

# Introduction to Quantum Toric Geometry (1st Lecture)

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Based on three papers: (1) The definition of a non-commutative toric variety (2014) (Contemp. Math)

Contemporary Mathematics  
Volume 620, 2014  
<http://dx.doi.org/10.1090/conm/620/12370>

### The definition of a non-commutative toric variety

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and Alberto Verjovsky

ABSTRACT. In this short note, we introduce a new family of non-commutative spaces that we call *non-commutative toric varieties*. We also briefly describe some of their main properties. The main technical tool in this investigation is a natural extension of LVM-theory for the irrational case. This note, aimed at graduate students, primarily consists of a brief survey of the required background for the definition of a non-commutative toric variety, plus a very short announcement of the future results at the end.

# (2) Non-commutative theory indomitable (2020). (Notices of the AMS, Jan 2021)

## High Energy Physics - Theory

[Submitted on 19 Aug 2020]

### Non-commutative Geometry Indomitable

[Ernesto Lupercio](#)

This paper is a very brief and gentle introduction to non-commutative geometry geared primarily towards physicists and geometers. It starts with a brief historical description of the motivation for non-commutative geometry and then goes on to motivate the subject from the point of view of the understanding of local symmetries afforded by the theory of groupoids. The paper ends with a very rapid survey of recent developments and applications such as non-commutative toric geometry, the standard model for particle physics and the study of the Riemann Hypothesis.

Comments: 21 pages, 2 figures, To appear in the Notices of the American Mathematical Society  
Subjects: **High Energy Physics - Theory (hep-th)**; Algebraic Geometry (math.AG); History and Overview (math.HO)  
MSC classes: 58B34, 14M25, 11M55  
Cite as: [arXiv:2008.08529 \[hep-th\]](#)  
(or [arXiv:2008.08529v1 \[hep-th\]](#) for this version)

# (3) Quantum (non-commutative) toric geometry: Foundations (2020).



Mathematics > Symplectic Geometry

[Submitted on 10 Feb 2020]

## Quantum (Non-commutative) Toric Geometry: Foundations

Ludmil Katzarkov, Ernesto Lupercio, Laurent Meersseman, Alberto Verjovsky

In this paper, we will introduce Quantum Toric Varieties which are (non-commutative) generalizations of ordinary toric varieties where all the tori of the classical theory are replaced by quantum tori. Quantum toric geometry is the non-commutative version of the classical theory; it generalizes non-trivially most of the theorems and properties of toric geometry. By considering quantum toric varieties as (non-algebraic) stacks, we define their category and show that it is equivalent to a category of quantum fans. We develop a Quantum Geometric Invariant Theory (QGIT) type construction of Quantum Toric Varieties.

Unlike classical toric varieties, quantum toric varieties admit moduli and we define their moduli spaces, prove that these spaces are orbifolds and, in favorable cases, up to homotopy, they admit a complex structure.

Comments: 93 pages, 4 figures

Subjects: **Symplectic Geometry (math.SG)**; Mathematical Physics (math-ph); Algebraic Geometry (math.AG)

MSC classes: 53D20 (Primary) 81S10, 53D37 (Secondary)

Cite as: arXiv:2002.03876 [math.SG]

(or arXiv:2002.03876v1 [math.SG] for this version)

# Recall the basic structure of a (compact) toric variety (over $\mathbb{C}$ )

- ▶ A classical complex toric variety:

$$X = \overline{\mathbb{T}^n}$$

$$\mathbb{T}^n := \mathbb{C}^* \times \dots \times \mathbb{C}^*$$

complex torus.

