Stability conditions and spectral networks

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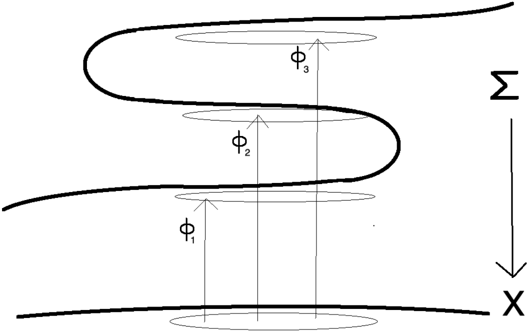
Joint work with Fabian Haiden, Ludmil Katzarkov, and Pranav Pandit

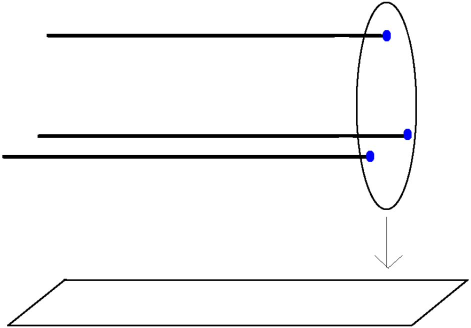
Consider a spectral curve $\Sigma \subset T^*X$ viewed as a multivalued differential 1-form that is locally the collection $\varphi_1, \ldots, \varphi_r$.



For us, it is going to be given by the equation

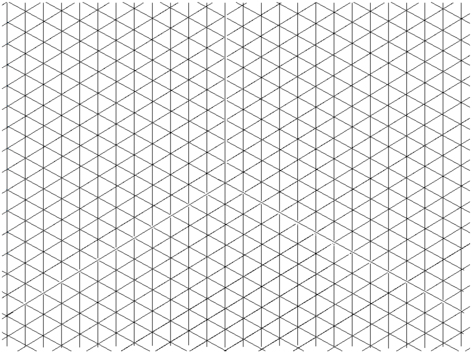
$$p^3 = (dz)^3$$

with solutions $\varphi_i = \omega^i dz$, $\omega = \frac{-1+\sqrt{-3}}{2}$

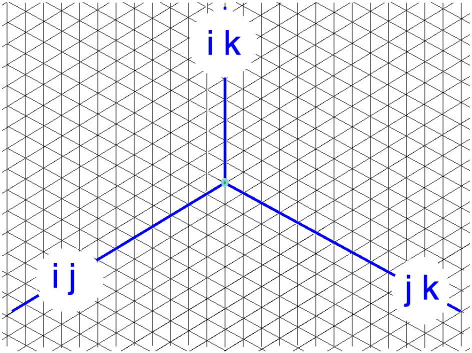


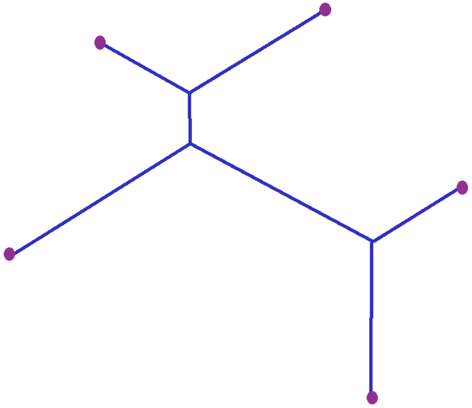
The differentials $\varphi_{ij} = \varphi_i - \varphi_j$ define real foliations $\Re\varphi_{ij} = 0$.

In our case there are three foliations whose leaves make angles of 60°.



