# HMS and Monodromy

JOHN MORGAN

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Institute of the Math. Sciences of the Americas

Motivated by HMS

Morrison formulated a conjecture about monodromy about max unipotent boundary points of moduli of compact CY manifolds

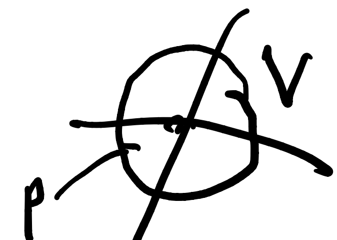
# Maximal Unipotent Bdry pts.

**General Picture:** $\mathcal{Y} \to \mathcal{M}$ smooth, proper family of alg. mflds of dim. $n$. Fiber $Y^n$.

$\overline{\mathcal{M}}$ "nice" completion of $\mathcal{M}$.

$p \in \overline{\mathcal{M}} \setminus \mathcal{M}$ pt of max depth, int. of $h^{n-1,1}(Y) = k$ divisors.

$V$ nbhd of $p$, $U = V \cap \mathcal{M}$



$\pi_1(U)$ free abelian, gen. $\gamma_1, \dots, \gamma_k$.

Monodromy $\gamma_i \longmapsto T_i \in \mathrm{Auto}(H^n(Y;\mathbb{Z}))$

Commuting ~~quasi~~ unipotent transformations

$N_i = \log T_i \qquad N = \sum_i a_i N_i, \quad a_i > 0 \ \forall i$

$N$ nilpotent $\rightsquigarrow$ monodromy wt filtration

Together w/ limiting Hodge filtration

gives limiting MHS.

Morrison Conj. for $Y$ Calabi-Yau $n$-fold.

$\exists\ p \in \overline{\mathcal{M}} - \mathcal{M}$, (1) max unipotent ($N^n \neq 0$)

(2) limiting MHS is HODGE-Tate

[$\Leftrightarrow f^{r,r}(\text{limit}) = h^{n-r,r}(Y)$]

(3) $\pi_1(U) \otimes W_{2n}/W_{2n-1} \xrightarrow{\cong} \dfrac{W_{2n-2}}{W_{2n-3}}$

Since MHS is HODGE-TATE

$\dfrac{W_{2n}}{W_{2n-1}}$ has rk 1 and $\dfrac{W_{2n-2}}{W_{2n-3}}$ has rk $= h^{n-1,1}(Y)$

HMS predicts deformations of complex str. (B-side) are iso to ($\cong$ to) deform. of symplectic structure (A-side)

(B-side)

1) Gauss-Manin connection
2) Flat lattice $H^*(Y;\mathbb{Z})$
3) non-deg. flat pairing [P.D.]
4) Hodge filtration $F^*$ satisfying Griffiths transversality $\nabla(F^p) \subset F^{p-1}$
5) limiting MHS at bdry points

## A-side: $X^n$ compact symplectic.

Holo V.B. $H^{2*}(X; \mathbb{C}) \times H^2(X; \mathbb{R}) \longrightarrow H^2(X; \mathbb{C})$

## FLAT CONNECTION:

Dubrovin connection, which comes from Q. cohomology.

On $H^{2*}(X; \mathbb{C})$ there is formal Q. coh.

$$\langle a \circ b, c \rangle = \sum_{\beta} \langle a, b, c \rangle_{\beta}\, q^{\beta}$$

Here $b_1, \dots, b_r$ basis for $H_2(X; \mathbb{Z})$

$q_i : H^2(X; \mathbb{C}) \longrightarrow \mathbb{C}$ is $\langle \,, b_i \rangle$

for $\beta = \sum \beta^i b_i$ we set $q^{\beta} = \prod_i q_i^{\beta^i}$

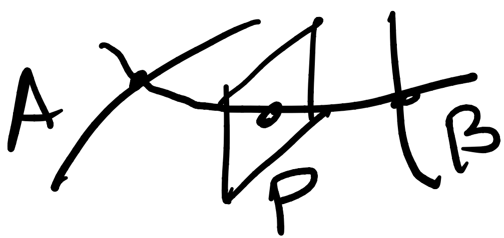
Ex. $X = \mathbb{P}^n$ Q.coh $QH(X) = \mathbb{Q}[x, q] / x^{n+1} = q$

<u>Properties:</u> (i) $a \circ b = b \circ a$, (ii) $\langle a \circ b, c \rangle = \langle a, b \circ c \rangle$

(iii) $(a \circ b) \circ c = a \circ (b \circ c)$

(iv) divisor eqn: $\langle a, p, b \rangle_\beta = \langle a, b \rangle_\beta \, p(\beta)$

$\forall \, p \in H^2(X; \mathbb{Z})$



For $\tau \in H^2(X,\mathbb{C})$ set $\langle a \circ_\tau b, c\rangle = \sum_\beta \langle a,b,c\rangle_\beta e^{\langle \tau,\beta\rangle}$

will converge on $U$ where $\langle \tau,\beta\rangle \ll 0$ $\forall \beta$ effective

$\log(q_i) = \tau_i$

## Dubrovin Connection

$$\nabla(\tau) = d - \frac{1}{z}\sum_i \frac{dq_i}{q_i} \cdot p^i \circ_\tau$$

flat for any $z \in \mathbb{C}^*$.