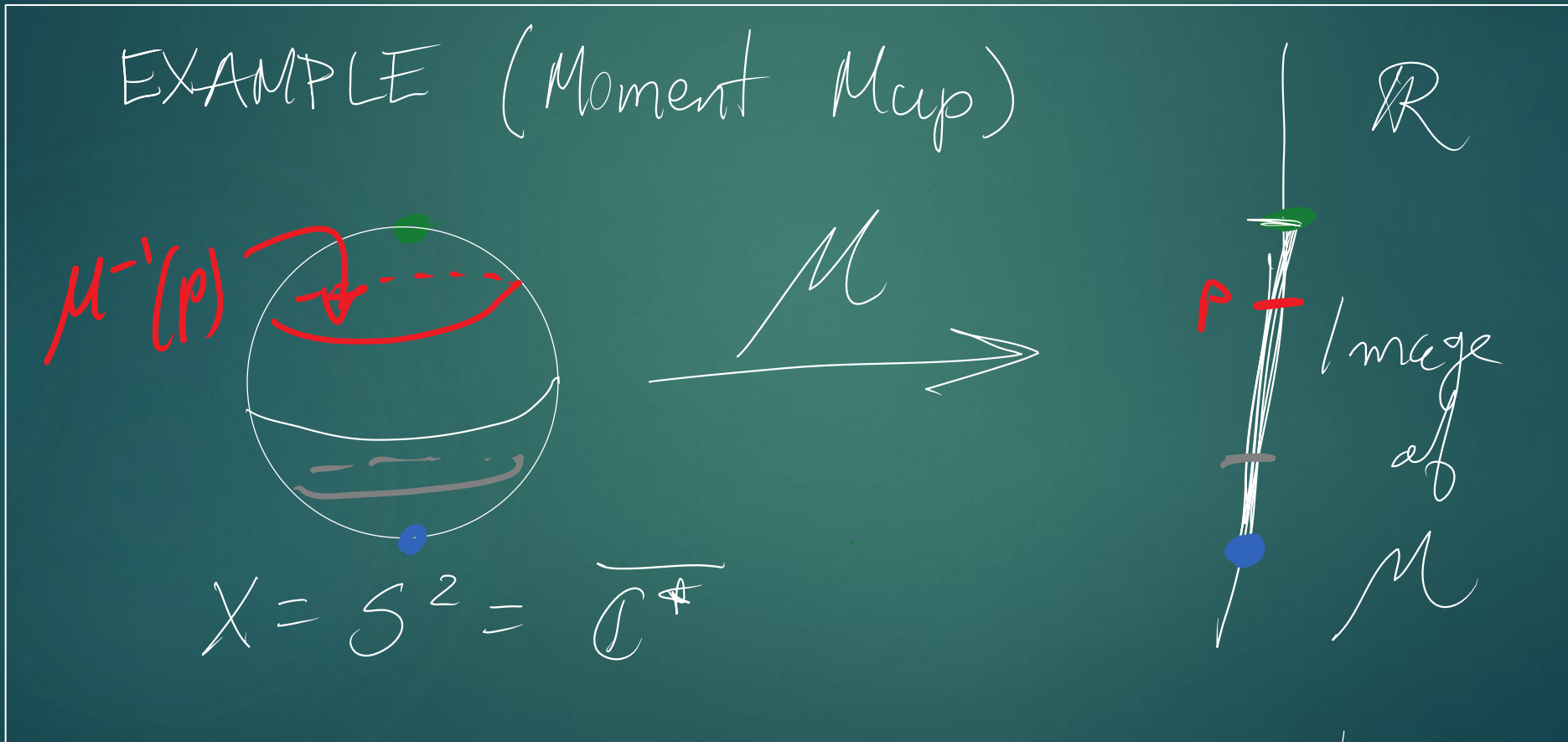
EQUIVARIANT

KÄHLER

COMPACTIFICATION

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

# Recall the basic structure of a (compact) toric variety (over $\mathbb{C}$ )



# In general...

## NON-COMMUTATIVE GEOMETRY INDOMITABLE

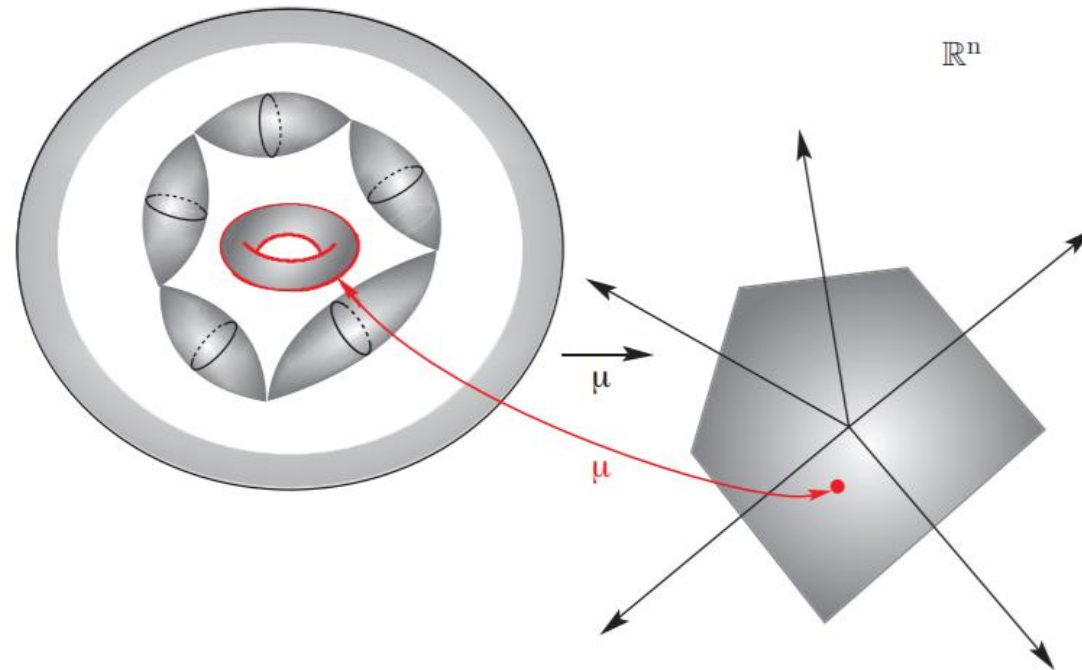


FIGURE 2. The moment map for a toric manifold: the inverse image of every point is a real torus of dimension equal to the dimension of the stratum of  $P$  where the point lands. The inverse image of edges are spheres made up of 1-tori (circles). In

# What are non-commutative spaces? (Gelfand duality)

**Theorem 5.1** (Gelfand Duality). *The categories **Spaces** of Hausdorff compact topological spaces and the opposite category to the category **Algebras** of commutative  $C^*$ -algebras are equivalent. Given a topological space  $X$ , its corresponding algebra is the algebra  $C(X)$  of continuous complex valued functions on  $X$  with pointwise multiplication.*

*Remark 5.2.* Given a category  $\mathcal{C}$ , its opposite category  $\mathcal{C}^{\text{op}}$  has the same objects and the same arrows but  $s$  and  $t$  exchange roles so that, in  $\mathcal{C}^{\text{op}}$ , we have that  $s^{\text{op}} = t$  and  $t^{\text{op}} = s$ .

In classical algebraic geometry, one starts with an affine variety  $X$  and one produces a commutative algebra  $\mathcal{O}(X)$  by taking its regular functions. Then, one can go back to  $X$  by taking the spectrum of maximal ideals of  $\mathcal{O}(X)$ . A similar but more delicate construction works in the case of a topological space  $X$ .



# What are non-commutative spaces?

- ▶ On a commutative manifold,  $M$ , we have:

$$x_1, \dots, x_n : U \longrightarrow \mathbb{R}, \quad U \subseteq M$$

$$x_i \circ x_j = x_j \circ x_i$$

- ▶ On a non-commutative manifold:



$$x_i \circ x_j \neq x_j \circ x_i$$

(in " $\mathcal{O}(U)$ ")

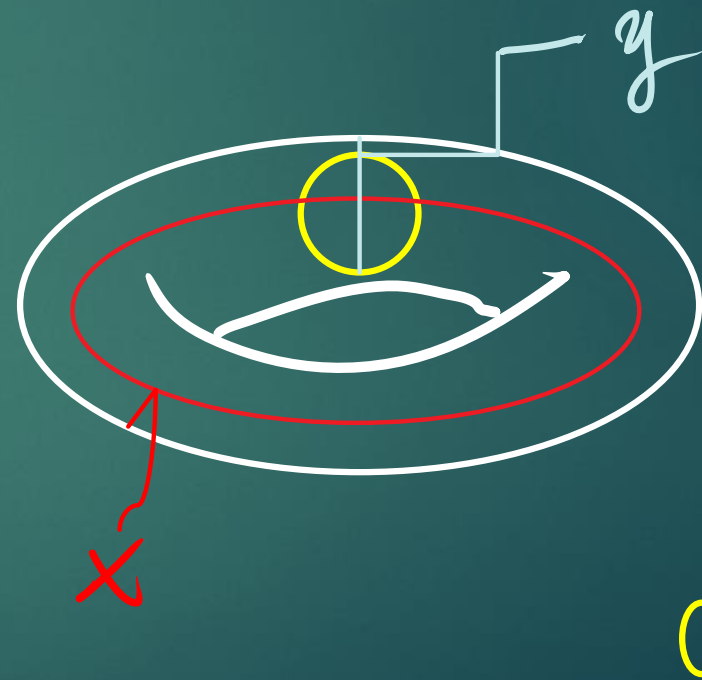
# Usual torus vs. Non-commutative torus aka quantum torus.

► Usual torus:

$$x \cdot y = y \cdot x$$

► Non-commutative torus:

