Two issues in partially hyperbolic dynamics

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(based on joint work with A. Avila and A. Wilkinson)

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Partially hyperbolic dynamics

A diffeomorphism $f : M \to M$ on a compact Riemannian manifold is partially hyperbolic if there exists a continuous decomposition

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x$$

which is invariant under the dynamics:

$$Df_x(E^*_x) = E^*_{f(x)} \text{ for all } * \in \{u, c, s\},$$

and ...

- E^s is uniformly contracting:

$$\|Df_x \mid_{E^s_x} \| \leq \lambda < 1$$

- E^u is uniformly expanding:

$$\|(Df_x \mid_{E^u_x})^{-1}\| \leq \lambda < 1$$

- E^c is "in between":

$$\frac{1}{\lambda}\frac{\|Df_x v^s\|}{\|v^s\|} \leq \frac{\|Df_x v^c\|}{\|v^c\|} \leq \lambda \frac{\|Df_x v^u\|}{\|v^u\|}$$

Uniformly hyperbolic dynamics

In the special case $E^c_x \equiv 0$, we say that f is uniformly hyperbolic (or Anosov), a notion that goes back to S. Smale, D. Anosov and Ya. Sinai in the 1960's.

Partial hyperbolicity is the most successful of the generalizations proposed in the 1970's, and has been a major topic in dynamics over the last 2–3 decades:

- it shares many of the geometric features of uniform hyperbolicity (e.g. invariant foliations);
- it includes many new interesting examples and phenomena;
- it is a good testing ground for outstanding issues in dynamics (e.g. interplay between ergodicity and KAM behavior);
- its implications on the dynamics and on the ambient space are not well understood.

Many new examples

Basic fact: partial hyperbolicity is an open property.

- Take $A \in \mathrm{SL}(d, \mathbb{Z})$ whose spectrum intersects the interior, the boundary, and the exterior of the unit disk in $\mathbb{C}$. Then the induced map is partially hyperbolic:

$$f_A : \mathbb{T}^d \to \mathbb{T}^d, \quad \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$$

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- Let $f^t : M \to M$, $t \in \mathbb{R}$ be an Anosov flow: there is an invariant decomposition

$$T_x M = E_x^u \oplus \mathbb{R}X(x) \oplus E_x^s, \quad X = \text{ associated vector field.}$$

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- Let $g : N \to N$ be Anosov. Then any isometry extension is partially hyperbolic:

$$f : N \times \mathbb{T}^d \to N \times \mathbb{T}^d, \quad f(x, v) = \big(g(x), v + \omega(x)\big).$$

Invariant foliations: absolute continuity

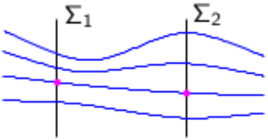
Theorem (Anosov, Sinai, Brin, Pesin, Hirsch, Pugh, Shub)

Assume that $f : M \to M$ is partially hyperbolic. Then:

- The stable and unstable bundles are uniquely integrable: there exist unique foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$T_x\mathcal{F}^s_x = E^s_x \text{ and } T_x\mathcal{F}^u_x = E^u_x \text{ everywhere.}$$

- Those foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are absolutely continuous: projections along the leaves send zero measure sets to zero measure sets.



Absolute continuity

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The geodesic flow on a compact manifold with negative curvature is ergodic.

In contrast, a center foliation $\mathcal{F}^c$, tangent to E^c,

- need not exist, nor be unique when it exists;
- need not be absolutely continuous.

A simple model

Consider the isometry extension $f_0 = g_A \times \text{id} : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$ of the map

$$g_A : \mathbb{T}^2 \to \mathbb{T}^2 \text{ is induced by } A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Known fact: every diffeomorphism f of $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T}^1$ close to f_0 is partially hyperbolic with a unique center foliation $\mathcal{F}^c$, and the center leaves are smooth circles.

In what follows, always take $f : \mathbb{T}^3 \to \mathbb{T}^3$ to be volume preserving.

