightarrow E_{-i-1} \otimes \mathcal{L}$

Then we can see that

$$\underbrace{E_0}_{\int} \otimes \oplus \underbrace{E_{-1}}_{\int} t^{-1} \oplus E_{-2} t^{-2} \cdots$$

$$0 \longrightarrow \mathcal{E}_0 \to \mathcal{E} \to \mathcal{E}_{-1}$$

$$\mathcal{E}'_1$$

$$0 \to t^{-1} \otimes \mathcal{E}'_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \to 0$$

$$0 \to t^{-1} \otimes \mathcal{E}'_2 \to \mathcal{E}'_1 \longrightarrow \mathcal{E}_1 \to 0$$

$$\vdots$$

$$\mathcal{E}'_{\ell(\lambda)-1} \subset \cdots \subset \mathcal{E}'_0 \subset \mathcal{E}$$

and the stability of $\mathcal{E}$ imposes $\mu_n(\mathcal{E}'_i) < \mu_n(\mathcal{E})$

Eventually

$$M^{\delta}_{h}(0)^{\mathbb{C}^*} = \coprod_{\lambda \vdash r} M^{\delta}_{h}(0)_{\lambda}^{\mathbb{C}^*}$$

Proposition Suppose that $r=2$

$$\left[M^{\delta}_{h}(0)^{\mathbb{C}^*}\right]^{vir}_{red} = \left[M^{\delta}_{h}(0)_{(2)}\right]^{vir}_{red} + \left[M^{\delta}_{h}(0)_{1^2}^{\mathbb{C}^*}\right]^{vir}_{red}$$

Monopole branch.

In full generality

$$\left[M^{\delta}_{h}(0)^{\mathbb{C}^*}\right]^{vir}_{red} = \left[M^{\delta}_{h}(0)_{(r)}\right]^{vir}_{red} + \left(\cdots\cdots + [M_{(1,\dots,1)}]^{vir}\right)$$

Instanton branch

all other partitions of r

Theorem The restriction of $\left[M^{\delta}_{h}(0)^{\mathbb{C}^*}\right]^{vir}_{red}$ to $(\underbrace{1,\dots,1}_{r})$

component is called the nested Hilbert scheme $\Longrightarrow$

denoted by $\mathcal{S}^{(n_1,n_2,\dots,n_r)}_{\beta_1,\dots,\beta_{r-1}}$ is identified with virtual class $\left[\mathcal{S}^{\vec{n}}_{\vec{\beta}}\right]^{vir}$ constructed in (Gholampour, –, Yau) 2017

<u>Thm</u> (GSY 2017) assume $\theta=(2,\gamma,m)$ then

residue integ

$$DT^{\mathcal{S}}_h(\theta;\alpha) = DT^{\mathcal{S}}_h(\theta;\alpha)_{Inst} + \sum_{n_1,n_2,\beta} \int_{\mathcal{S}^{n_1,n_2}_{\beta}} \frac{\alpha}{\gamma}$$

when $\beta=0$

$$\int_{\mathcal{S}^{n_1}\times\mathcal{S}^{n_2}}$$

Remark When $\mathcal{S}$ is Fano complete intersection or a K3 surface $\mathcal{M}^{\mathcal{S}}_h(\theta)^{\mathbb{C}^*}_{1,1} \cong \phi$.

What happens for higher rank?

if $r=3$. Richard Thomas shows that (using cosection localization)

<u>Thm</u> Consider a connected component P of the monopole branch. At any closed point $\in$ the maps $\phi : E_i \longrightarrow E_{i+1}$ are both injective and generically surjective on S $\forall\, i = 0, \ldots, r-2$. In particular

if Monopole branch is $\neq \phi$ then $\operatorname{rk} E_0 = \operatorname{rk} E_1 = \cdots = \operatorname{rk} E_{r-1}$

Cor. When r is prime then the only surviving components are (r) and $\underbrace{(1,1,\ldots,1)}_{r}$. Hence in $r=3$ case

We are done!

first interesting case!

in $r=4$ case (4) $(1,1,1,1)$, $(2,2)$

in $r=5$ (5) , $(1,\ldots,1)$

$\gamma = 6$ $\qquad (6) \quad \underbrace{(1,\dots,1)}_{6} \quad \underbrace{(2,2,2)}_{?} \quad \underbrace{(3,3)}_{?}$

Question: How to capture invariant theory for moduli spaces of flags of higher rk sheaves on S.