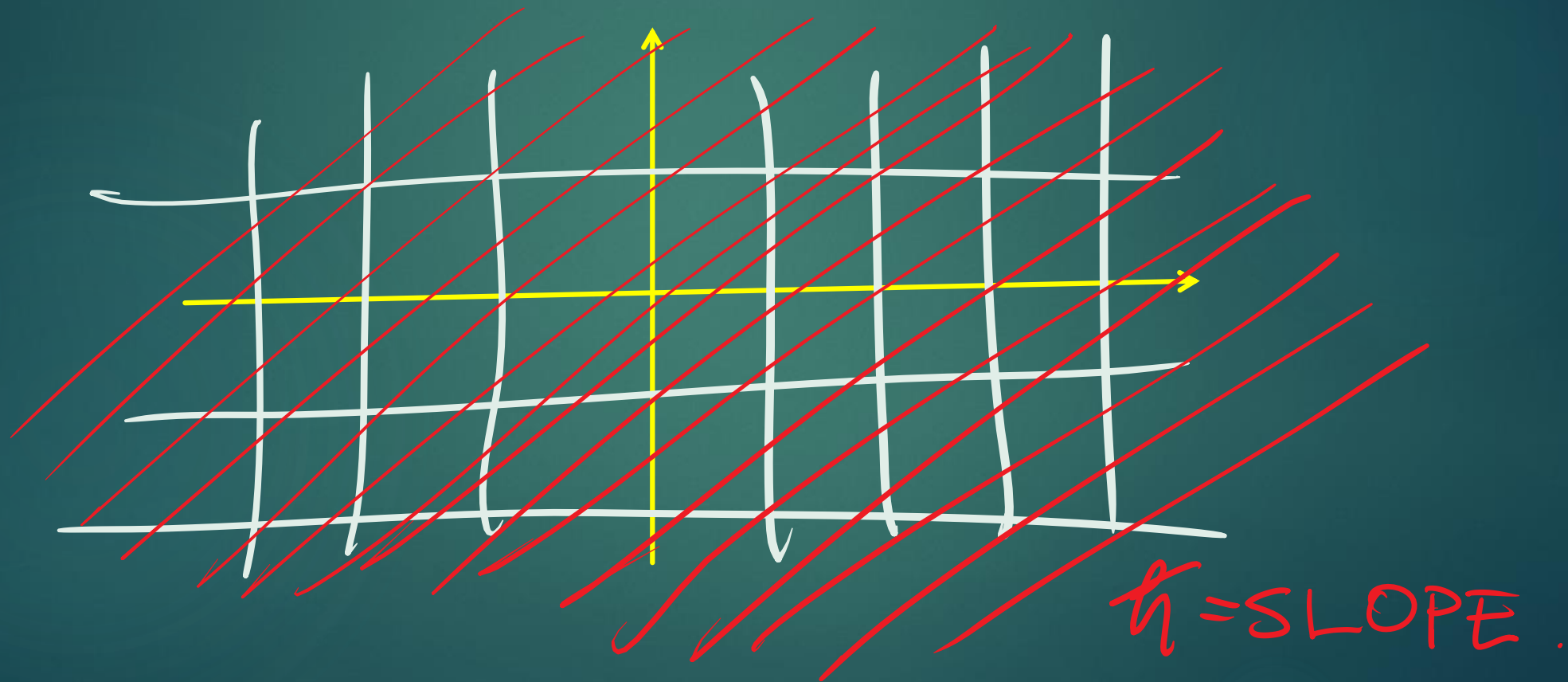
$$x \cdot y = e^{2\pi i k} y \cdot x$$



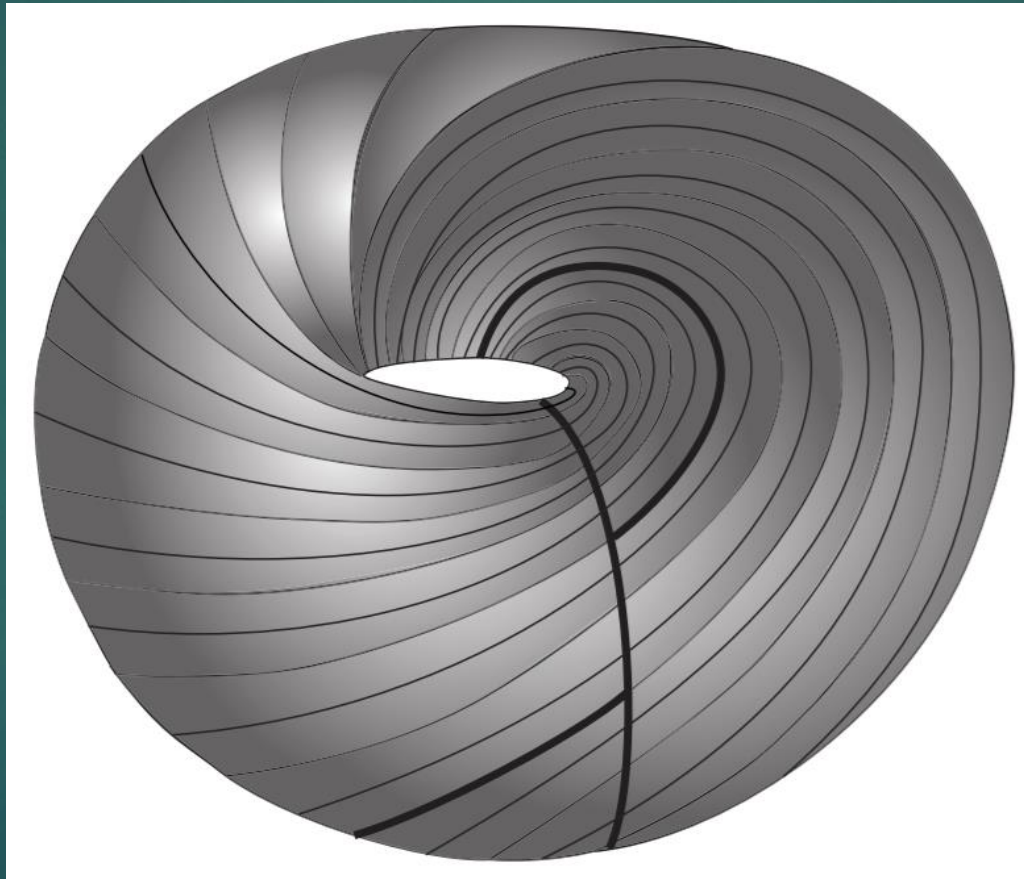
# Non-commutative spaces from nature: bad quotient spaces.

- ▶ A perfectly good topological Hausdorff space can have a non-Hausdorff quotient.
- ▶ This is typical behavior for  $M/G$  with  $G$  non-compact.
- ▶ Take, for example, the Kronecker foliation...

# Constructing the Kronecker foliation.



# The Kronecker foliation.



# Exercise.

- ▶ For RATIONAL slopes, the quotient is Hausdorff, and moreover, it is a circle.
- ▶ For IRRATIONAL slopes, the quotient is non-Hausdorff, no longer a manifold.
- ▶ We can profit from thinking of the second quotient as a **non-commutative space**.

# We need **Groupoids**, objects that generalize groups actions (groups).

**Definition 3.1.** A *category*  $\mathcal{C} = (C_0, C_1, s, t)$  consists of objects  $C_0$  and arrows  $C_1$  together with maps:

- a) Two maps,  $s : C_1 \rightarrow C_0$  and  $t : C_1 \rightarrow C_0$ , called *the source map* and *the target map* so that if for an arrow  $\alpha \in C_1$  we have,  $s(\alpha) = x$  and  $t(\alpha) = y$ , then we write

$$x \xrightarrow{\alpha} y$$

or

$$\alpha : x \longrightarrow y.$$

- b) The *identity-arrow* map

$$i : C_0 \rightarrow C_1$$

assigning to every object  $x \in C_0$  its identity arrow

$$i(x) : x \longrightarrow x,$$

$$1_x := i(x).$$

- c) A composition law for arrows:

$$m : C_1 \times_s C_1 \longrightarrow C_1$$

Here  $C_1 \times_s C_1$  consists of pairs of arrows  $(\alpha, \beta)$  so that  $t(\alpha) = s(\beta)$ . The composition law is only *partially defined*, namely, the domain of  $m$  is not all of  $C_1 \times C_1$  but only the subset  $C_1 \times_s C_1$ . For the composition law, we often write:

$$\beta \circ \alpha := m(\alpha, \beta).$$

# Associativity and commutativity

c) A composition law for arrows:

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Here  $C_1 \times_s C_1$  consists of pairs of arrows  $(\alpha, \beta)$  so that  $t(\alpha) = s(\beta)$ . The composition law is only *partially defined*, namely, the domain of  $m$  is not all of  $C_1 \times C_1$  but only the subset  $C_1 \times_s C_1$ . For the composition law, we often write:

$$\beta \circ \alpha := m(\alpha, \beta).$$

These data satisfy the two strong algebraic conditions:

i) Associativity:

$$\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma,$$

whenever  $s(\alpha) = t(\beta)$  and also  $s(\beta) = t(\gamma)$ . In other words, in the following commutative diagram,  $\delta$  is well defined:

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \uparrow \delta & & \downarrow \beta \\ w & \xleftarrow{\gamma} & z \end{array}$$