Gaiotto-Moore-Neitzke define a *spectral network* to be a graph on X whose edges follow the leaves of these foliations, with appropriate labels, and allowing *collisions* :





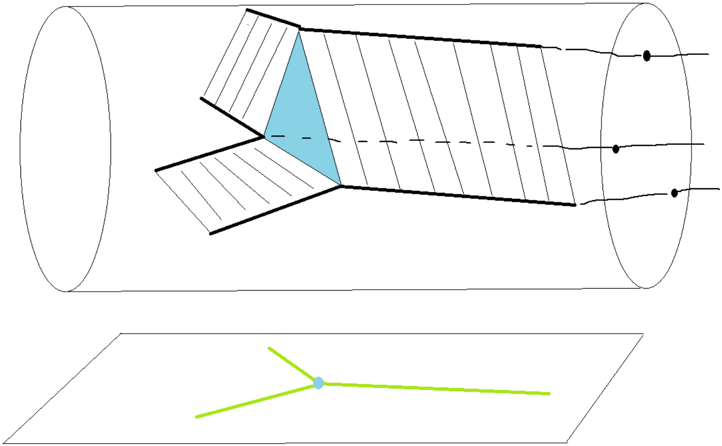
A spectral network with 3 collisions

Kontsevich program : interpret spectral networks as special Lagrangians.
The notion of *special Lagrangian* is defined when we have a holomorphic form of top degree.
In our case, it is the tautological 2-form on T^*X.

A second aspect : we take a limit of rescaling the symplectic form on the fibers (Kontsevich, StringMath-2016).
Then, special Lagrangians look locally like special Lagrangians in the fibers, over special Lagrangians in the base.

We look at Lagrangians with boundary on Σ.

Locally on a spectral network away from a collision, over an edge labeled ij we put the segment from the i-th to the j-th point in the spectral covering.



The Lagrangian upstairs

The limiting theory becomes the Fukaya category of Lagrangians in X with coefficients in the Fukaya-Seidel category of the fiber.

In the spectral curve case the fiber is $(\mathbb{C}, n \text{ points})$.

For us $n = 3$, the fiber category is therefore A_2, the category of representations of the "quiver" with a single arrow.

Furthermore, we are going to restrict to a special case where the base is just the complex plane, relative to a finite collection of boundary points

$$X = (\mathbb{C}, \{p_1, \ldots, p_k\})$$

Fukaya categories with coefficients

Suppose we are given a fiber category $\mathscr{D}$ that is a dg-category.
Suppose we are given a graph $G \subset X$ with endpoints on the given subset of marked points.
Let $\mathcal{O} = \mathbf{k}[[t^a]]_{a \in \mathbb{R}, a \geq 0}$ so $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$ is the Novikov field.

We get the Fukaya category $\mathcal{F}(G, \mathscr{D})$ of objects over G with coefficients in $\mathscr{D}$, an A_∞-category over the ring $\mathcal{O}$. We are interested in $\mathcal{F}(G, \mathscr{D}) \otimes_{\mathcal{O}} \mathcal{K}$.

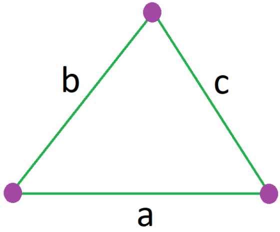
For later, the family of categories—that is here just a constant family with fiber $\mathscr{D}$—could really be a “perverse schober” over X. It happens for example if the spectral curve Σ/X has ramification.

The other basic ingredient is a *stability condition* on $\mathscr{D}$ with coefficients in the cotangent bundle of X.

In the present case it is just a usual stability condition constant over the plane, but one should bear in mind that the phases of fiber objects need to be considered relative to directions in the base.

In practice it means that the "total phase" is the sum of the fiber phase minus the angle of a curve in the base.

Our fiber category is $\mathscr{D} = A_2$. It is already a Fukaya-Seidel category of Lagrangians in the plane relative to three points.



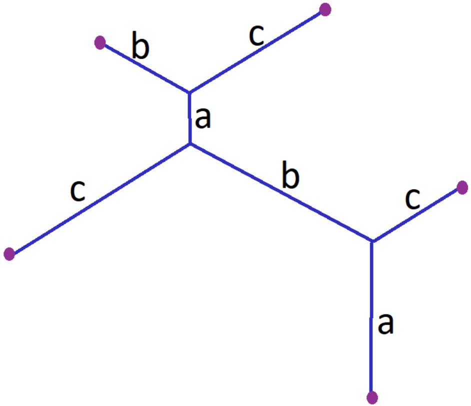
The objects of $\mathscr{D}$ are direct sums of shifts of three basic objects a, b, c that fit into an exact triangle

$$a \to b \to c \to a[1].$$

The standard stability condition on $\mathscr{D}$ has a, b, c stable objects of phases 0, $\pi/3$, and $2\pi/3$ respectively.
The maps in the exact triangle increase phase by $\pi/3$.