Failure of absolute continuity

The (average) center Lyapunov exponent is the number

$$\lambda(f) = \int_{\mathbb{T}^3} \log |Df \mid_{E^c} |.$$

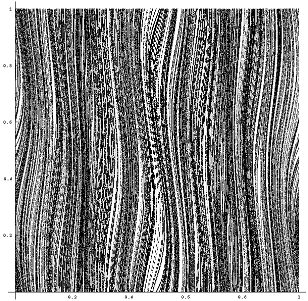
Theorem (Shub, Wilkinson)

There are ergodic diffeomorphisms f close to f_0 such that $\lambda(f) \neq 0$. Then the center foliation $\mathcal{F}^c$ cannot be absolutely continuous.

Atomic disintegration

Theorem (Ruelle, Wilkinson)

If $\lambda(f) \neq 0$ then there exist $k \geq 1$ and a full volume subset of $\mathbb{T}^3$ that intersects every center leaf at exactly k points.



(there exists a full area subset of the square consisting of exactly 1 point on each of these curves)

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be any C^∞ (volume preserving) diffeomorphism close to $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$.

Theorem (Avila, Viana, Wilkinson)

If the center foliation $\mathcal{F}^c$ is absolutely continuous, then f is C^∞-conjugate to a rotation extension

$$\mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1, \quad (x, v) \mapsto (g(x), v + \omega(x))$$

of some Anosov $g : \mathbb{T}^2 \to \mathbb{T}^2$, and $\mathcal{F}^c$ is actually a C^∞ foliation.

Assume also that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is accessible: any two points of $\mathbb{T}^3$ may be joined by a piecewise smooth path whose legs are contained in $\mathcal{F}^s$ or $\mathcal{F}^u$ leaves.

Theorem (Avila, Viana, Wilkinson)

Either the center foliation $\mathcal{F}^c$ is absolutely continuous or there exists $k \geq 1$ and a full volume subset that intersects every center leaf at exactly k points. Generically, $k = 1$.

Accessibility is a mild assumption: it is known to be an open and dense in this setting (Niţică, Török).

Let $f : T^1S \to T^1S$ be any diffeomorphism close to the time–one map of the geodesic flow on a surface S with negative curvature.

Theorem (Avila, Viana, Wilkinson)

1. Either the foliation $\mathcal{F}^c$ is C^∞, or there exists some full volume subset that intersects every center leaf at exactly one point.
2. In the first case, f is the time–one map of a C^∞ flow, whose trajectories coincide with the leaves of $\mathcal{F}^c$.

Geodesic flows

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The proofs are based on a machinery (Invariance Principle etc) developed jointly with C. Bonatti, A. Avila and J. Santamaria, with roots going back to Furstenberg and Ledrappier.

The Invariance Principle is also at the basis of the following result.
The center Lyapunov exponents of f are the numbers

$$\lambda(v^c) = \lim_n \frac{1}{n} \log \|Df_x^n(v^c)\| \text{ of vectors } v^c \in E_x^c$$

They are well defined almost everywhere (Oseledets theorem).

Question: Can we always perturb f to make all the center Lyapunov exponents non-zero?

Symplectic diffeomorphisms

Let $f_A : \mathbb{T}^4 \to \mathbb{T}^4$ be induced by some $A \in SL(4, \mathbb{Z})$ with two eigenvalues in the unit circle.

Basic facts:

- f_A preserves some (constant) symplectic form ω.
- f_A preserves volume.
- Assuming that no eigenvalue is a root of unit, f_A is ergodic.

F. Rodriguez-Hertz: every volume preserving diffeomorphism f close to f_A is ergodic (stable ergodicity).

Symplectic diffeomorphisms

In fact, f is stably Bernoulli among symplectic diffeomorphisms:

Theorem (Avila, Viana)

Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be any ω-symplectic diffeomorphism close to f_A. Then:

- either f has all center Lyapunov exponents non-zero,
- or f is conjugate to f_A by a volume preserving diffeomorphism.

In either case, f is ergodically equivalent to a Bernoulli shift.

Direct perturbation of Lyapunov exponents

A different application of the Invariance Principle yields a direct proof that vanishing exponents can be disposed of, in some cases:

Let $f : M \to M$ be a partially hyperbolic, symplectic, C^∞ diffeomorphism. Under suitable additional assumptions:

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f is C^∞-approximated by symplectic diffeomorphisms with non-vanishing Lyapunov exponents.