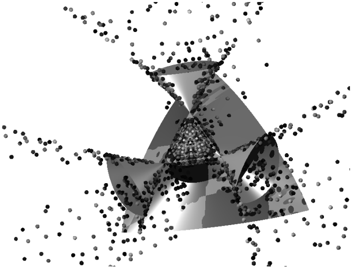
Rational points, rational curves, rational varieties



We study solutions of diophantine equations:

- rational points = (nontrivial) rational solutions of equations, e.g.,

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Rational and integral points

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- integral points = integral solutions of nonhomogeneous equations, e.g.,

$$\underbrace{x^2 + y^2 + z^2 = 3xyz}_{\text{log-K3 surface}}, \qquad \underbrace{x^2 + y^2 + z^2 = c}_{\text{log-Fano surface}}, \ c \in \mathbb{N}.$$

Balakrishnan, ... (2018)

The only rational points on the curve

$$y^2 = -4x^7 + 24x^6 - 56x^5 + 72x^4 - 56x^3 + 28x^2 - 8x + 1.$$

are

$$(0, -1), (0, 1), (1, -1), (1, 1).$$

Euler's conjecture (1769)

$$\underbrace{x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n}_{\text{Calabi-Yau}}, \qquad n \geq 4,$$

has no nontrivial solutions in $\mathbb{Q}$.

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Counterexample $n = 5$:

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For $n = 4$, rational points are dense. The smallest solution is

$$(95800, 217519, 414560, 422481).$$

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23 core-years. No solution is known for $c = 42$.

Rational and integral points

Sporadic, appear out of nowhere.

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Close connections to complexity theory and computer science:

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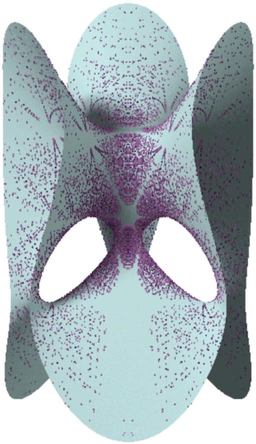
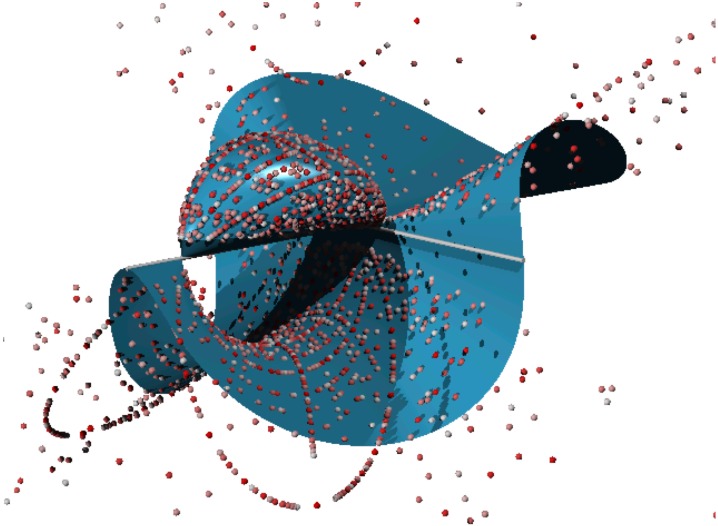
Close connections to complexity theory and computer science:

- A solution is easy to verify but hard to find
- Hidden structures (group law on elliptic curves)
- Lattices interacting with geometry



Rational points on cubic surfaces



Rational points on cubic surfaces

How to navigate?

There is an abundance of concrete, computational results concerning specific equations.

We need an organizing principle: geometry.

