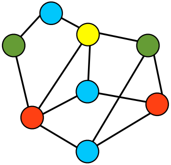
From Eulerian Polynomials and Chromatic Polynomials to Hessenberg Varieties

Michelle Wachs
University of Miami

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1



Eulerian polynomials - Euler's definition

$$\sum_{i \geq 1} t^i = \frac{t}{1-t}$$

Eulerian polynomials - Euler's definition

$$\sum_{i \geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i \geq 1} i t^i = \frac{t}{(1-t)^2}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} it^i = \frac{t}{(1-t)^2}$$

$$\sum_{i\geq 1} i^2t^i = \frac{t(1+t)}{(1-t)^3}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} it^i = \frac{t}{(1-t)^2}$$

$$\sum_{i\geq 1} i^2 t^i = \frac{t(1+t)}{(1-t)^3}$$

$$\sum_{i\geq 1} i^3 t^i = \frac{t(1+4t+t^2)}{(1-t)^4}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} i t^i = \frac{t}{(1-t)^2}$$

$$\sum_{i\geq 1} i^2 t^i = \frac{t(1+t)}{(1-t)^3}$$

$$\sum_{i\geq 1} i^3 t^i = \frac{t(1+4t+t^2)}{(1-t)^4}$$

Leonhard Euler
(1707-1783)

Euler's triangle

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\,A_n(t)}{(1-t)^{n+1}}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} it^i = \frac{t}{(1-t)^2}$$

$$\sum_{i\geq 1} i^2t^i = \frac{t(1+t)}{(1-t)^3}$$

$$\sum_{i\geq 1} i^3t^i = \frac{t(1+4t+t^2)}{(1-t)^4}$$

Leonhard Euler
(1707-1783)

Euler's triangle

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\,A_n(t)}{(1-t)^{n+1}}$$

Euler's exponential generating function formula

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t}$$

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

$$\sum_{i\geq 1} i t^i = \frac{t}{(1-t)^2}$$

$$\sum_{i\geq 1} i^2 t^i = \frac{t(1+t)}{(1-t)^3}$$

$$\sum_{i\geq 1} i^3 t^i = \frac{t(1+4t+t^2)}{(1-t)^4}$$

Leonhard Euler
(1707-1783)

Euler's triangle

1				
1	1			
1	4	1		
1	11	11	1	
1	26	66	26	1

Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\, A_n(t)}{(1-t)^{n+1}}$$

Euler's exponential generating function formula

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t} = \frac{(1-t)e^z}{e^{tz}-te^z}$$

Eulerian polynomials - combinatorial interpretation

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

Eulerian polynomials - combinatorial interpretation

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i\}$

$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1, 3\}$$

Define $\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$. So

$$\mathrm{exc}(32541) = 2$$

Eulerian polynomials - combinatorial interpretation

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1+4t+t^2$$

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1+4t+t^2$$

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Eulerian polynomials - combinatorial interpretation

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma \in \mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1 + 4t + t^2$$

$$\sum_{\sigma \in \mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1 + 4t + t^2$$

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

Eulerian polynomials - combinatorial interpretation

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1+4t+t^2$$

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1+4t+t^2$$

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

MacMahon (1905) showed equidistribution of des and exc.
Carlitz and Riordin (1955) showed these are Eulerian polynomials.

Mahonian Permutation Statistics - q-analogs

Let $\sigma \in \mathfrak{S}_n$.

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \le i < j \le n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(3142) = 3$$

Major Index:

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(3142) = \mathrm{maj}(3.14.2) = 1 + 3 = 4$$

Major Percy Alexander MacMahon

(1854 - 1929)

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$\sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{maj}(\sigma)}$$

$$= 1 + 2q + 2q^2 + q^3$$

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} &= \sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} \\ &= 1+2q+2q^2+q^3 \\ &= (1+q+q^2)(1+q)\end{aligned}$$

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$\sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} = 1+2q+2q^2+q^3 = (1+q+q^2)(1+q)$$

Theorem (MacMahon 1905)

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1+q+\cdots+q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

Theorem (MacMahon, 1916; Carlitz, 1954)

$$\sum_{i\geq 1} [i]_q^n \, t^i = \frac{t A_n^{\mathrm{maj},\mathrm{des}}(q,t)}{\prod_{i=0}^{n}(1-tq^i)}$$

q-analogs of Euler's exp. generating function formula

Theorem (Stanley, 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

q-analogs of Euler's exp. generating function formula

Theorem (Stanley, 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

Theorem (Shareshian-W., 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}$$

q-Eulerian polynomials and q-Eulerian numbers

Theorem (Shareshian-W., 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q, tq^{-1})\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

We use symmetric function theory and bijective combinatorics to prove this.

q-Eulerian polynomials and q-Eulerian numbers

Theorem (Shareshian-W., 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q, tq^{-1})\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

We use symmetric function theory and bijective combinatorics to prove this.