Flat extension to.

$$\begin{array}{c} H^{2*}(X;\mathbb{C}) \times H(X;\mathbb{C}) \times \mathbb{C}^* = F \\ \downarrow \\ H^2(X;\mathbb{C}) \times \mathbb{C}^* \end{array}$$

There is a flat non-degenerate pairing on F.

$$\langle a, b \rangle_F (\tau, u) = \langle a(\tau, z), b(\tau, -z) \rangle_{P.D.}$$

# Flat Lattice. (IRITANI & Katzarkov–Kontsevich–Pantev)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \, ; \quad \Gamma(n) = (n-1)!$$

$$\hat{\Gamma}(V) = \prod_i \Gamma(1+\lambda_i(V)) \qquad \lambda_i \text{ Chern roots}$$

$$K(X) \longrightarrow H^{2*}(X;\mathbb{C})$$

$$V \longmapsto (2\pi)^{-n/2} \hat{\Gamma}(TX) \cup (2\pi i)^{\deg/2} ch(V)$$

Embeds $K(X)/\text{Tor}$ as a lattice in $H^{2*}(X;\mathbb{C})$

Any flat section of $F$ is asymptotic to $e^{-\tau/z}\alpha$ for some $\alpha \in H^{2*}(X;\mathbb{C})$.
Identifies flat sections w/ $H^{2*}(X;\mathbb{C})$ & hence embeds $K(X)/_{Tor}$ as a lattice of flat sections.

Prop (Iritani): Under this embedding the pairing on flat sections pulls back to P.D. on $K(X)$

At this point on the A-side we have a hol. V.B. over $H^2(X;\mathbb{C}) \times \mathbb{C}^*$ a flat connection with non-deg. pairing and a flat lattice identified with $K(X)$ on which the pairing is non-degenerate.

No Hodge structure. In some sense the extra factor of $\mathbb{C}^*$ replaces the HS. as explained by both Iritani and Katzarkov-Kontsevich-Pantev.

Monodromy is an action of $H^2(X;\mathbb{Z})$
The induced action on $K(X)$ sends
$\xi \in H^2(X;\mathbb{Z})$ to $\otimes L^*_\xi$.

Dividing out gives a hol V. B. with flat connection
flat lattice and non-degenerate pairing
over $\left( H^2(X;\mathbb{C}) / 2\pi i H^2(X;\mathbb{Z}) \right) \times \mathbb{C}^*$

Monodromy assoc. with $2\pi i \xi$ is $\otimes L_\xi^{-1}$

MORRISON-TYPE Conjecture holds for A-SIDE & K-theory: The monodromy consists of $\operatorname{rk} H^2(X)$ commuting unipotent transformations. The assoc. wt filtration is:

$$W_{2k} \cap K(X) = \{ \alpha \in K(X) \mid ch_i(\alpha) = 0 \text{ for } i < n-k \}$$

$$W_{2n}/W_{2n-1}(K(X)) \xrightarrow[\cong]{c_0} \mathbb{Z}.$$

$$W_{2n-2}/W_{2n-3}(K(X)) \xrightarrow[\cong]{c_1} 2\pi i\, H^2(X;\mathbb{Z})$$

and monodromy IS $H^2(X;\mathbb{Z}) \otimes \dfrac{W_{2n}}{W_{2n-1}} \to \dfrac{W_{2n-2}}{W_{2n-3}}$

# HMS

We have similar types of structure on A-side & B-side. It is time to introduce the mirror isomorphism. In general, it is conjectural. There is one class of examples where there are theorems: Toric varieties assoc with reflexive polytopes & hypersurfaces in them. This relies on Givental's work

**Toric Fanos:** $N$ lattice; $\Sigma < N \otimes \mathbb{R}$ reflex. poly

$X_\Sigma$ assoc. toric variety (automatically FANO)