ii) Identity: if

$$\alpha : x \longrightarrow y$$

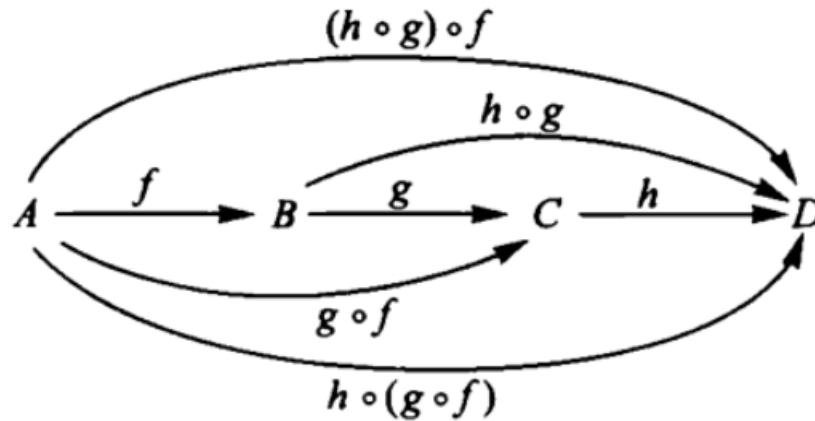
Then

$$\alpha \circ 1_x = 1_y \circ \alpha = \alpha.$$

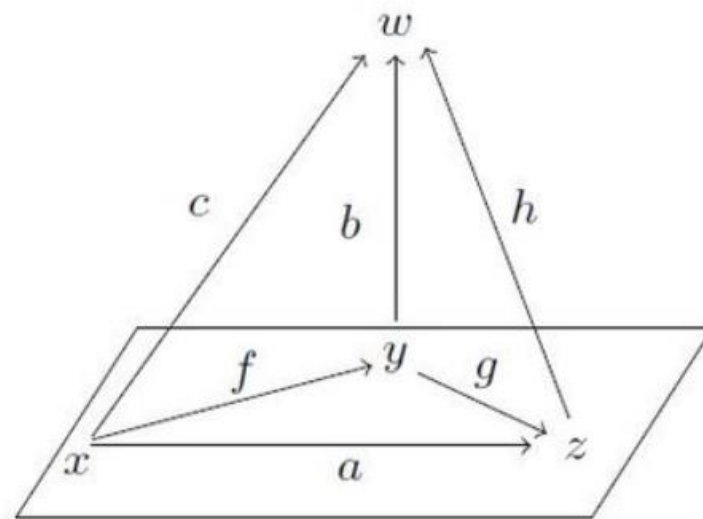
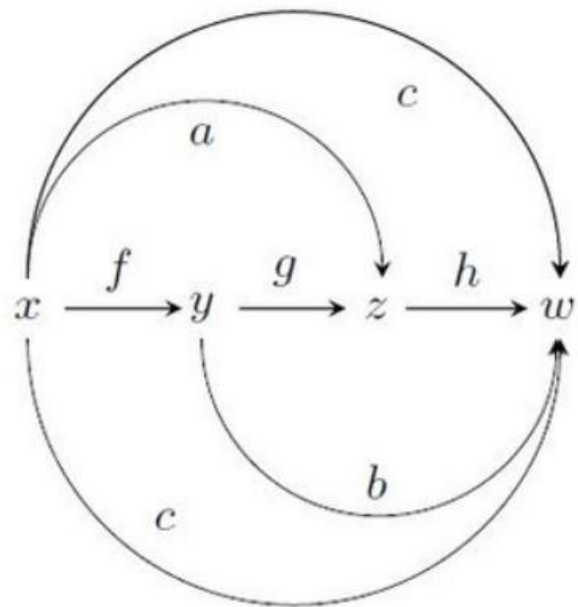


# Associativity (from n-Lab):

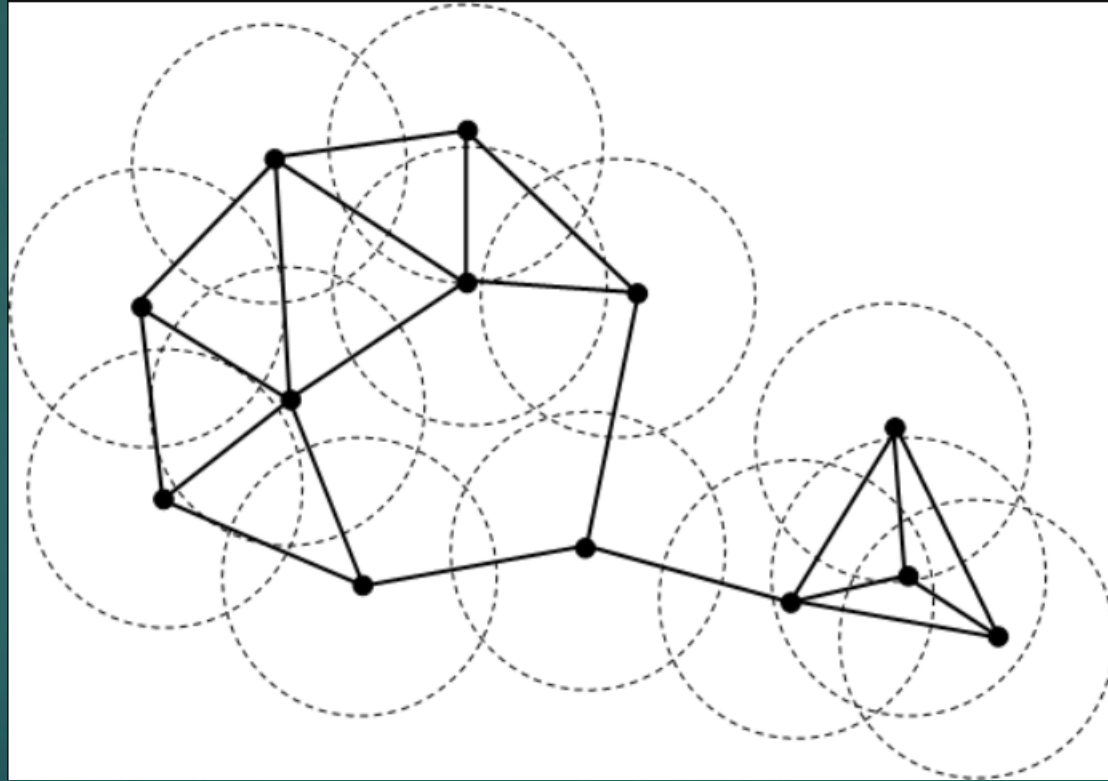
$$h \circ (g \circ f) = (h \circ g) \circ f$$



# Associativity...



# Motivation: Nerve of an open cover...



# Functors.

**Definition 3.2.** Given two categories  $(\mathcal{C}, C_0, C_1, s, t)$  and  $(\mathcal{C}', C'_0, C'_1, s', t')$  a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is a rule assigning objects in  $\mathcal{C}$  to objects in  $\mathcal{C}'$  and arrows in  $\mathcal{C}$  to arrows in  $\mathcal{C}'$  and satisfying  $F \circ s = s' \circ F$ ,  $F \circ t = t' \circ F$ ,  $F \circ i = i' \circ F$  and:

$$F(\alpha \circ \beta) = F(\alpha) \circ F(\beta).$$

# Internal facts... (Logic).

**Definition 3.3.** Given objects  $x, y \in C_0$ , we say that  $x$  is isomorphic to  $y$  in  $\mathcal{C}$  and write  $x \cong y$  if there is an arrow  $\alpha : x \rightarrow y$  together with an inverse arrow  $\alpha^{-1} : y \rightarrow x$ , that is to say, we have:  $\alpha \circ \alpha^{-1} = 1_y$ ,  $\alpha^{-1} \circ \alpha = 1_x$ .

**Example 3.4.** The category  $\mathcal{S} = (S_0, S_1, s, t)$  of all sets where  $S_0$  is the class of all sets and  $S_1$  the class of all mappings of sets. For two sets  $x, y$ , and a mapping:

$$\alpha : x \longrightarrow y,$$

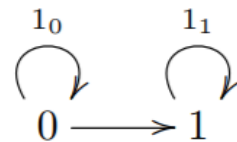
we set  $s(\alpha) = x$  and  $t(\alpha) = y$ . It is immediate to verify all the necessary algebraic conditions, and so,  $\mathcal{S}$  is a category.

This category has many subcategories of importance, for example, the category of all groups  $\mathcal{G} = (G_0, G_1)$ , where  $G_0$  is the class of all groups and  $G_1$  is the class of all group homomorphisms.

# Natural transformations.

Category theory was discovered by S. Eilenberg and S. Mac Lane in the years of 1942-1945 and first appeared fully formed in their 1945 classical paper *General Theory of Natural Equivalences* [11] (very much under the influence of E. Noether, one of Mac Lane's teachers). In this work, the concept of category was mostly auxiliary, for the developments of homological algebra in algebraic topology had motivated Eilenberg and Mac Lane to understand systematically the concept of natural transformation.