From now on the q-Eulerian polynomials and the q-Eulerian numbers are

$$A_n(q,t) := A_n^{\mathrm{maj,exc}}(q, tq^{-1}) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$\left\langle {n \atop j} \right\rangle_q := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

So the result with Shareshian becomes

$$\sum_{n\geq 0} A_n(q,t)\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

Palindromicity and unimodality of the q-Eulerian numbers

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\ldots$	$6+6q+11q^2+\ldots$	$4+3q+5q^2+\ldots$	1

Theorem (Shareshian-W., 2006)

The q-Eulerian polynomial $A_n(q,t) = \sum_{t=0}^{n-1} \left\langle {n \atop j} \right\rangle_q t^j$ is

- *palindromic in the sense that* $\left\langle {n \atop j} \right\rangle_q = \left\langle {n \atop n-1-j} \right\rangle_q$ *for* $0 \leq j \leq \frac{n-1}{2}$
- *q-unimodal in the sense that* $\left\langle {n \atop j} \right\rangle_q - \left\langle {n \atop j-1} \right\rangle_q \in \mathbb{N}[q]$ *for* $1 \leq j \leq \frac{n-1}{2}$

A symmetric function analog of the Eulerian polynomials

- Let ω be the involution on the ring of symmetric functions that takes the elementary symmetric functions e_n to the complete homogeneous symmetric functions h_n.
- For a homogeneous symmetric function $f(x_1, x_2, \dots)$ of degree n with coefficients in ring R, the stable principal specialization of f is

$$\mathrm{ps}_q\left(f(x_1, x_2, \dots)\right) = f(1, q, q^2, \dots) \prod_{i=1}^{n} (1 - q^i) \in R[q].$$

A symmetric function analog of the Eulerian polynomials

- Let ω be the involution on the ring of symmetric functions that takes the elementary symmetric functions e_n to the complete homogeneous symmetric functions h_n.
- For a homogeneous symmetric function $f(x_1, x_2, \dots)$ of degree n with coefficients in ring R, the stable principal specialization of f is

$$\mathrm{ps}_q\left(f(x_1, x_2, \dots)\right) = f(1, q, q^2, \dots) \prod_{i=1}^{n} (1 - q^i) \in R[q].$$

Let $W_n := \{w \in \mathbb{Z}^n_{>0} : w_i \neq w_{i+1}\ \forall i\}$ (Smirnov words) and let

$$W_n(\mathbf{x}, t) := \sum_{w \in W_n} t^{\mathrm{des}(w)} x_{w_1} \cdots x_{w_n}.$$

Example: $37572 \in W_5$ contributes $t^2 x_2 x_3 x_5 x_7^2$ to $W_5(\mathbf{x}, t)$.

A symmetric function analog of the Eulerian polynomials

- Let ω be the involution on the ring of symmetric functions that takes the elementary symmetric functions e_n to the complete homogeneous symmetric functions h_n.
- For a homogeneous symmetric function $f(x_1, x_2, \dots)$ of degree n with coefficients in ring R, the stable principal specialization of f is

$$\mathrm{ps}_q\left(f(x_1, x_2, \dots)\right) = f(1, q, q^2, \dots) \prod_{i=1}^{n} (1 - q^i) \in R[q].$$

Let $W_n := \{w \in \mathbb{Z}_{>0}^n : w_i \neq w_{i+1} \ \forall i\}$ (Smirnov words) and let

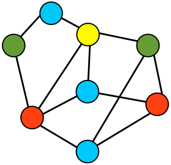
$$W_n(\mathbf{x}, t) := \sum_{w \in W_n} t^{\mathrm{des}(w)} x_{w_1} \cdots x_{w_n}.$$

Theorem (Shareshian-W., 2006)

$$A_n(q, t) = \mathrm{ps}_q(\omega W_n(\mathbf{x}, t))$$

Chromatic polynomials

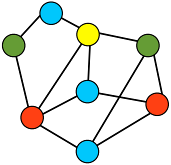
A proper coloring of a graph $G = (V, E)$ is a map $c : V \to C$ such that $c(u) \neq c(v)$ if $\{u, v\} \in E$.



The chromatic polynomial $\chi_G(m)$ of a graph G is defined to be the number of proper colorings $c : V \to C$ where $|C| = m$.

Chromatic polynomials

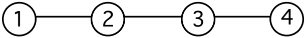
A proper coloring of a graph $G = (V, E)$ is a map $c : V \to C$ such that $c(u) \neq c(v)$ if $\{u, v\} \in E$.