Rational curves

$\mathbb{P}^1 = (x : y)$

Rational curves

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$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$$\{x^2 + y^2 = z^2\}$$

$$x(t) := t^2 - 1, \quad y(t) = 2t, \quad z(t) = t^2 + 1$$

Rational surfaces

$$xyz = w^3$$

can be parametrized by two independent variables

$$x = s, \quad y = t, \quad z = s^2 t^2, \quad w = st$$

contains lines, e.g., $x = w = 0$.

$$X \subset \mathbb{P}^n$$

algebraic variety = system of homogeneous polynomial equations in $n+1$ variables, with coefficients in a field k, e.g.,

$$X := \left\{ \sum_{i=0}^{n} c_j x_i^d = 0, \quad c_i \in k \right\}, \qquad d = 2, 3, 4, \ldots$$

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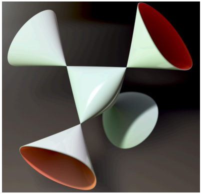
Geometry concerns properties over algebraically closed fields, e.g., $k = \mathbb{C}$.

Basic algebraic geometry

Main questions:

- Invariants (dimension, degree)
- Classification
  - Fano, Calabi-Yau, general type
  - rational, stably rational, unirational, rationally connected
  - homogeneous, ...
- Singularities
- Fibrations, families of subvarieties, e.g., lines

Surfaces of degree 2, 3, and 4



Study of

$$X(k),$$

the set of k-rational points of X, i.e., nontrivial solutions of the system of defining equations, when k is not algebraically closed:

$$k = \mathbb{F}_p, \quad \mathbb{Q}, \quad \mathbb{F}_p(t), \quad \mathbb{C}(t), \ldots$$

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Main questions:

- Existence of points
- Density in various topologies

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$$\{\text{Arithmetic}\} \quad \Leftrightarrow \quad \{\text{Geometry}\}$$

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Goal today: Discuss examples, at the interface of these fields.

Classification via degree

1. low degree: del Pezzo surfaces, Fano threefolds, ...
2. high degree: general type
3. intermediate type

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Basic examples:

1. $X_d \subset \mathbb{P}^n$, with $d \leq n$: quadrics, cubic surfaces
2. X_d with $d \geq n + 2$
3. X_d with $d = n + 1$: K3 surfaces and their higher dimensional analogs, Calabi-Yau varieties

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(RC) rational connectedness = for all $x_1, x_2 \in X(k)$ there is a rational curve C/k such that $x_1, x_2 \in C(k)$

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These properties depend on the ground field k

$$\underbrace{x^2 + y^2 + z^2 = 0}_{\text{not rational over } \mathbb{Q},\ X(\mathbb{Q}) = \emptyset} \qquad \underbrace{x^2 + y^2 - z^2 = 0}_{\text{rational over } \mathbb{Q}}$$

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- Over $\mathbb{C}$, in dimension ≤ 2, the notions coincide.

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- Over $\mathbb{C}$, in dimension ≤ 2, the notions coincide.
- Over $\mathbb{Q}$, in dimension ≥ 2, and over $\mathbb{C}$, in dimension ≥ 3,

$$(\mathrm{R}) \neq (\mathrm{S}) \neq (\mathrm{U}) \stackrel{?}{=} (\mathrm{RC})$$

Rationality of surfaces

- There exist intersections of two quadrics $Q_1 \cap Q_2 \subset \mathbb{P}^4$, over $\mathbb{Q}$, which are stably rational but not rational.

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A minimal nonrational cubic surface is not stably rational.

- There are effective procedures to determine rationality of a cubic surface over $\mathbb{Q}$.
- There is no effective procedure to determine whether a cubic surface over $\mathbb{Q}$ has a $\mathbb{Q}$-rational point, at present.

Unirationality over k implies Zariski density of $X(k)$.

Smooth quartic threefolds $X_4 \subset \mathbb{P}^4$ are not rational, some are known to be unirational.

Fano threefolds: Quartics

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Harris–T. (1998)

Rational points on X_4 over number fields k are potentially dense, i.e., Zariski dense after a finite extension of k.

Intersections of two quadrics

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A very general X is not stably rational over $k = \mathbb{C}(t)$.

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This surface contains 48 lines, over $\bar{\mathbb{Q}}$.

Rational curves on K3 surfaces

Let $N(d)$ be the number of rational d-nodal curves on a K3 surface.