Perhaps the best way to think of a natural transformation  $\eta : F \Rightarrow F'$  from a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  to a functor  $F' : \mathcal{A} \rightarrow \mathcal{B}$  is as a homotopy from  $F$  to  $F'$ . To make sense of this, it is useful to define the *unit interval category*  $\mathcal{I}$  as the category having two objects 0 and 1 and three arrows (only one being a non-identity arrow) depicted below:



# Natural transformations as homotopies.

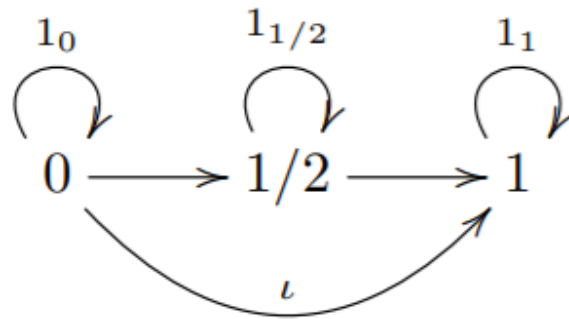
**Definition 3.5.** A *natural transformation*  $\eta$  from  $F$  to  $F'$  is a functor

$$\eta : \mathcal{A} \times \mathcal{J} \longrightarrow \mathcal{B},$$

so that  $F = \eta|_{\mathcal{A} \times 0}$  and  $F' = \eta|_{\mathcal{A} \times 1}$ . When such a natural transformation exists, we write  $\eta : F \Longrightarrow F'$ .

# Natural equivalences (think of homotopies).

It is natural to define the composition (concatenation)  $\eta \circ \eta'$  of natural transformations by considering the *double interval category* which contains  $\mathcal{J}$ :



**Definition 3.6.** We say that  $(\eta : F \rightarrow F', \eta' : F' \rightarrow F)$  are a *natural equivalence* (homotopy equivalence) of categories, and we write  $\mathcal{A} \simeq \mathcal{B}$  if the concatenations  $\eta \circ \eta'$  and  $\eta' \circ \eta$  send  $\iota$  to the identity transformations  $1_F$  and  $1_{F'}$  respectively.



# Equivalent categories are like homotopy equivalent spaces...

have vastly different number of objects. In fact, intuitively, if  $\mathcal{A} \simeq \mathcal{B}$ , then  $\mathcal{B}$  can be obtained from  $\mathcal{A}$  by means of an intermediate category  $\mathcal{C}$ :

$$\mathcal{A} \longleftarrow \mathcal{C} \longrightarrow \mathcal{B}$$

Both arrows induce equivalences, and the left arrow (resp. the right arrow) can be obtained from  $\mathcal{A}$  (resp.  $\mathcal{B}$ ) by deleting objects of  $\mathcal{A}$  (and all arrows starting or ending in the deleted object) (resp. deleting objects of  $\mathcal{B}$ ) making sure that  $\mathcal{C}$  still has, at least, one object in every isomorphism class of objects in  $\mathcal{A}$  (resp.  $\mathcal{B}$ ); the left arrow thins out  $\mathcal{A}$ , and the second arrow fats up  $\mathcal{C}$  to obtain  $\mathcal{B}$ . The diagram above is important, for it is an archetype for non-commutative geometry: we will see this later, when we talk about bi-bundles.

**Example 3.7.** Consider the category  $\mathcal{V}$  of complex  $n$ -dimensional vector spaces together with linear isomorphisms. It is not hard to see that this category is equivalent to the category  $[\bullet/\mathrm{GL}_n(\mathbb{C})]$  which has just one (abstract) object  $\bullet$ , and  $n \times n$  invertible matrices as arrows with multiplication as its composition law. Notice that (by definition) every arrow in both categories has an inverse.

# Groupoids

**Definition 3.8.** A category  $\mathcal{G} = (G_0, G_1, s, t)$  in which for every arrow  $\alpha : x \rightarrow y$  there exists an inverse arrow  $\alpha^{-1} : y \rightarrow x$ , namely an arrow so that:

$$\alpha \circ \alpha^{-1} = 1_y, \quad \alpha^{-1} \circ \alpha = 1_x,$$

is called a *groupoid*.

**Example 3.9.** Every group  $G$  can be made into a groupoid  $[\bullet/G] := (\{\bullet\}, G, s, t)$  (for  $s$  and  $t$  the constant maps  $G \rightarrow \{\bullet\}$ ) by considering the category  $[\bullet/G]$  with one (abstract) object  $\bullet$  and an arrow  $\tilde{g}$  for every element  $g \in G$ . Given two arrows  $\tilde{g} : \bullet \rightarrow \bullet$  and  $\tilde{h} : \bullet \rightarrow \bullet$  (for  $h, g \in G$ ), we define:

$$\tilde{g} \circ \tilde{h} := g \cdot h.$$

**Example 3.10.** Every equivalence relation can be made into a groupoid. Consider a set  $I$  and  $R \subseteq I \times I$  an equivalence relation on  $I$  ( $R$  is the set of pairs  $(i, j)$  so that  $i$  is related to  $j$ ). Then, we can define a groupoid  $[I/R] := (I, R, s, t)$  writing  $s(i, j) := i$ ,  $t(i, j) := j$  and

$$(i, j) \circ (j, k) := (i, k).$$

The verification of the claim that  $[I/R]$  is a groupoid is immediate.

# Group Actions produce Groupoids

**Example 3.11.** Every group action  $G \times M \rightarrow M$  of  $G$  on  $M$  can be made into a *translation groupoid*  $[M/G] := (M, M \times G, s, t)$  by writing  $s(m, g) = m$ ,  $t(m, g) := g \cdot m$  and

$$(gm, h) \circ (m, g) := (m, hg).$$

For the purposes of geometry, it is useful to restrict our attention to small categories (which do not include the category of sets).

# Lie Groupoids

**Definition 3.12.** We say that a category  $\mathcal{C} = (C_0, C_1, s, t)$  is small if both  $C_0$  and  $C_1$  are sets.

**Definition 3.13.** Given an object  $x$  in  $C_0$  for  $\mathcal{C}$  a small category, the set of invertible arrows  $g : x \rightarrow x$  forms a group called *the automorphism group* of  $x$  in  $\mathcal{C}$ .

The main source of non-commutative spaces are groupoids that have a geometric structure, namely, topological and Lie groupoids.

**Definition 3.14.** A topological (resp. Lie) groupoid is a small groupoid  $\mathcal{G} = (G_0, G_1, s, t)$  so that  $G_0$  and  $G_1$  are topological spaces (resp. Hausdorff smooth manifolds) and all structure maps  $s, t, m, i$  are continuous (resp. smooth).

# Étale Groupoids.

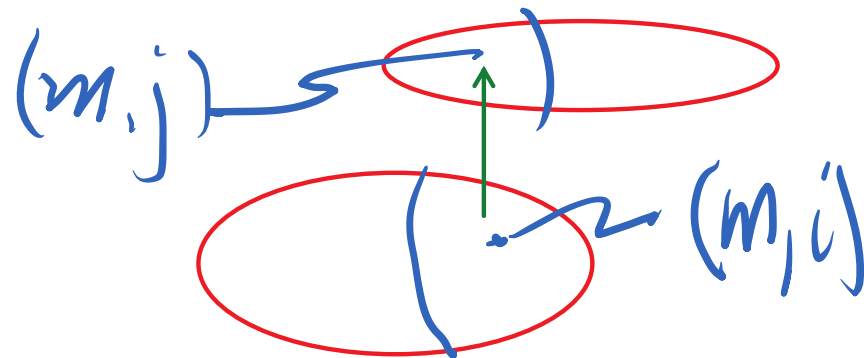
**Definition 3.16.** We say that a smooth map of manifolds  $f : M \rightarrow N$  is *étale* if it is a local diffeomorphism; that is to say  $f$  is both a submersion and an immersion. We say that  $\mathcal{G} = (G_0, G_1, s, t)$  is an *étale* Lie groupoid if  $s$  is étale.