The chromatic polynomial $\chi_G(m)$ of a graph G is defined to be the number of proper colorings $c : V \to C$ where $|C| = m$.

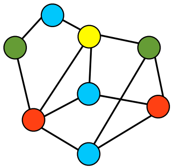
$V = [n] := \{1, \ldots, n\}$
$E = \{\{i, i+1\} : i \in [n-1]\}$

$\chi_G(m) = m(m-1)^{n-1} \in \mathbb{Z}[m]$



Chromatic polynomials

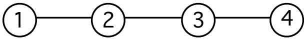
A proper coloring of a graph $G = (V, E)$ is a map $c : V \to C$ such that $c(u) \neq c(v)$ if $\{u, v\} \in E$.



The chromatic polynomial $\chi_G(m)$ of a graph G is defined to be the number of proper colorings $c : V \to C$ where $|C| = m$.

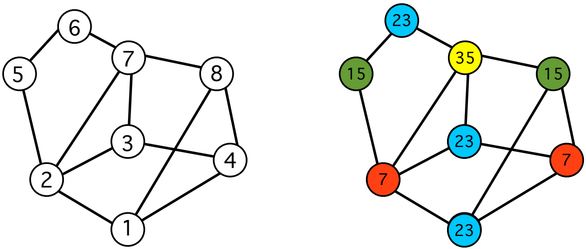
$V = [n] := \{1, \ldots, n\}$
$E = \{\{i, i+1\} : i \in [n-1]\}$

$\chi_G(m) = m(m-1)^{n-1} \in \mathbb{Z}[m]$



Birkhoff introduced this for planar graphs in 1912 as a means of proving the four color theorem. Whitney generalized this to all graphs in 1932.

Stanley's chromatic symmetric function -1995

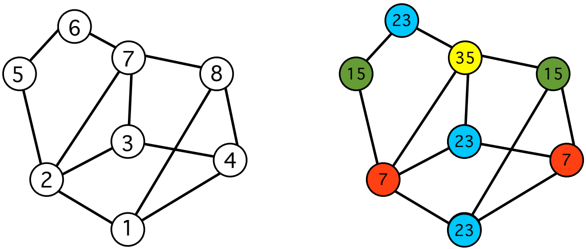


Let $C(G)$ be set of proper colorings $c : [n] \to \mathbb{Z}_{>0}$ of graph $G = ([n], E)$.

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \dots x_{c(n)}$$

$$X_G(\underbrace{1, 1, \dots, 1}_{m}, 0, 0, \dots) = \chi_G(m)$$

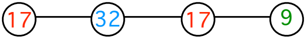
Stanley's chromatic symmetric function -1995



Let $C(G)$ be set of proper colorings $c : [n] \to \mathbb{Z}_{>0}$ of graph $G = ([n], E)$.

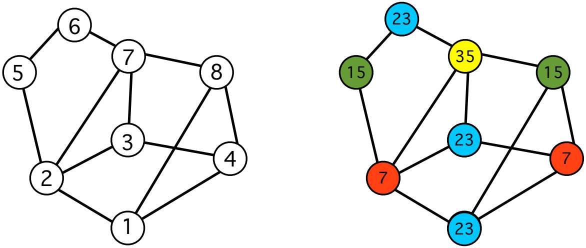
$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

$$X_G(\underbrace{1, 1, \ldots, 1}_{m}, 0, 0, \ldots) = \chi_G(m)$$



When G is the path with n nodes, $X_G(\mathbf{x}) = W_n(\mathbf{x}, 1)$.

A refinement



Chromatic quasisymmetric function (Shareshian-W., 2011)

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}_G(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}_G(c) := |\{\{i,j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

When G is the path with n nodes, $X_G(\mathbf{x}, t) = W_n(\mathbf{x}, t)$ and so

$$A_n(q,t) = \mathrm{ps}_q(\omega X_G(\mathbf{x}, t))$$

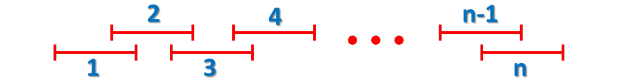
When is $X_G(\mathbf{x}, t)$ symmetric?

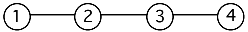
Given a collection of n unit intervals $I_1, \ldots, I_n$ on $\mathbb{R}$, labeled from left to right, form a labeled graph $G = ([n], E)$, where

$$E = \{\{i, j\} : I_i \cap I_j \neq \emptyset\}.$$

This is called a natural unit interval graph.

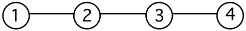
Example.

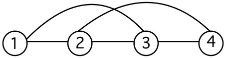




When is $X_G(\mathbf{x}, t)$ symmetric?