A spectral network is now considered as an object of $\mathcal{F}(G, \mathscr{D})$. The underlying graph is contained in G and we label those edges by the objects a, b, c depending on the labeling of the spectral network.

The "special Lagrangian" condition says that the *total phase*, sum of the phase of the edge in the base minus the phase of the edge in the fiber, should stay constant.


b
c
a
c
b
c
a

At a 3-fold vertex of the graph, we have an exact triangle relating the 3 labeling objects.

For an object of $\mathcal{F}(G, \mathscr{D})$ one requires the information of a *strict triangle* at each vertex. More generally it could be an A_∞ functor from the category A_2 to $\mathscr{D}$.

Construction of $\mathcal{F}(G, \mathscr{D})$—heuristics :

–Use direct sums of morphism complexes at the vertices of the graph. The thin-disk version counts only infinitely thin disks along G, getting a Morse-theory complex that calculates the colimit of $\mathscr{D}$ over the graph.

–The higher-order corrections from big disks, with coefficient t^{Area}, can be written down explicitly in the thin-disk complex.

–Abouzaid's verification of the A_∞ relations carries over here.

Construction of $\mathcal{F}(G, \mathscr{D})$—current version :

–Choose a full system of arcs for the graph G. The arcs don't intersect away from the boundary points.

–An initial dg-category has objects : arcs provided with objects of the fiber category. Morphisms include wrapping maps going counterclockwise when arcs intersect at the boundary.

–Then look at twisted objects provided with Maurer-Cartan elements with coefficients in the Novikov ring $\mathcal{O}$.
–The MC coefficients are allowed to have nonzero constant terms in the wrapping maps.
–These express the triangles at vertices of the graph as multiply iterated cones.

More details tomorrow!

Set

$$\mathcal{F}(X, \mathscr{D}) = \varinjlim \mathcal{F}(G, \mathscr{D}).$$

This is an A_∞ category over the Novikov ring $\mathcal{O}$. It has two projections

$$\begin{tikzcd} & \mathcal{F}(X, \mathscr{D}) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(X, \mathscr{D}) \otimes_{\mathcal{O}} \mathbf{k} & & \mathcal{F}(X, \mathscr{D}) \otimes_{\mathcal{O}} \mathcal{K} \end{tikzcd}$$

Conjecture : *There exists a stability condition on*

$$\mathcal{F}(X, \mathscr{D}) \otimes_{\mathcal{O}} \mathcal{K}$$

whose semistable objects of phase φ *are exactly the objects admitting liftings to* $\mathcal{F}(X, \mathscr{D})$ *whose projections to the residue field are spectral networks of total phase* φ.

This is viewed as a generalization of the Bridgeland-Smith theorem, their theorem concerns the case of fiber category A_1.

This viewpoint is inspired by the IHES paper of Haiden, Katzarkov and Kontsevich.

We are going to look at the case where the fiber category is A_2 and the base space is

$$X = \left(\mathbb{C}^2, \{1, \mu, \mu^2, \mu^3, \mu^4, \mu^5\}\right)$$

with $\mu = \frac{1+\sqrt{-3}}{2}$.

That is to say, we have 6 points arranged on the vertices of a regular hexagon.

The category $\mathcal{F}(X, \mathscr{D}) \otimes_{\mathcal{O}} \mathcal{K}$ is the category of representations of the A_5 quiver, in representations of the A_2 quiver, over $\mathcal{K}$. That is why we call this "$A_5 \otimes A_2$".