Yau-Zaslow formula (1996)

$$\sum_{d\geq 0} N(d)t^d = \prod_{d\geq 1}\left(\frac{1}{1-t^d}\right)^{24}.$$

BOGOMOLOV-T. (2000)

Let $X \to \mathbb{P}^1$ be an elliptic K3 surface over a field k of characteristic zero. Then

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Rational points and curves on K3 surfaces

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Technique: deformation and specialization

Rational curves on K3 surfaces

Let X be a K3 surface over an algebraically closed field k of characteristic zero. Then X contains infinitely many rational curves.

- Bogomolov-Hassett-T. (2010): $\deg(X) = 2$, i.e.,

$$w^2 = f_6(x, y, z),$$

- Li-Liedtke (2011): $\mathrm{Pic}(X) \simeq \mathbb{Z}$
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Technique: Reduction modulo p, deformation and specialization

Kamenova-Vafa (2019)

Let X be a Calabi-Yau variety over $\mathbb{C}$ of dimension ≥ 3 (whose mirror-dual exists and is not Hodge-degenerate). Then X contains rational or elliptic curves.

Yau-Zaslow exhibited an abelian fibration

$$X^{[n]} \to \mathbb{P}^n,$$

n-th punctual Hilbert scheme (n-th symmetric power) of the K3 surface X, a holomorphic symplectic variety.

Zariski density of rational points

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Let X be a K3 surface over a field. Then there exists an n such that rational points on $X^{[n]}$ are potentially dense.

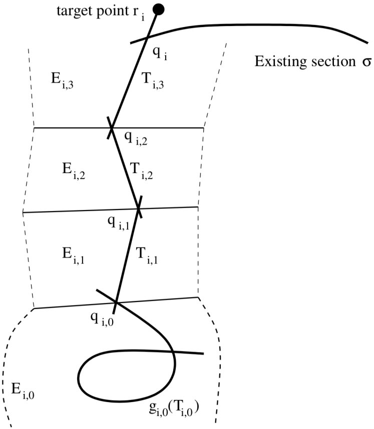
Rational curves on $K3^{[n]}$

- Conjectural description of ample and effective divisors and of birational fibration structures (Hassett-T. 1999)
- Examples with $\mathrm{Aut}(X)$ trivial but $\mathrm{Bir}(X)$ infinite (Hassett–T. 2009)
- Proof of conjectures by Bayer–Macri (2013), Bayer–Hassett–T. (2015)

Zariski density over $k = \mathbb{C}(B)$ / Hassett–T.

- Examples of general K3 surfaces X with $X(k)$ dense
- Examples of Calabi-Yau: hypersurfaces of degree $n+1$ in $\mathbb{P}^n$, with $n \geq 4$
- Integral points on log-Fano varieties
- Integral points on log-K3 surfaces over number fields are also potentially dense

Technique: Broken teeth



Techniques

Managing rational curves:

- comb constructions
- deformation theory
- degenerations (bend and break)
- producing rational curves in prescribed homology classes

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How to parametrize $x^3 + y^3 + z^3 + w^3 = 0$? Elkies:

$$\begin{aligned} x &= -(s+r)t^2 + (s^2+2r^2)t - s^3 + rs^2 - 2r^2s - r^3 \\ y &= t^3 - (s+r)t^2 + (s^2+2r^2)t + rs^2 - 2r^2s + r^3 \\ z &= -t^3 + (s+r)t^2 - (s^2+2r^2)t + 2rs^2 - r^2s + 2r^3 \\ w &= (s-2r)t^2 + (r^2-s^2)t + s^3 - rs^2 + 2r^2s - 2r^3 \end{aligned}$$

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z &= -t^3 + (s+r)t^2 - (s^2+2r^2)t + 2rs^2 - r^2s + 2r^3 \\
w &= (s-2r)t^2 + (r^2-s^2)t + s^3 - rs^2 + 2r^2s - 2r^3
\end{aligned}
$$

What about $x^3 + y^3 + z^3 + 2w^3 = 0$?