Examples: Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| \leq r\}$.

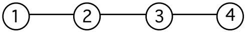
$G_{4,1}$ is the path: 

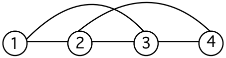
$G_{4,2}$ is the graph: 

$G_{n,n-1}$ is the complete graph K_n.

When is $X_G(\mathbf{x}, t)$ symmetric?

Examples: Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| \leq r\}$.

$G_{4,1}$ is the path: 

$G_{4,2}$ is the graph: 

$G_{n,n-1}$ is the complete graph K_n.

Theorem (Shareshian-W., 2011)

If G is a natural unit interval graph then $X_G(\mathbf{x}, t)$ is symmetric in $\mathbf{x}$ and palindromic (as a polynomial in t).

$$X_{G_{3,1}} = e_3 + (e_3 + e_{2,1})t + e_3 t^2$$

$$X_{G_{4,1}} = e_4 + (e_4 + e_{3,1} + e_{2,2})t + (e_4 + e_{3,1} + e_{2,2})t^2 + e_4 t^3$$

Refinement of Stanley-Stembridge e-positivity conjecture

Let G be a natural unit interval graph.

Conjecture (Shareshian-W., '11)

$X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.

True for

- $G_{n,1}$ and $G_{n,r}$, $r \geq n-3$ (Shareshian-W., 2011)
- various infinite classes (Shareshian-W., 2014; Cho-Huh, 2018)
- computer verification up to $n = 9$

Refinement of Stanley-Stembridge e-positivity conjecture

Let G be a natural unit interval graph.

Conjecture (Shareshian-W., '11)

$X_G(\mathbf{x}, t)$ *is e-positive and e-unimodal.*

True for

- $G_{n,1}$ and $G_{n,r}$, $r \geq n-3$ (Shareshian-W., 2011)
- various infinite classes (Shareshian-W., 2014; Cho-Huh, 2018)
- computer verification up to $n = 9$

Theorem (Shareshian-W., '14)

$X_G(\mathbf{x}, t)$ *is Schur-positive.*

Refinement of Stanley-Stembridge e-positivity conjecture

Let G be a natural unit interval graph.

Conjecture (Shareshian-W., '11)

$X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.

True for

- $G_{n,1}$ and $G_{n,r}$, $r \geq n-3$ (Shareshian-W., 2011)
- various infinite classes (Shareshian-W., 2014; Cho-Huh, 2018)
- computer verification up to $n = 9$

Theorem (Shareshian-W., '14)

$X_G(\mathbf{x}, t)$ is Schur-positive.

Theorem (Shareshian-W. '14, Athanasiadis,'15)

$\omega X_G(\mathbf{x}, t)$ is p-positive.

($t = 1$ Schur positivity: Haiman, 1993, Gasharov, 1993; $t = 1$ p-positivity: Stanley for all graphs.)

Specializing $\omega X_{G_{n,r}}(\mathbf{x}, t)$

Let $1 \leq r \leq n-1$. Our refinement of the Stanley-Stembridge conjecture implies: $\mathrm{ps}_q(\omega X_{G_{n,r}}(\mathbf{x}, t))$ is palindromic and q-unimodal.

Specializing $\omega X_{G_{n,r}}(\mathbf{x}, t)$

Let $1 \leq r \leq n-1$. Our refinement of the Stanley-Stembridge conjecture implies: $\mathrm{ps}_q(\omega X_{G_{n,r}}(\mathbf{x}, t))$ is palindromic and q-unimodal.

Let $A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{>r}(\sigma)} t^{\mathrm{inv}_{\leq r}(\sigma)}$ where

$$\begin{aligned}
\mathrm{inv}_{\leq r}(\sigma) &:= |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) \leq r\}| \\
\mathrm{DES}_{>r}(\sigma) &:= \{i \in [n-1] : \sigma(i) - \sigma(i+1) > r\} \\
\mathrm{maj}_{>r}(\sigma) &:= \sum_{i \in \mathrm{DES}_{>r}} i
\end{aligned}$$

Theorem (Shareshian-W., 2011)

$$\mathrm{ps}_q(\omega X_{G_{n,r}}(\mathbf{x}, t)) = A_n^{(r)}(q,t)$$

Consequently $A_n^{(1)}(q,t) = A_n(q,t)$.

Proof involves quasisymmetric function theory.

$A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{>r}(\sigma)} t^{\mathrm{inv}_{\leq r}(\sigma)}$

Exercise (Stanley EC1, 1.50 f): Prove that $\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\leq r}(\sigma)}$ is palindromic and unimodal.