In fact, the main examples that we will consider in this note (foliation groupoids) can be made to be étale [10, 17] (e.g. the non-commutative torus below).

**Example 3.17.** A Lie groupoid  $\mathcal{G} := [M/G]$  (usually called a *translation groupoid*) is étale whenever  $G$  is discrete.

**Example 3.18.** A choice of an atlas  $(U_i)_i$  for a manifold  $M$ , gives rise to an étale groupoid  $\mathcal{U} := (\amalg_i U_i, \amalg_{(i,j)} U_{ij}, s, t)$ , where

- $\amalg_i U_i := \{(m, i) \mid m \in U_i\}$ ,
- $\amalg_{(i,j)} U_{ij} := \{(m, i, j) \mid m \in U_i \cap U_j\}$ ,
- $s(m, i, j) := (m, i)$ ,
- $t(m, i, j) := (m, j)$ ,
- $(m, j, k) \circ (m, i, j) := (m, i, k)$ .



# Morita equivalence.

We need a geometric version of the equivalence of groupoids that corresponds to the equivalence of categories of the previous section:

**Definition 3.19.** Given two Lie groupoids  $\mathcal{H} = (H_0, H_1, s, t)$  and  $\mathcal{G} = (G_0, G_1, s, t)$ , a morphism  $\phi_i : H_i \rightarrow G_i$ ,  $i = 0, 1$ , is an *essential equivalence* if

i)  $\phi$  induces a surjective submersion  $(y, g) \mapsto t(g)$  from

$$H_0 \times_{G_0} G_1 = \{(y, g) \mid \phi(y) = s(g)\}$$

onto  $H_0$ ; and

ii)  $\phi$  induces a diffeomorphism  $h \mapsto (s(h)\phi(h), t(h))$  from  $H_1$  to the pullback  $H_0 \times_{G_0} G_1 \times_{G_0} H_0$ .

# Finally... Stacks!

We say that two Lie groupoids  $\mathcal{G}'$  and  $\mathcal{G}$  are *Morita equivalent* if there exists a Lie groupoid  $\mathcal{H}$  and two essential equivalences  $\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'$  (and we will say that  $\mathcal{H}$  is a  $\mathcal{G}$ - $\mathcal{G}'$ -bi-bundle). The equivalence class  $\bar{\mathcal{G}}$  of the groupoid  $\mathcal{G}$  under Morita equivalence is called the  $C^\infty$ -*stack associated to  $\mathcal{G}$* .

**Example 3.20.** Given a fixed manifold  $M$  and two atlases  $(U_i)$  and  $(V_j)$ , then the two associated étale groupoids  $\mathcal{U}$  and  $\mathcal{V}$  are Morita equivalent if and only if the atlases are equivalent in the atlas sense. Thus,  $M$  itself is the stack associated to  $\mathcal{U}$  (and  $\mathcal{V}$ ):

$$M \cong \bar{\mathcal{U}} \cong \bar{\mathcal{V}}.$$

# Stacks associated to foliations. (Think of the Kronecker foliation).

**Example 3.21.** Consider a foliated manifold  $(M, \mathcal{F})$  with  $q$  the codimension of the foliation. The holonomy (or foliation) groupoid  $\mathcal{H} = \text{Holo}(M, \mathcal{F})$  has as objects  $H_0 = M$ , and two objects  $x, y$  in  $M$  are connected by an arrow if and only if they belong to the same leaf  $L$ ; arrows from  $x$  to  $y$  are in correspondance to homotopy classes of paths lying on  $L$  starting at  $x$  and ending at  $y$ . The foliation groupoid  $\mathcal{H} = \text{Holo}(M, \mathcal{F})$  is always Morita equivalent to an étale groupoid for if we take an embedded  $q$ -dimensional transversal manifold  $T$  to the foliation that hits each leaf at least once then the restricted groupoid  $\mathcal{H}|_T$  is an étale groupoid, and, moreover, it is Morita equivalent to  $\mathcal{H} = \text{Holo}(M, \mathcal{F})$  [17].

[17] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR 2012261



# Convolution algebras.

It is time to explain how to obtain a non-commutative algebra out of a groupoid.

**Definition 4.1.** Given an étale groupoid  $\mathcal{G}$ , we associate to it a non-commutative algebra  $A_{\mathcal{G}}$ , the *convolution algebra* of  $\mathcal{G}$ ; its elements are compactly supported smooth complex valued functions on the manifold  $G_1$  of arrows of  $\mathcal{G}$ ,  $f: G_1 \rightarrow \mathbb{C}$ . The *convolution product*  $f * g$  of two functions is given by:

$$(f * g)(\alpha) = \sum_{\beta \circ \gamma = \alpha} f(\beta)g(\gamma),$$

where the sum is well defined because it ranges over a discrete space ( $\mathcal{G}$  is étale) and finite because the functions are required to be compactly supported. The algebra  $A_{\mathcal{G}}$  can be made into a  $C^*$ -algebra. In general,  $A_{\mathcal{G}}$  is a non-commutative algebra.

# Examples.

**Example 4.2.** Consider a (discrete) group  $G$ , the convolution algebra of the groupoid  $[\bullet/G]$  is exactly the same as the group algebra of  $G$ .

**Example 4.3.** Consider now the Heisenberg groupoid  $[I/I \times I]$  from matrix mechanics. Its convolution algebra is a matrix algebra:

$$A_{[I/I \times I]} \cong \text{Mat}_{n \times n}(\mathbb{C}),$$

where  $n$  is the cardinality of  $I$ .

# 2-categories

The category of all categories is actually a *2-category*: it has objects, and for every pair of objects  $x, y$ , the family of arrows going from  $x$  to  $y$  is itself a category. An arrow  $\eta : \alpha \rightarrow \beta$  between arrows is referred to as a 2-arrow.

There are two 2-categories that are of great importance in non-commutative geometry: the 2-category of groupoids and the 2-category of algebras. Due to space considerations, I am all but ignoring the analytical issues concerning  $C^*$ -algebras, which is too bad for it is a very important ingredient in the field; in any case, we will be working only at the formal level from now on.

# The 2-category of groupoids.

- G1) Objects: groupoids.
- G2) Arrows: (smooth) functors.
- G3) 2-arrows: natural transformations.

# The 2-category of non-commutative algebras.

- A1) Objects: associative (possibly non-commutative) algebras.
- A2) Arrows: bimodules over algebras.
- A3) 2-arrows: bimodule morphisms.

# Morita equivalence of algebras.

Observe that a morphism  $A \rightarrow B$  of algebras is not an algebra homomorphism but rather a bi-module  ${}_A M_B$ . The composition of two arrows (bimodules) is given by:

$${}_B M_C \circ {}_A M_B := {}_A M_B \otimes {}_B M_C.$$

The notion of isomorphism of algebras in this category is called Morita equivalence of algebras.

**Definition 4.4.** Two algebras  $A$  and  $B$  are *Morita equivalent* iff there is an  $A$ - $B$ -bimodule  $M$ , and a  $B$ - $A$ -bimodule  $N$  so that  $M \otimes_B N \cong A$  (as  $A$ - $A$ -bimodules), and  $N \otimes_A M \cong B$  (as  $B$ - $B$ -bimodules). Equivalently,  $A$  and  $B$  are Morita equivalent if and only if their categories of modules  $A\text{-Mod}$  and  $B\text{-Mod}$  are equivalent.

**Example 4.5.** Two commutative algebras are Morita equivalent iff they are isomorphic.

# Non-commutative spaces from Stacks.

The important point [18] is that there is a convolution 2-functor:

$$\mathbf{Groupoids} \longrightarrow \mathbf{NCAlgebras},$$

that, when restricted to objects, sends  $\mathcal{G}$  to its convolution algebra  $A_{\mathcal{G}}$ .