Remedy (Holomorphic triples and their generalizations)

$$E_1 \xrightarrow{\phi} E_2 \quad \text{together with} \quad \begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \wr\wr & \circlearrowright & \wr\wr \\ E_1' & \xrightarrow{\phi} & E_2' \end{array}$$

<u>Definition</u>

Stability for hol triples. Let $q(m)$ be given by a polynomial of degree at ~~least~~ most 2 with numerical coefficients such that it~~s~~ leading coefficient is positive. A holomorphic torsion free triple of

$(\mathcal{E}_1, \mathcal{E}_2, \phi)$ of type $(\vec{v}_1, \vec{v}_2)$ is $q(m)$ stable iff the following two conditions are satisfied

(1) For all proper nonzero subsheaves $E_2' \subset E_2$ for which ϕ does not factor through E_2' we have

$$r_1 r_2 P_{E_2'} < r_1 \mathrm{rk}(E_2') P_{E_2} + r_1 \mathrm{rk}(E_2') q(m)$$

(2) For all proper subsheaves $E_2' \subset E_2$ for which the map ϕ factors throug

$$r_2 \left(P_{E_2'} + q(m) \right) < \mathrm{rk} E_2' \left(P_{E_2} + q(m) \right)$$

$$r_2 \left(r_1 \cdot P_{E_1'} - r_{E_1'} P_{E_1} + r_{E_1'} \cdot q(m) \right)$$

$$+ r_1 \left(r_2 P_2' - r_2' P_2 - r_2' q(m) \right) \leq 0 < 0$$

Moduli Stack of Hol. triples: There is construction) due to Schmitt $\mathcal{M}^{q}_{\vartheta_1,\vartheta_2}$

Lemma. If $E_1 \xrightarrow{\phi} E_2$ is q-stable then

$$\underset{q_{\text{min}}}{\underbrace{\frac{r_1 r_2}{r_1+r_2}\left(\frac{P_1}{r_1}-\frac{P_2}{r_2}\right)}} \leqslant q \leqslant \underset{q_{+>>}}{\frac{2P_2}{2r_2-r_1}}$$

$q_{\cancel{\text{crit}}}$

<u>Lemma</u> Given a holom triple (E_1,E_2,ϕ) of type $(\vartheta_1,\vartheta_2)$ the $q_{+>>}$-stability implies that the morphism ϕ is injective

<u>Lemma</u> Consider an $\mathcal{L}$-twisted torsion-free hol triple $E_1 \xrightarrow{\phi} E_1\otimes\mathcal{L}$, where $\mathcal{L}$ is a line bundle on S. Then the $\mathcal{L}$-twisted triple is q_{crit}-stable iff the induced split Higgs pair

$$Q_L = m(H.c_1(\mathcal{L})) + \frac{c_1(\mathcal{L})^2}{2}$$

$$E_1 \oplus E_2 \xrightarrow{\begin{pmatrix}0 & 0\\ \phi & 0\end{pmatrix}} (E_1 + E_2)\otimes \mathcal{L}$$

is Higgs stable $\left(q_{crit} = \frac{r_2 P_1 - r_1 P_2}{r_1+r_2} + \frac{r_1 r_2}{r_1+r_2} Q_L \right)$

Higgs stability says that given $G \to G \otimes \mathcal{L}$ the pair is Higgs stable with respect to an ample line bundle $\mathcal{O}_S(1)$ iff

$$\frac{P(G')}{rk(G')} < \frac{P(G)}{rk(G)}$$

for any proper subsheaf G' of G which is invariant under the Higgs map.