Solution:

$A_n^{(r)}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{>r}(\sigma)} t^{\mathrm{inv}_{\le r}(\sigma)}$

Exercise (Stanley EC1, 1.50 f): Prove that $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)}$ is palindromic and unimodal.

Solution:

Theorem (De Mari and Shayman - 1988)

Let $\mathcal{H}_{n,r}$ be the type A_{n-1} regular semisimple Hessenberg variety of degree r. Then

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)} = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)}$ is palindromic and unimodal.

Stanley: Is there a more elementary proof of unimodality?

$A_n^{(r)}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{>r}(\sigma)} t^{\mathrm{inv}_{\le r}(\sigma)}$

Exercise (Stanley EC1, 1.50 f): Prove that $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)}$ is palindromic and unimodal.

Solution:

Theorem (De Mari and Shayman - 1988)

Let $\mathcal{H}_{n,r}$ be the type A_{n-1} regular semisimple Hessenberg variety of degree r. Then

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)} = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\le r}(\sigma)}$ is palindromic and unimodal.

Stanley: Is there a more elementary proof of unimodality?

Shareshian-W.: Can we find a q-analog or a symmetric function analog?

Symmetric function analog

The Frobenius characteristic is a linear map

$$ch : \{\text{virtual } \mathfrak{S}_n\text{-modules}\} \longrightarrow \Lambda_n,$$

where Λ_n is the vector space of homogeneous symmetric functions of degree n.

The image of the set of (actual) $\mathfrak{S}_n$-modules equals the set of Schur-positive symmetric functions of degree n.

We need a representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_{n,r})$ whose Frobenius characteristic is the coefficient of t^j in $\omega X_{G_{n,r}}(\mathbf{x}, t)$. It also has to commute with the hard Lefshetz map.

$$H^{2j}(\mathcal{H}_{n,r}) \overset{\mathrm{ch}}{\longmapsto} \omega X_{G_{n,r}}(\mathbf{x}, t)|_{t^j} \overset{\mathrm{ps}_q}{\longmapsto} A_n^{(r)}(q, t)|_{t^j}$$

Symmetric function analog

The Frobenius characteristic is a linear map

$$ch : \{\text{virtual } \mathfrak{S}_n\text{-modules}\} \longrightarrow \Lambda_n,$$

where Λ_n is the vector space of homogeneous symmetric functions of degree n.

The image of the set of (actual) $\mathfrak{S}_n$-modules equals the set of Schur-positive symmetric functions of degree n.

We need a representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_{n,r})$ whose Frobenius characteristic is the coefficient of t^j in $\omega X_{G_{n,r}}(\mathbf{x}, t)$. It also has to commute with the hard Lefshetz map.

$$H^{2j}(\mathcal{H}_{n,r}) \overset{\mathrm{ch}}{\longmapsto} \omega X_{G_{n,r}}(\mathbf{x}, t)|_{t^j} \overset{\mathrm{ps}_q}{\longmapsto} A_n^{(r)}(q, t)|_{t^j}$$

Tymoczko (2008) used GKM theory (Goresky, Kottwitz, MacPherson) to obtain a representation of $\mathfrak{S}_n$ on each cohomology.

Does this representation work for us?

First - more general Hessenberg variety

- De Mari, Procesi, Shayman (1992) extended the notion of semisimple Hessenberg variety so that $\mathcal{H}_{\mathbf{m}}$ is defined for each sequence $\mathbf{m} = (m_1 \leq \cdots \leq m_n)$ of integers satisfying $1 \leq i \leq m_i \leq n$. (Call these Hessenberg sequences.)

- There is a bijection between natural unit interval graphs and Hessenberg seqences. Let

$$\mathcal{H}_G := \mathcal{H}_{\mathbf{m}(G)}$$

  where $\mathbf{m}(G)$ is the Hessenberg sequence associated with natural unit interval graph G.

Symmetric function analog

Conjecture (Shareshian and W., 2011)

Let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ *be the Frobenius characteristic of Tymoczko's representation of* $\mathfrak{S}_n$ *on* $H^{2j}(\mathcal{H}_G)$*. Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

Consequenlty, by the Hard Lefschetz Theorem $\omega X_G(\mathbf{x}, t)$ *is Schur unimodal.*

Symmetric function analog

Conjecture (Shareshian and W., 2011)

Let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ *be the Frobenius characteristic of Tymoczko's representation of* $\mathfrak{S}_n$ *on* $H^{2j}(\mathcal{H}_G)$. *Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

Consequenlty, by the Hard Lefschetz Theorem $\omega X_G(\mathbf{x}, t)$ *is Schur unimodal.*

If this conjecture is true then our refinement of the Stanley-Stembridge e-positivity conjecture is equivalent to:

Conjecture

Tymoczko's representation of $\mathfrak{S}_n$ *on* $H^{2j}(\mathcal{H}_G)$ *is a permutation representation for which each point stabilizer is a Young subgroup.*

Hessenberg varieties (De Mari-Shayman (1988), De Mari-Procesi-Shayman (1992))

Let $\mathcal{F}_n$ be the set of all flags of subspaces of $\mathbb{C}^n$

$$F : F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$$

where $\dim F_i = i$.