Ueda : we know mirror symmetry for this case.
The mirror (on the B-side) is the stack quotient of a special elliptic curve by its automorphism group $\mathbb{Z}/6\mathbb{Z}$.
The stack quotient is $\mathbb{P}^1$ with three orbifold points of degrees $2, 3, 6$ respectively.

Therefore, we expect the moduli space of stable objects of a given phase to be a $\mathbb{P}^1$ with three special points.

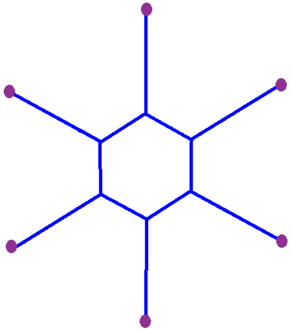
This is the behavior we'll see with spectral networks.

–One can draw all the spectral networks.
–They correspond in a nice way to the stable objects on the B-side.
–There are spherical functors transporting objects between phases.
–These functors on the A-side transport the pictures in the same way.

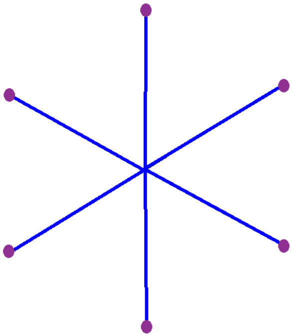
–This can be made into a proof of the conjecture for our example $A_5 \otimes A_2$, we are currently finishing to check and write it up.

More on the proof tomorrow.

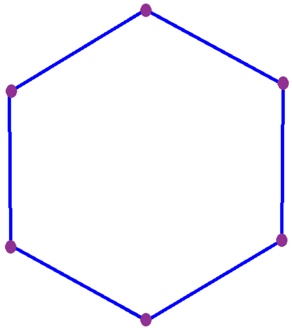
Today we'll look at how the pictures work.



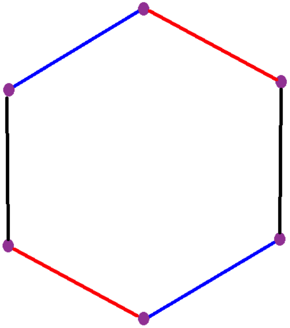
A general spectral network at the principal phase



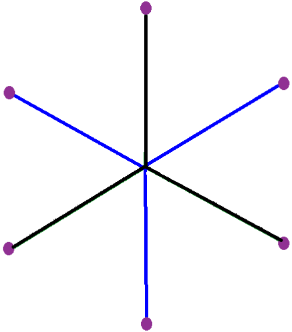
Limit as the area of the hexagon goes to 0



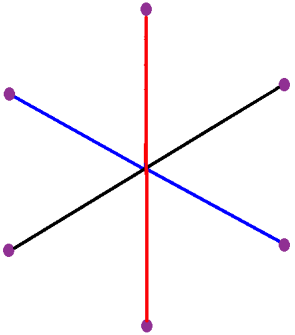
Limit as the area of the hexagon becomes maximal



At the maximal point it splits into 6 spectral networks with extension classes relating them



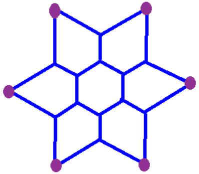
At the minimal point, one way is to split as a sum of two collisions with an extension class



At the minimal point, the other way is to split as a sum of three straight lines

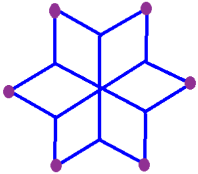
These splittings at the special points correspond to the fact that on the B-side, the limit of an indecomposable skyscraper sheaf going towards an orbifold point will split up into 2, 3 or 6 summands.

The same occurs in the moduli spaces for all the phases, indeed these moduli spaces are isomorphic by the spherical functors.