This implies immediately that (for étale groupoids) if the groupoid  $\mathcal{G}$  is Morita equivalent to  $\mathcal{G}'$  (as groupoids), then the algebra  $A_{\mathcal{G}}$  is Morita equivalent to  $A_{\mathcal{G}'}$  (as algebras): the Morita equivalence class  $\bar{A}_{\mathcal{G}}$  only depends on the stack  $\bar{\mathcal{G}}$  and not on the groupoid. But two completely different stacks could have the same convolution algebra.

# Examples.

**Example 4.6.** Given a compact manifold  $M$  and an atlas  $(U_i)$ , the (non-commutative) convolution algebra  $A_{\mathcal{U}}$  of the groupoid  $\mathcal{U}$  associated to the atlas is Morita equivalent to  $C(M)$  the algebra of smooth complex valued functions on  $M$  which is commutative.

**Example 4.7.** Consider the groupoids  $\mathcal{G}_1 = [\bullet/\mathbb{Z}]$  and  $\mathcal{G}_2 = [\mathbb{Z}/\{1\}]$ . The first one is connected, while the second has infinitely many components; therefore, the groupoids  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are not Morita equivalent; nevertheless the Fourier transform  $\mathcal{F}: A_{\mathcal{G}_1} \rightarrow A_{\mathcal{G}_2}$  is an isomorphism and, therefore, a Morita equivalence. This shows that the convolution 2-functor forgets information. This is a feature rather than a bug in non-commutative geometry.

**Example 4.8.** The Heisenberg groupoid  $[I/I \times I]$  is Morita equivalent to the trivial groupoid  $[\bullet/\{1\}]$ ; therefore, the non-commutative matrix algebra  $\text{Mat}_{n \times n}(\mathbb{C})$  is Morita equivalent to the 1-dimensional commutative algebra  $\mathbb{C}$ .



# The category of non-commutative spaces and its relation to Stacks.

*Remark 5.3.* The category **Algebras** is the same as the category  $\mathbf{Algebras}/\sim_M$  where we have inverted Morita equivalences as two commutative algebras are Morita equivalent iff they are isomorphic.

We are finally ready to define non-commutative spaces.

**Definition 5.4.** The category of non-commutative spaces **NCSpaces** is the opposite to the category  $\mathbf{NCAlgebras}/\sim_M$  of possibly non-commutative algebras up to Morita equivalence.

This definition extends Gelfand duality into the non-commutative realm:

$$\begin{array}{ccc} \mathbf{Spaces} & \xleftarrow{\cong} & \mathbf{Algebras}^{\text{op}} \\ \downarrow & & \downarrow \\ \mathbf{NCSpaces} & \xleftarrow{\cong} & (\mathbf{NCAlgebras}/\sim_M)^{\text{op}} \end{array}$$

Also, the convolution functor becomes a well defined functor:

$$\mathbf{Stacks} \longrightarrow \mathbf{NCSpaces}.$$

In fact, we have:

$$\begin{array}{ccc} \mathbf{Groupoids} & \xrightarrow{\mathcal{C}} & \mathbf{NCAlgebras} \\ \downarrow & & \downarrow \\ \mathbf{Stacks} & \xrightarrow{\mathcal{C}} & \mathbf{NCSpaces} \end{array}$$

where  $\mathbf{Stacks} \cong \mathbf{Groupoids}/\sim_M$  and  $\mathbf{NCSpaces} \cong \mathbf{NCAlgebras}/\sim_M$ .

# Non-commutative rational topology.

**Definition 6.1.** The Hochschild complex  $C_\bullet(A, A)$  of  $A$  is a negatively graded complex (we will have all differentials of degree +1):

$$\xrightarrow{\partial} A \otimes A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\partial} A,$$

where  $A^{\otimes k}$  lives on degree  $-k + 1$ . The differential  $\partial$  is given by

$$\begin{aligned} \partial(a_0 \otimes \cdots \otimes a_n) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \\ &+ \dots + (-1)^{n-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The terms of this formula are meant to be written cyclically:

$$(6.1) \quad \begin{array}{ccc} & & a_0 \\ & \otimes & \otimes \\ a_n & & a_1 \\ \otimes & & \otimes \\ \vdots & & \vdots \\ & \otimes & \otimes \\ & & a_i \end{array}$$

for  $a_0 \otimes \cdots \otimes a_n$ . It is immediate to check that  $\partial^2 = 0$ . We write

$$HH(A, A) := \text{Ker } \partial / \text{Im } \partial.$$

# Hochschild-Konstant-Rosenberg

We can interpret the homology of the Hochschild complex in terms of homological algebra:

$$HH(A, A) = \mathrm{Tor}_{\bullet}^{A \otimes_k A^{\mathrm{op}} - \mathrm{mod}}(A, A).$$

It is an idea of A. Connes that, in non-commutative geometry, the Hochschild homology of  $A$  can be interpreted as the complex of differential forms:

**Theorem 6.2** (Hochschild-Konstant-Rosenberg, 1961, [13]). *Let  $X$  be a smooth affine algebraic variety, then if  $A = \mathcal{O}(X)$ , we have:*

$$HH_i(X) := H^{-i}(C_{\bullet}(A, A); \partial) \cong \Omega^i(X)$$

where  $\Omega^i(X)$  is the space of  $i$ -forms on  $X$ .

*Proof.* Write the diagonal embedding  $X \xrightarrow{\Delta} X \times X$  and, because the normal bundle of  $\Delta$  is the tangent bundle of  $X$ , we have:

$$HH_{\bullet}(X) = \mathrm{Tor}_{\bullet}^{\mathrm{Quasi-coherent}(X \times X)}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}).$$

A local calculation finishes the proof. □

The Hochschild-Konstant-Rosenberg theorem allows us to interpret  $HH_i(A)$  as the space of differential forms of degree  $i$  on a non-commutative space.

# Connes B differential.

Whenever  $A$  is non-commutative, we have:

$$H^0(C_\bullet(A, A); \partial) = A/[A, A].$$

In the commutative case  $A = \mathcal{O}(X)$ , to an element  $a_0 \otimes \cdots \otimes a_n$  in  $C_\bullet(A, A)$ , the corresponding differential form is:  $\frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n$ .

It is convenient to mention a reduced version of the complex  $C_\bullet^{\text{red}}(A, A)$  that computes the same cohomology; it is obtained by reducing modulo constants all terms but the first:

$$\longrightarrow A \otimes A/(\mathbf{k} \cdot 1) \otimes A/(\mathbf{k} \cdot 1) \longrightarrow A \otimes A/(\mathbf{k} \cdot 1) \longrightarrow A.$$

Alain Connes' observed that we can write a formula for an additional differential  $B$  on  $C_\bullet(A, A)$  of degree  $-1$ , inducing a differential on  $HH_\bullet(A)$  that is meant to be the de Rham differential:

$$B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma} (-1)^{\sigma} 1 \otimes a_{\sigma(0)} \otimes \cdots \otimes a_{\sigma(n)}$$

# Cyclic Homology

$$B^2 = 0, \quad B\partial + \partial B = 0, \quad \partial^2 = 0,$$

this we write as:

$$\dots \begin{array}{c} \xleftarrow{B} \\ \xrightarrow{\partial} \end{array} A \otimes A/1 \otimes A/1 \begin{array}{c} \xleftarrow{B} \\ \xrightarrow{\partial} \end{array} A \otimes A/1 \begin{array}{c} \xleftarrow{B} \\ \xrightarrow{\partial} \end{array} A$$

and by computing the cohomology, this gives us a complex  $(\text{Ker } \partial / \text{Im } \partial; B)$ . A naive definition on the de Rham cohomology in this context is the homology of this complex  $\text{Ker } B / \text{Im } B$ .