The type A regular semisimple Hessenberg variety associated with natural unit interval graph G is

$$\mathcal{H}_G := \{F \in \mathcal{F}_n \mid DF_i \subseteq F_{m_i(G)} \ \forall i \in [n]\},$$

where

- D is the $n \times n$ diagonal matrix whose diagonal entries are $1, 2, \ldots, n$
- $\mathbf{m}(G) = (m_1(G), m_2(G), \ldots, m_n(G))$ is the Hessenberg sequence associated with G.

GKM theory and moment graphs

Goresky, Kottwitz, MacPherson (1998): Construction of equivariant cohomology ring of smooth complex projective varieties with a torus action. From this, one gets ordinary cohomology ring.

The group T of nonsingular $n \times n$ diagonal matrices acts on

$$\mathcal{H}_G := \{F \in \mathcal{F}_n \mid DF_i \subseteq F_{m_i(G)} \ \forall i \in [n]\}.$$

by left multiplication.

Moment graph: graph whose vertices are T-fixed points and whose edges are one-dimensional orbits.

Fixed points of the torus action:

$$F_\sigma : \langle e_{\sigma(1)} \rangle \subset \langle e_{\sigma(1)}, e_{\sigma(2)} \rangle \subset \cdots \subset \langle e_{\sigma(1)}, \ldots, e_{\sigma(n)} \rangle$$

where σ is a permutation.

So the vertices of the moment graph can be represented by permutations.

Combinatorial description of the moment graph

Let $G = ([n], E)$ be a natural unit interval graph. The moment graph $\Gamma(G)$ for the Hessenberg variety $\mathcal{H}_G$ has vertex set $\mathfrak{S}_n$ and edge set

Combinatorial description of the moment graph

Let $G = ([n], E)$ be a natural unit interval graph. The moment graph $\Gamma(G)$ for the Hessenberg variety $\mathcal{H}_G$ has vertex set $\mathfrak{S}_n$ and edge set

$$\{\{\sigma, \sigma(i,j)\} : \sigma \in \mathfrak{S}_n \text{ and } \{i,j\} \in E\}.$$

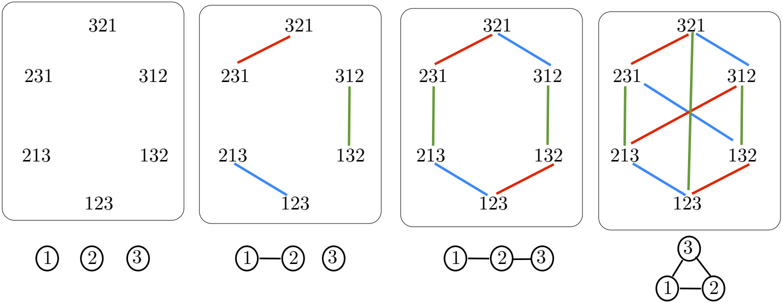
Combinatorial description of the moment graph

Let $G = ([n], E)$ be a natural unit interval graph. The moment graph $\Gamma(G)$ for the Hessenberg variety $\mathcal{H}_G$ has vertex set $\mathfrak{S}_n$ and edge set

$$\{\{\sigma, \sigma(i,j)\} : \sigma \in \mathfrak{S}_n \text{ and } \{i,j\} \in E\}.$$

Example: $n = 3$.

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**

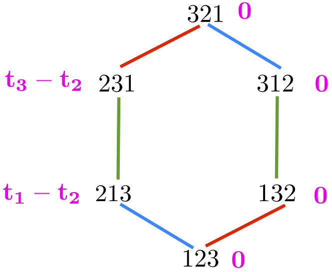


The equivariant cohomology ring $H_T^*(\mathcal{H}_G)$

$H_T^*(\mathcal{H}_G)$ is isomorphic to a subring of $R_n := \prod_{\sigma \in \mathfrak{S}_n} \mathbb{C}[t_1, \ldots, t_n]$.

For $p \in R_n$, let $p_\sigma(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$ denote the σ-component of p, where $\sigma \in \mathfrak{S}_n$.