- **Larsen–Lunts (2003):** $\mathrm{K}_0(Var_k)/\mathbb{L}$ = free abelian group spanned by classes of algebraic varieties over k, modulo stable rationality.
- **Nicaise–Shinder (2017):** motivic reduction – formula for the homomorphism

$$\mathrm{K}_0(Var_K)/\mathbb{L} \to \mathrm{K}_0(Var_k)/\mathbb{L}, \quad K = k((t)),$$

  in motivic integration, as in Kontsevich, Denef–Loeser, ...
- **Kontsevich–T. (2017):** Same formula for

$$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

  the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.

Specialization (Kontsevich-T. 2017)

- Let $\mathfrak{o} \simeq k[[t]]$, $K \simeq k((t))$, $\mathrm{char}(k) = 0$.
- Let X/K be a smooth proper (or projective) variety of dimension n, with function field $L = K(X)$.
- Choose a regular model

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o}),$$

such that π is proper and the special fiber $\mathcal{X}_0$ over $\mathrm{Spec}(k)$ is a simple normal crossings (snc) divisor:

$$\mathcal{X}_0 = \cup_{\alpha \in \mathcal{A}} d_\alpha D_\alpha, \quad d_\alpha \in \mathbb{Z}_{\geq 1}.$$

- Put

$$\rho_n([L/K]) := \sum_{\emptyset \neq A \subseteq \mathcal{A}} (-1)^{\# A - 1} [D_A \times \mathbb{A}^{\# A - 1}/k],$$

How to apply?

Exhibit a family

$$\mathcal{X} \to B$$

such that some, mildly singular, special fibers admit (cohomological) obstructions to (stable) rationality.

Then a very general member of this family will also fail (stable) rationality.

Sample application

Smooth cubic threefolds $X/\mathbb{C}$ are not rational.

Via analysis of the geometry of the corresponding intermediate Jacobian $\mathrm{IJ}(X)$, Clemens-Griffiths (1972).

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Nonrationality of the smooth Klein cubic threefold $X \subset \mathbb{P}^4$

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0,$$

is easier to prove: $\mathrm{PSL}_2(\mathbb{F}_{11})$ acts on X and on $\mathrm{IJ}(X)$; this action is not compatible with a decomposition of $\mathrm{IJ}(X)$ into a product of Jacobians of curves.

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Specialization of rationality implies that a very general smooth cubic threefold is also not rational.

Applications of specialization, over $\mathbb{C}$

Hassett–Kresch–T. (2015)

Very general conic bundles $\pi : X \to S$ over rational surfaces with discriminant of sufficiently large degree are not stably rational.

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A very general nonrational Del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$, which is not birational to a cubic threefold, is not stably rational.

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A very general nonrational Fano threefold X which is not birational to a cubic threefold is not stably rational.

Rationality in dimension 3

The stable rationality problem in dimension 3, over $\mathbb{C}$, is essentially settled, with the exception of cubic threefolds.

Now the focus is on (stable) rationality over nonclosed fields.

Let X and Y be birational varieties with (birational) actions of a (finite) group G. Is there a G-equivariant birational isomorphism between X and Y?

Equivariant birational geometry

Let X and Y be birational varieties with (birational) actions of a (finite) group G. Is there a G-equivariant birational isomorphism between X and Y?

Extensive literature on classification of (conjugacy classes of) finite subgroups of the Cremona group.

Main tool: explicit analysis of birational transformations.

Equivariant birational types

- G - finite abelian group, $A = G^\vee = \mathrm{Hom}(G, \mathbb{C})$
- X - smooth projective variety, with G-action
- $$\beta : X \mapsto \sum_\alpha [F_\alpha, [\ldots]], \quad X^G = \sqcup F_\alpha.$$

Equivariant birational types

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- X - smooth projective variety, with G-action
- $$\beta : X \mapsto \sum_\alpha [F_\alpha, [\ldots]], \quad X^G = \sqcup F_\alpha.$$