Here any torsion-free sheaf $E_1' \oplus E_2'$ is ϕ inv. if $E_1' \to E_2' \otimes \mathcal{L}$ is a subtriple of $E_1 \to E_2 \otimes \mathcal{L}$ the the argument follows easily.

Back to DW invts.

Lemma. Under the equivalence of abelian Categories $\mathrm{Coh}_c(X) \cong \mathrm{Higgs}_{\mathcal{L}}(S)$ the φ_n-stability of $\mathcal{E} \in \mathrm{Coh}_c(X)$ is equivalent to Higgs stability of $q_*\mathcal{E} \to q_*\mathcal{E} \otimes \mathcal{L}$. When $\mathrm{ch}(q_*\mathcal{E}) = (2, \gamma, n) \Rightarrow$ That implies q_{crit} Stability of $E_1 \to E_2 \otimes \mathcal{L}$ where $q_*\mathcal{E} = E_1 \oplus E_2$

We can also generalize and relate Higgs stability of $E_1 \oplus \cdots \oplus E_n \xrightarrow{\begin{pmatrix} 0 & \cdots & & 0 \\ \Phi_1 & 0 & \cdots & \\ 0 & \Phi_2 & \cdots & \\ \cdots & & \Phi_{n-1} & 0 \end{pmatrix}} (E_1 \oplus E_2 \oplus \cdots \oplus E_n) \otimes \mathcal{L}$ to Stability q_{crit} Stability of System of $E_i \xrightarrow{\Phi_i} E_{i+1} \otimes \mathcal{L}$ $i = 1, \ldots, n-1$

Def-obs Thy for $q_{+>>}$-stability. $\left\{ q_{+>>} < \frac{r_1 p_{E_2}}{2r_2 - r_1} \right\}$

Thm: Let us assume the holomorphic triple $E_1 \xrightarrow{\phi} E_2$ ($ch = v_1$, $ch = v_2$) is given such that E_1 is stable sheaf and $K_S \leq 0$. Then $M^{q_{+>>}}_{v_1, v_2}$ is equipped with a perfect def-obs Theory

$$\mathbb{F}^{\bullet} \longrightarrow \mathbb{L}^{\bullet}_{M^{q_{+>>}}_{v_1,v_2}}$$

where

$$F^{\bullet\vee} = \mathrm{Cone}\left(\begin{array}{c} \left(R\mathcal{H}om_{\pi}(\mathcal{E}_1, \mathcal{E}_1) \oplus R\mathcal{H}om_{\pi}(\mathcal{E}_2, \mathcal{E}_2) \right)_0 \\ \downarrow \\ R\mathcal{H}om_{\pi}(\mathcal{E}_1, \mathcal{E}_2) \end{array} \right)$$

Constructed inductively

$$M^{q_{+>>}}_{v_1 v_2} \underset{\Delta_1}{\times} M^{q_{+>>}}_{v_2 v_3} \underset{\Delta_2}{\times} \cdots \underset{\Delta_{n-1}}{\times} M^{q}_{v_{n-1} v_n}$$

More generally on $M^{q_{+>>}}_{\vec{v} = (v_1, v_2, \ldots v_n)}$ when E_1 is stable and S is such that $K_S \leq 0$ we get an obstruction

Theory

$$F^{\bullet\vee} = \mathrm{Cone}\left[\left(\bigoplus_{i=1}^{r} R\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i)\right)_0 \longrightarrow \bigoplus_{i=1}^{r-1} R\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1})\right]$$

Now in order to capture VW invts we can use Mochizuki's wall crossing in the master space by perturbing $q_{\text{ess}} \longrightarrow q_{\text{crit}}$ and compute invts of Tanaka–Thomas from invts of q_{ess}-stab flags.