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**

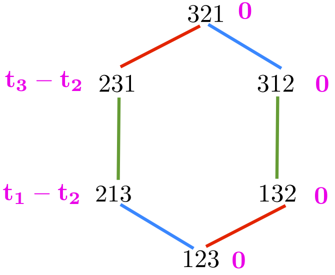


The equivariant cohomology ring $H_T^*(\mathcal{H}_G)$

$H_T^*(\mathcal{H}_G)$ is isomorphic to a subring of $R_n := \prod_{\sigma \in \mathfrak{S}_n} \mathbb{C}[t_1, \ldots, t_n]$.

For $p \in R_n$, let $p_\sigma(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$ denote the σ-component of p, where $\sigma \in \mathfrak{S}_n$.

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**



$p \in R_n$ satisfies the edge condition for the moment graph Γ_G if for all edges $\{\sigma, \tau\}$ of $\Gamma(G)$ with label (i, j), the polynomial

$$p_\sigma(t_1, \ldots, t_n) - p_\tau(t_1, \ldots, t_n)$$

is divisible by $t_i - t_j$.

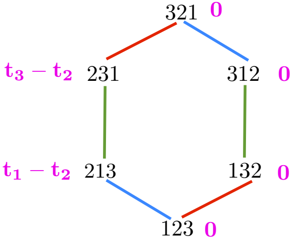
$H_T^*(\mathcal{H}_G)$ is isomorphic to the subring of R_n whose elements satisfy the edge condition for Γ_G.

Tymoczko's representation

$\sigma \in \mathfrak{S}_n$ acts on $p \in H_T^*(\mathcal{H}_G)$ by

$$(\sigma p)_\tau(t_1, \ldots, t_n) = p_{\sigma^{-1}\tau}(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**

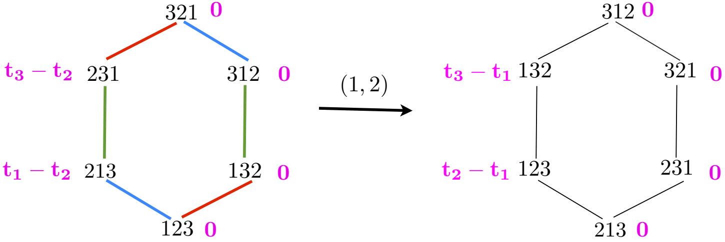


Tymoczko's representation

$\sigma \in \mathfrak{S}_n$ acts on $p \in H_T^*(\mathcal{H}_G)$ by

$$(\sigma p)_\tau(t_1, \ldots, t_n) = p_{\sigma^{-1}\tau}(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$

Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**

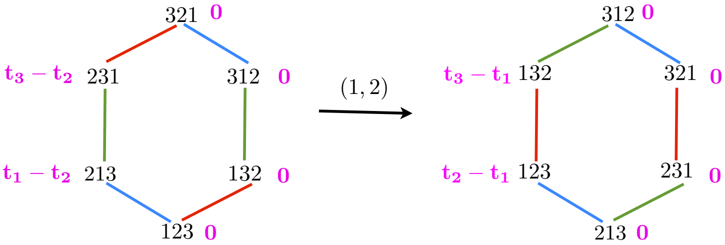


Tymoczko's representation

$\sigma \in \mathfrak{S}_n$ acts on $p \in H_T^*(\mathcal{H}_G)$ by

$$(\sigma p)_\tau(t_1, \ldots, t_n) = p_{\sigma^{-1}\tau}(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$

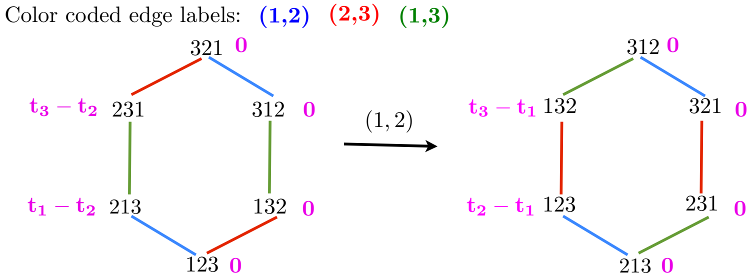
Color coded edge labels: **(1,2)** **(2,3)** **(1,3)**



Tymoczko's representation

$\sigma \in \mathfrak{S}_n$ acts on $p \in H_T^*(\mathcal{H}_G)$ by

$$(\sigma p)_\tau(t_1, \ldots, t_n) = p_{\sigma^{-1}\tau}(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$$



$$H^*(\mathcal{H}_G) \cong H_T^*(\mathcal{H}_G)/\langle t_1, \ldots, t_n \rangle H_T^*(\mathcal{H}_G)$$

The representation of $\mathfrak{S}_n$ on $H_T^*(\mathcal{H}_G)$ induces a representation on the graded ring $H^*(\mathcal{H}_G)$.