# Periodic cyclic homology

We can improve this by considering the negative cyclic complex  $C_{\bullet}^{-}(A)$ , which is a projective limit (here  $u$  is just a formal variable of degree  $\deg(u) = +2$ ):

$$C_{\bullet}^{-} := (C_{\bullet}^{\text{red}}(A, A)[[u]]; \partial + uB) = \varprojlim_N (C_{\bullet}^{\text{red}}(A, A)[u]/u^N; \partial + uB).$$

**Definition 6.3.** The *periodic complex* is defined as the inductive limit:

$$C_{\bullet}^{\text{per}} := (C_{\bullet}^{\text{red}}(A, A)((u)); \partial + uB) = \varinjlim_i (u^{-i} C_{\bullet}^{\text{red}}(A, A)[[u]]; \partial + uB).$$

It is a  $\mathbf{k}((u))$ -module, and this implies that multiplication by  $u$  induces a kind of Bott periodicity. The resulting cohomology groups called (even, odd) periodic cyclic homology and are written (respectively):

$$HP_{\text{even}}(A), \quad HP_{\text{odd}}(A).$$

This is the desired replacement for de Rham cohomology.

# Examples.

For example, when  $A = C^\infty(X)$  is considered with its nuclear Fréchet algebra structure, and taking  $\otimes$  to be the topological tensor product, then we obtain the canonical isomorphisms:

$$HP_{\text{even}}(A) \cong H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \oplus \dots$$

$$HP_{\text{odd}}(A) \cong H^1(X, \mathbb{C}) \oplus H^3(X, \mathbb{C}) \oplus \dots$$

**Theorem 6.4** (Connes, [5], cf. Feigin-Tsygan, [12]). *If  $X$  is a possibly singular affine algebraic variety and  $X_{\text{top}}$  is its underlying topological space then:*

$$HP_{\text{even}}(A) \cong H^{\text{even}}(X_{\text{top}}, \mathbb{C})$$

and

$$HP_{\text{odd}}(A) \cong H^{\text{odd}}(X_{\text{top}}, \mathbb{C})$$

*and these homologies are finite-dimensional.*

As expected, whenever  $A$  is Morita equivalent to  $B$ , then  $HP_\bullet(A) \cong HP_\bullet(B)$ ; in other words,  $HP_\bullet(A)$  only depends on the non-commutative space represented by  $A$ .

# The non-commutative torus.

- ▶ As a non-commutative algebra:

The quantum 2-torus  $\mathcal{T}_{\hbar}^2 \in \mathbf{NCSpaces} \cong \mathbf{NCAlgebras}/\sim_M$  corresponds under Gelfand duality to the algebra  $A_{\hbar}$  generated by two (periodic) generators  $X, Y$  that don't commute but rather satisfy the relation:

$$XY = e^{2\pi i \hbar} YX.$$

- ▶ This is the exponential of the classical Born-Heisenberg-Jordan relation in classical quantum mechanics.



# Quantum Torus

The algebra  $A_{\hbar}$  is only truly non-commutative when  $\hbar$  is irrational; When  $\hbar$  is rational, while  $A_{\hbar}$  is non-commutative on the nose (except for  $\hbar = 0$ ), it is, in reality, Morita equivalent to the commutative algebra of an ordinary torus ( $XY = YX$ ).

**Theorem 7.1** (Alain Connes [4], cf. Marc Rieffel, [20]).  *$A_{\hbar}$  is Morita equivalent to  $A_{\hbar'}$  if and only if:*

$$\hbar' = \frac{a\hbar + b}{c\hbar + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

One can also prove that:

$$HP_{\mathrm{even}}(A_{\hbar}) = H^0(T^2) \oplus H^2(T^2),$$

and

$$HP_{\mathrm{odd}}(A_{\hbar}) = H^1(T^2, \mathbb{C}).$$

# The Quantum Torus and the Kronecker Foliation.

It is a beautiful discovery of Connes [9] that the non-commutative torus can be thought as the non-commutative space that models the space of leaves of the Kronecker foliation. The universal covering of the classical torus is the Euclidean plane, by taking the foliation of all lines of slope  $\hbar$  on the plane and projecting it into the torus by the covering map, we obtain the Kronecker foliation of slope  $\hbar$  on  $T^2$  (Figure 1). By taking a vertical transversal circle to the foliation, it is easy to see that the holonomy groupoid of this foliation is  $[S^1 / \langle \rho_{\hbar} \rangle]$  where  $\rho_{\hbar}$  acts on  $S^1$  by a rotation of angle  $\hbar$  (cf. the thick line in Figure 1). In section 6 of [9], it is shown that the convolution algebra of  $[S^1 / \langle \rho_{\hbar} \rangle]$  is  $A_{\hbar}$  (it is a nice exercise using Fourier series that the interested reader may try).

# GO TO LECTURE 2

▶ END OF LECTURE 1

# PREVIEW OF LECTURE 2

▶ [GO TO LECTURE 2](#)

# Quantum Toric Geometry (joint with Katzarkov, Meersseman and Verjovsky.

Classical  $n$ -complex dimensional compact, projective Kähler toric manifolds  $X$  are defined as equivariant, projective compactifications of the  $n$ -complex dimensional torus  $\mathbb{T}_{\mathbb{C}}^n := \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ :

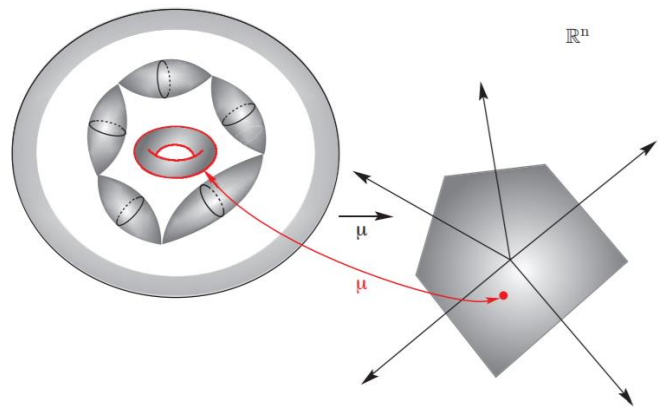
$$X := \overline{\mathbb{T}_{\mathbb{C}}^n}.$$

An interesting question, even from a classical point of view, would be: How to define a meaningful moduli space of toric manifolds? The main problem being that toric manifolds are rigid as equivariant objects. Non-commutative geometry helps elucidate this question in a surprising beautiful way.

# The moment map of a toric variety is made up of tori.

$$\mu: X \longrightarrow P \subset \mathbb{R}^d \cong \text{LieAlgebra}(\mathbb{T}_{\mathbb{R}}^d)^*.$$

For a toric variety  $X$ ,  $P$  happens to be a convex, rational, Delzant polytope: in other words, the combinatorial dual of  $P$  is a triangulation of the sphere  $S^{d-1}$ , and all the slopes of all the edges of  $P$  are rational. By taking cones over the origin of the dual to the polytope, we get the *fan* associated to the toric manifold



# The gist of quantum toric geometry.

In [15], classical toric geometry is generalized: *by replacing all the classical tori in toric geometry for non-commutative tori, one can obtain non-commutative toric varieties.* Now the (possibly irrational) fan (or possibly irrational polytope) no longer lives in  $\mathbb{Q}^n$ , but rather lives in (a possibly irrational) quantum lattice  $\Gamma \subset \mathbb{R}^n$  ( $\Gamma$  is a finitely generated possibly dense Abelian subgroup of  $\mathbb{R}^n$  as it may have more than  $n$  generators over  $\mathbb{Z}$ ).

- [15] Ludmil Katzarkov, Ernesto Lupercio, Laurent Meersseman, and Alberto Verjovsky, *Quantum (non-commutative) toric geometry: Foundations*, arXiv preprint arXiv:2002.03876 (2020).

# The moduli space.

Then, a moduli space of toric varieties  $\mathcal{M}$  can be defined (fixing the combinatorics of the polytope or fan). In a large family of favorable cases the moduli space  $\mathcal{M}$  is a complex orbifold: its rational points are precisely the classical toric varieties, and *its irrational points are precisely the truly non-commutative toric varieties*. Non-commutative geometry is precisely what is needed to define a nice moduli space of toric varieties.

Just as classical toric geometry has been used in the solution of multiple problems in geometry, physics and combinatorics, non-commutative toric geometry allows many of these solutions to generalize to wider settings.