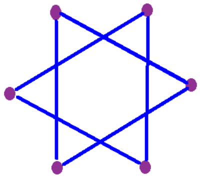


We now start to modify towards spectral networks of other phases

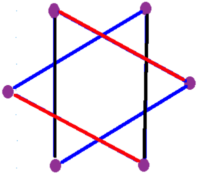
(rotate the 6 endpoints instead of the edge angles)



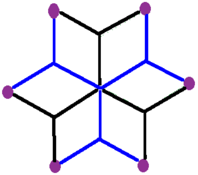
The modified one has a corresponding minimal limit



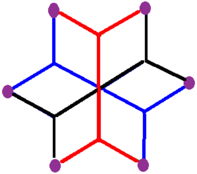
And a corresponding maximal limit



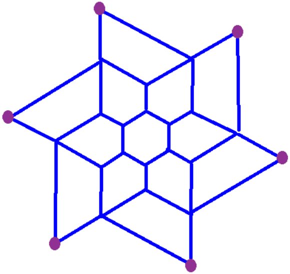
At the maximal limit we can again split into 6 spectral networks



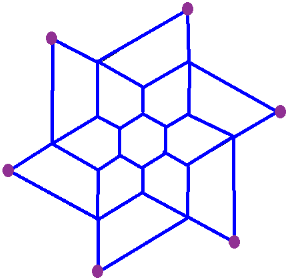
At the minimal limit we can again split into 2 spectral networks stemming from collisions



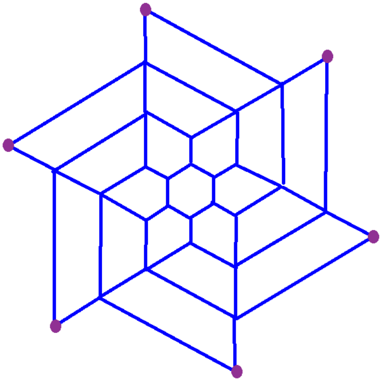
Or 3 spectral networks stemming from lines



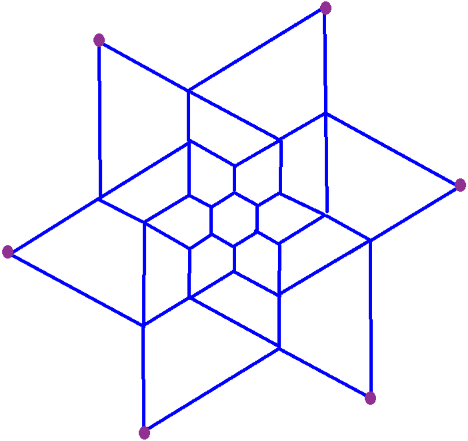
For the next step there is a choice of direction, here we go two steps to the left and one to the right



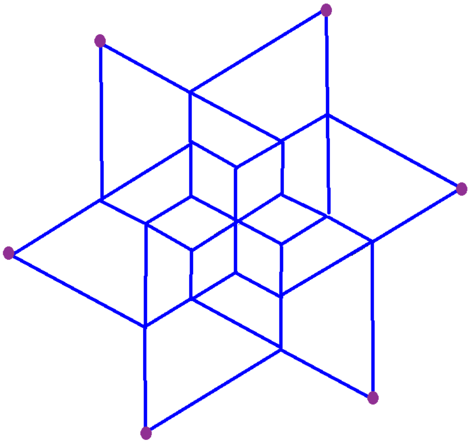
Alternatively we can go two steps to the right and one to the left



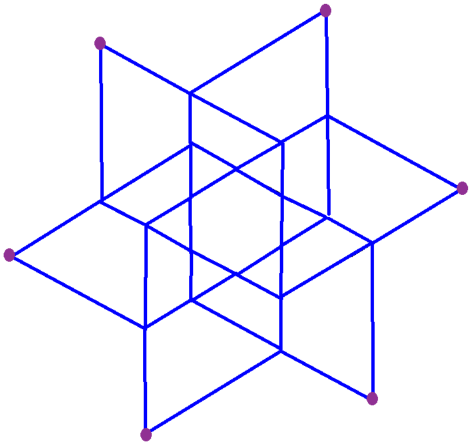
Here is the left development followed by another leftward step