The hard Lefschetz map commutes with the action of $\mathfrak{S}_n$.

Consequences of our conjecture

Let G be a natural unit interval graph. The conjecture

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j$$

has the following consequences.

Combinatorial consequences:

- $X_G(\mathbf{x}, t)$ is Schur-positive and Schur-unimodal.
- Generalized q-Eulerian polynomials $A_n^{(r)}(q, t)$ are q-unimodal.

Algebro-geometric consequences:

- Multiplicity of irreducibles in Tymoczko's representation can be obtained from our expansion of $X_G(\mathbf{x}, t)$ in Schur basis.
- Character of Tymoczko's representation can be obtained from our expansion of $X_G(\mathbf{x}, t)$ in power-sum basis.

So our conjecture is a two-way bridge between combinatorics and algebraic geometry.

Brosnan and Chow prove our conjecture!

Theorem (Brosnan and Chow (2015), Guay-Paquet (2016))

Let G be a natural unit interval graph and let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ be the Frobenius characteristic of Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

Brosnan and Chow prove our conjecture!

Theorem (Brosnan and Chow (2015), Guay-Paquet (2016))

Let G be a natural unit interval graph and let $\mathrm{ch} H^{2j}(\mathcal{H}_G)$ be the Frobenius characteristic of Tymoczko's representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_G)$. Then

$$\omega X_G(\mathbf{x}, t) = \sum_{j \geq 0} \mathrm{ch} H^{2j}(\mathcal{H}_G) t^j.$$

- Brosnan and Chow reduce the problem of computing Tymaczko's representation of $\mathfrak{S}_n$ on regular semisimple Hessenberg varieties to that of computing the Betti numbers of regular (but not nec. semisimple) Hessenberg varieties. To do this they use results from the theory of local systems and perverse sheaves. In particular they use the local invariant cycle theorem of Beilinson-Bernstein-Deligne
- Guay-Paquet introduces a new Hopf algebra on labeled graphs to recursivley decompose the regular semisimple Hessenberg varieties.

Other recently discovered connections with $X_G(\mathbf{x}, t)$

- Hecke algebra characters evaluated at Kazhdan-Lusztig basis elements: Clearman-Hyatt-Shelton-Skandera (2015). This is a t-analog of work of Haiman (1993).
- Macdonald polynomials: Haglund-Wilson ('17).
- LLT polynomials: Carlsson-Mellit ('15), Haglund-Wilson ('17).

Other symmetric $X_G(\mathbf{x}, t)$

Extension of p-positivity result.

Theorem (Ellzey (2016))

If G is a labeled graph for which $X_G(\mathbf{x}, t)$ is symmetric then $\omega X_G(\mathbf{x}, t)$ is p-positive.

Other symmetric $X_G(\mathbf{x}, t)$

Extension of p-positivity result.

Theorem (Ellzey (2016))

If G is a labeled graph for which $X_G(\mathbf{x}, t)$ is symmetric then $\omega X_G(\mathbf{x}, t)$ is p-positive.

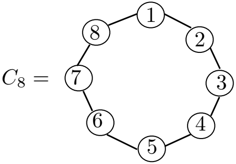
Quasisymmetric power-sum functions: Ballantine, Daugherty, Hicks, Mason, and Niese (2017)

Theorem (Alexandersson-Sulzgruber (2018))

For any labeled graph G, the chromatic quasisymmetric function $\omega X_G(\mathbf{x}, t)$ is quasisymmetric p-positive.

The proof uses Ellzey's techniques.

Other symmetric $X_G(\mathbf{x}, t)$



not a unit interval graph

Theorem (Stanley (1995))

$X_{C_n}(\mathbf{x})$ is e-positive for all $n \geq 2$.

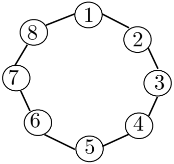
Theorem (Ellzey-W. (2018))

$X_{C_n}(\mathbf{x}, t)$ is e-positive for all $n \geq 2$.

Are there any other labeled connected graphs whose chromatic quasisymmetric function is symmetric besides for the natural unit interval graphs and the naturally labeled cycle?

Other symmetric $X_G(\mathbf{x}, t)$

$C_8 =$



not a unit interval graph

Theorem (Stanley (1995))

$X_{C_n}(\mathbf{x})$ is e-positive for all $n \geq 2$.

Theorem (Ellzey-W. (2018))

$X_{C_n}(\mathbf{x}, t)$ is e-positive for all $n \geq 2$.

Are there any other labeled connected graphs whose chromatic quasisymmetric function is symmetric besides for the natural unit interval graphs and the naturally labeled cycle?

Ellzey, 2018 UM Ph.D thesis: directed graph version.