Let $\tilde{X} \to X$ be a G-equivariant blowup. Consider relations

$$\beta(\tilde{X}) - \beta(X) = 0.$$

Birational types

Fix an integer $n \geq 2$ (dimension of X). Consider the $\mathbb{Z}$-module

$$\mathcal{B}_n(G) \quad \text{generated by} \quad [a_1, \dots, a_n], \quad a_i \in A,$$

such that $\sum_i \mathbb{Z} a_i = A$, and

(S) for all $\sigma \in \mathfrak{S}_n$, $a_1, \dots, a_n \in A$ we have

$$[a_{\sigma(1)}, \dots, a_{\sigma(n)}] = [a_1, \dots, a_n],$$

(B) for all $2 \leq k \leq n$, all $a_1, \dots, a_k \in A$, $b_1, \dots, b_{n-k} \in A$ with

$$\sum_i \mathbb{Z} a_i + \sum_j \mathbb{Z} b_j = A$$

we have

$$[a_1, \dots, a_k, b_1, \dots b_{n-k}] =$$

$$= \sum_{1 \leq i \leq k,\ a_i \neq a_{i'}, \forall i' < i} [a_1 - a_i, \dots, a_i, \dots, a_k - a_i, b_1, \dots, b_{n-k}]$$

Birational types

Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Birational types

Assume that

$$G = \mathbb{Z}/p\mathbb{Z} \simeq A.$$

Then $\mathcal{B}_2(G)$ is generated by symbols $[a_1, a_2]$ such that

$$a_1, a_2 \in \mathbb{Z}/p\mathbb{Z}, \quad \gcd(a_1, a_2, p) = 1,$$

and

- $[a_1, a_2] = [a_2, a_1]$,
- $[a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2]$, where $a_1 \neq a_2$,
- $[a, a] = [a, 0]$, for all $a \in \mathbb{Z}/p\mathbb{Z}$, $\gcd(a, p) = 1$.

Birational types

This gives $\binom{p}{2}$ linear equations in the same number of variables.

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Jumps at

$$p = 43, 59, 67, 83, \ldots$$

Birational types

Consider the $\mathbb{Z}$-module

$$\mathcal{M}_n(G) \quad \text{generated by} \quad \langle a_1, \ldots, a_n \rangle, \quad a_i \in A,$$

such that $\sum_i \mathbb{Z} a_i = A$, and

(S) for all $\sigma \in \mathfrak{S}_n$, $a_1, \ldots, a_n \in A$ we have

$$\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle = \langle a_1, \ldots, a_n \rangle,$$

(M)

$$\langle a_1, a_2, a_3, \ldots, a_n \rangle =$$

$$\langle a_1, a_2 - a_1, a_3, \ldots, a_n \rangle + \langle a_1 - a_2, a_2, a_3, \ldots, a_n \rangle$$

Birational types

The natural homomorphism

$$\mathcal{B}_n(G) \to \mathcal{M}_n(G)$$

is a surjection (modulo 2-torsion), and conjecturally an isomorphism, modulo torsion.

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is a surjection (modulo 2-torsion), and conjecturally an isomorphism, modulo torsion.

Imposing an additional relation on symbols

$$\langle -a_1, a_2, \ldots, a_n \rangle = -\langle a_1, a_2, \ldots, a_n \rangle$$

we obtain a surjection

$$\mathcal{M}_n(G) \to \mathcal{M}_n^-(G).$$

Hecke operators on $\mathcal{M}_n(G)$

The modular groups carry (commuting) Hecke operators:

$$T_{\ell,r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G) \quad 1 \leq r \leq n-1$$