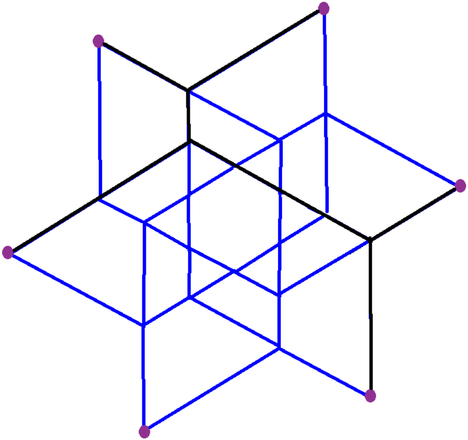
Here is the left development followed by a rightward step



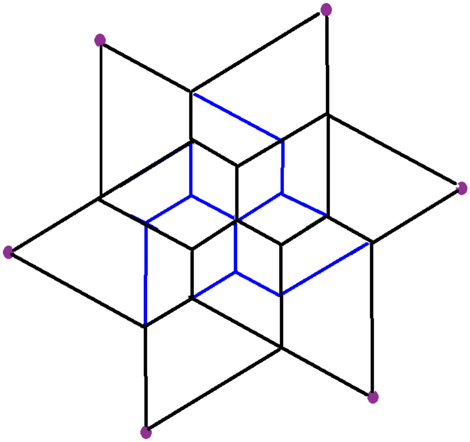
At each stage there is the minimal limiting picture



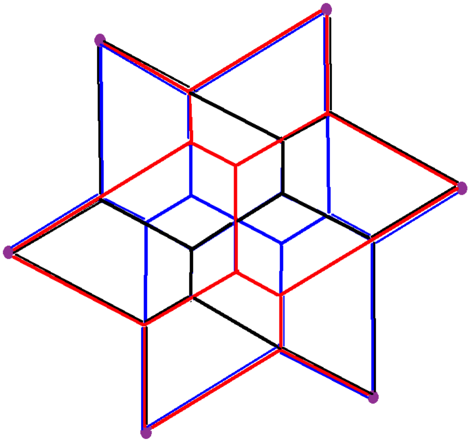
And the maximal limiting picture



Again the maximal limiting picture allows us to split off 6 spectral networks like the one pictured here

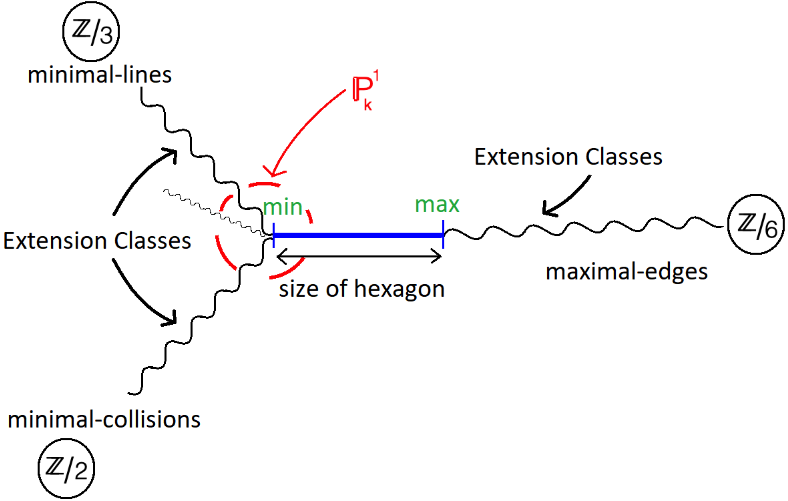


Similarly the minimal limiting picture allows us to split off two spectral networks with a collision at the center



And three spectral networks with a straight line at the center—note the multiplicities that are highlighted by coloring everything

We close with a picture of how the various spectral networks fit into pictures of the Berkovich spaces for the moduli $\mathbb{P}^1$'s



ℤ/3
minimal-lines
ℙ¹ₖ
Extension Classes
min
max
Extension Classes
ℤ/6
maximal-edges
size of hexagon
minimal-collisions
ℤ/2