Remark when the base is a curve (1,2) components are interesting: Donagi - Ein - Lazarsfeld conjecture

↳ Joint work with Michael McBreen

A bit on Construction of abs def-obs Theory:

$\phi_T : \tilde{\mathcal{E}}_{1_T} \to \tilde{\mathcal{E}}_{2_T}$ when $\mathcal{E}_1$ is fixed and $\mathcal{E}_2$ is deforming the obs $(\tilde{\phi}, J) \in \mathrm{Ext}^2_{S\times T}(\mathrm{coker}\,\phi_T, \pi_T^* J \otimes \mathcal{E}_2)$

This is cup product of Illusie's reduced Atiyah class $At_{red}(\phi_T) : \mathrm{Ext}^1_{S\times T}(\mathrm{coker}\,\phi_T, \pi_T^\alpha \mathbb{L}^0 \otimes \mathcal{E}_2)$

and the pull back of $\mathrm{Ext}^1_T(\mathbb{L}^0_a, J)$

Thm : The complex

$$F^\bullet_{rel} = \mathrm{Cone}\left(R\mathcal{H}om_\pi(\mathcal{E}_2, \mathcal{E}_2) \to R\mathcal{H}_\pi(\mathcal{E}_1, \mathcal{E}_2)\right)^\vee$$

$$\cong R\mathcal{H}om_\pi(\mathrm{Coker}\,\phi, \mathcal{E}_2)^\vee[-1]$$

defines a perfect relative obstruction theory for the morphism $\mathcal{M}^{q_{f>>}}_{\vec{v}_1, \vec{v}_2} \to \mathcal{M}_{v_1}$ in other words $F^\bullet_{rel}$ is perfect of amplitude $[-1,0]$ and there is a map α, $\mathbb{F}^{\circ}_{rel} \to \mathbb{L}^{\circ}_{\pi}$ such that $h^{\circ}(\alpha)$ is an isomorphism and $h^{-1}(\alpha)$ is epimorphism.

Now in order to show perfectness it suffices to show $h^{i}\left(Lt^{*} F^{\circ}_{rel}\right)^{\vee}=0$ $\forall i\neq 0,1$ for $t: p \hookrightarrow M^{q,>>}_{\theta_1 U_2}$ where $p=(E_1 \xrightarrow{\phi} E_2)$ is an arbitrary point of the moduli space.

It is easy to see that

$$\cdots \longrightarrow \mathrm{Ext}^{i}_{S}(E_1,E_2) \to h^{i}(Lt^{*}F^{\circ\vee}_{rel}) \to \mathrm{Ext}^{i+1}_{S}(E_2,E_2) \to \cdots$$

All the $\mathrm{Ext}^{i}=0$ for $i\neq -1,0,1,2$

$$h^{-1}(Lt^{*}F^{\circ\vee}_{rel}) = \ker\left(\mathrm{Hom}(E_2,E_2) \to \mathrm{Hom}_{S}(E_1,E_2)\right)$$

Suppose $\mathrm{Hom}_{S}(E_2,E_2) \longrightarrow \mathrm{Hom}_{S}(E_1,E_2)$ is not injective

then there exists a nonzero map $f: E_2 \to E_2$

Such that the composition $E_1 \xrightarrow{f_2} F_2 \xrightarrow{f} E_2$ is zero

f_2 or $\ker(f)$ $\qquad$ $\operatorname{im}(f)$ $\to 0$

Since $\ker(f) \subset E_2$ by $\frac{q}{f_{>>}}$-stability and

$$\frac{P_{\ker f}}{rk(\ker(f))} + \frac{q}{rk(\ker f)} < \frac{P_{E_2}}{rk E_2} + \frac{q}{rk(E_2)}$$

On the other hand

$$\frac{P_{Im(f)}}{rk(Im(f))} + \frac{q}{rk(Im(f))} < \frac{P_{E_2}}{rk E_2} + \frac{q}{rk E_2}$$

①

Now $P_{\ker} = P_{E_2} - P_{Im f}$ and $rk(\ker f) = rk(E_2) - rk(Im f)$

$$\frac{P_{E_2}}{rk(E_2)} + \frac{q_{cm}}{rk(E_2)} < \frac{P_{Im f}}{rk(Im f)} \Longrightarrow$$

②

①,② $\Longrightarrow$ $\frac{q}{rk(Im f)} < 0$ ↯ $\Longrightarrow$ Perfectness follows!