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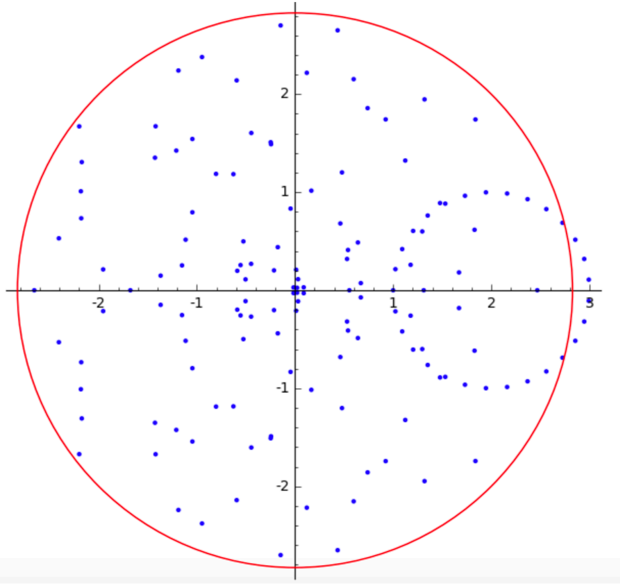
Example:

$$T_2(\langle a_1, a_2 \rangle) = \langle 2a_2, a_2 \rangle + \big(\langle a_1 - a_2, 2a_2 \rangle + \langle 2a_1, a_2 - a_1 \rangle\big) + \langle a_1, 2a_2 \rangle.$$

Eigenvalues of T_2 on $\mathcal{M}_2(\mathbb{Z}/59\mathbb{Z})$



Computations of $\mathbb{Q}$-ranks of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$

	(2, 2)	(3, 2)	(3, 3)	(4, 2)	(4, 3)	(4, 4)	(5, 2)
2	0	0	1	0	0	0	0
3	1	0	2	0	0	3	0
4	1	0	2	0	0	2	0
5	2	0	4	0	0	6	0
6	3	0	7	0	0	14	0
7	3	0	6	0	0	9	0
8	4	0	14	0	0	17	0
9	6	1	13	0	0	45	0
10	6	0	18	0	0	17	0
11	6	1	12	0	0	17	0
12	11	2	44	0	1	117	0
13	8	2	15	0	0	20	0
14	10	1	28	0	0	28	0
15	16	5	40	0	1	141	0
16	14	3	81	0	1	121	0
17	13	5	22	0	0	29	0
18	20	6	68	0	2	313	0
19	16	7	27	0	0	35	0
20	24	7	138	0	3	228	0
21	27	11	70	0	0	313	0
22	22	8	70	0	1	68	0
23	23	12	35	0	0	45	0
24	37	15	256	0	19	904	0
25	32	16	66	0	2	116	0
26	30	14	100	0	2	84	0
27	40	22	100	1	5	665	0
28	42	18	268	0	7	519	0
29	36	22	56	0	0	64	0
30	55	27	253	0	15	1243	0
31	41	26	58	0	0	71	0
32	51	27	419	0	26	877	0
33	58	35	153	2	7	980	0
34	50	31	166	0	5	142	0
35	66	37	161	0	3	346	0
36	76	46	573	3	66	2931	0
37	58	40	81	0	0	94	0
38	62	42	210	0	7	188	0
39	78	53	208	4	14	1508	0
40	88	51	769	0	64	1914	0
41	71	51	94	0	0	111	0
42	97	63	475	2	36	3040	0
43	78	58	119	1	1	130	0
44	94	63	694	2	41	1709	0

Operations

Consider exact sequences of finite abelian groups

$$0 \to G' \to G \to G'' \to 0.$$

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We have operations

$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_{n'+n''}(G)$$

$$\Delta : \mathcal{M}_{n'+n''}(G) \to \mathcal{M}_{n'}(G') \otimes \mathcal{M}^{-}_{n''}(G'')$$

Structure

The resulting homomorphism

$$\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \to \mathcal{M}_2^-(\mathbb{Z}/p\mathbb{Z}) \oplus \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, up to torsion.

Structure

The resulting homomorphism

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is an isomorphism, up to torsion.

We have

$$\dim(\mathcal{M}_2^-(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = \mathsf{g}(X_1(p)),$$

where

$$X_1(p) = \Gamma_1(p)\backslash\mathcal{H}$$

is the modular curve for the congruence subgroup $\Gamma_1(p)$.

This is the tip of the iceberg – there is an unexpected connection between birational geometry and cohomology of arithmetic groups.

Arithmetic geometry today:

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- source of intuition and new approaches to classical problems in complex geometry