What is its mirror? Landau-Ginzburg model

**Example:**

$\mathbb{P}^n$ ; $\Sigma_n$ = convex hull $\{e_1, \ldots, e_n, -e_1 \ldots -e_n\}$

$$W_q : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

$$(T_1, \ldots, T_n) \longmapsto T_1 + \cdots + T_n + \frac{q}{\prod_i T_i}$$

The L-G model consists of $(\mathbb{C}^*)^{n+k} \xrightarrow{f} H^2(X_\Sigma; \mathbb{C}^*)$ with fibers $Y_q$ are isomorphic to $M \otimes \mathbb{C}^*$ [M dual lattice to N]

and $W : (\mathbb{C}^*)^{n+k} \longrightarrow \mathbb{C}$

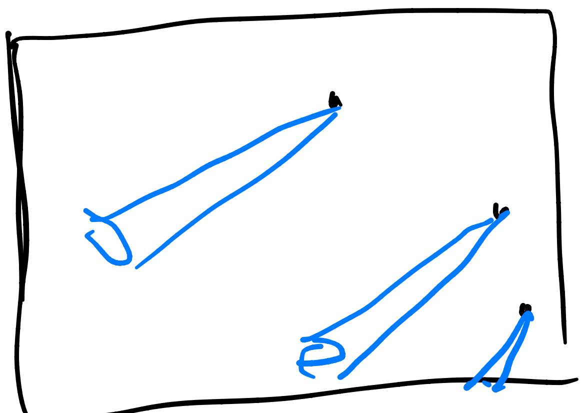
$$W = z_1 + \cdots + z_{n+k}.$$

For each $q \in H^2(X_\Sigma; \mathbb{C}^*)$ we denote by $W_q$ the restriction $W|_{Y_q}$

$$W_q : Y_q \longrightarrow \mathbb{C}.$$

$$f_{q,z} = \mathrm{Re}(W_q/z) ; Y_q \longrightarrow \mathbb{R}$$

$$H_n(Y_q, f_{q,z} \ll 0; \mathbb{Z}) = R_n^\vee(q,z)$$



vanishing cycles.

$R_n(q,z)$ dual lattice to $R_n^\vee(q,z)$

Intersection pairing is $R_n^\vee(q,z) \otimes R_n^\vee(q,-z) \to \mathbb{Z}$

Thm (Iritani) Mirror $\cong$ between Flat Dubrovin connection for $X_\Sigma$ and the flat V. bundle $R_n \otimes \mathcal{O}_{Y \times \mathbb{C}^*}$ on L-G mirror. This isomorphism sends the lattice $K(X_\Sigma)$ to $R_n = H^n(Y_q, f_{q,n}^{<<0}; \mathbb{Z})$

Compatible with monodromy action, the lattices and the pairings.

# CY hypersurfaces:

$Y \subset X_\Sigma$ anti-canonical hypersurface.

$W_q : Y'_q \to \mathbb{C}$ the L-G mirror.

Mirror of $Y$ is compactification $\check{Z}$ of $Z = W_q^{-1}(1) \subset Y'_q$, a compactification coming from dual polytope $\Sigma' \subset M \otimes \mathbb{R}$ to $\Sigma$ in $N \otimes \mathbb{R}$.

We DEFINE

(a) $K_{amb}(Y) \subset K(Y)$ to be image under pullback by inclusion. $K(X_\Sigma) \xrightarrow{i^*} K(Y)$

(b) $H^{n-1}_{res}(\check{Z}) \subset H^{n-1}(\check{Z}; \mathbb{Z})$ is P.D. to the subgroup of $H_{n-1}(\check{Z}; \mathbb{Z})$ generated by the V.C. for $Z \subset Y'_q = N \otimes \mathbb{C}^*$

**Thm** (Iritani) Let $U \subset H^2(X_\Sigma; \mathbb{C})$ be a domain near $\tau \to -\infty$ on which the Q product converges. Suppose $\iota^*: H^2(X_\Sigma; \mathbb{Q}) \to H^2(Y; \mathbb{Q})$ is an $\cong$.

Then there is a mirror isomorphism from

$$U / 2\pi i H^2(Y; \mathbb{Z}) \longleftrightarrow \mathcal{M}_{poly}(\check{Z}).$$

This mirror isomorphism sends the bundle

$$H^{2*}(Y; \mathbb{C}) \times \left(U / 2\pi i H^2(Y; \mathbb{Z})\right) \to U / 2\pi i H^2(Y, \mathbb{Z})$$

to the bundle of
of relative $(n-1)$ coh. of the fibers
of $\check{Z} \longrightarrow \mathcal{M}_{\check{Z}}$ sending the Dubrovin
connection to the Gauss-Manin conn.
and the flat lattice $\mathrm{Kamb}(Y)$ to relative

$H^{n-1}_{res}(\check{Z})$ and preserves the pairings
and monodromy

$$\begin{tikzcd}
K(X_\Sigma) \arrow[r] \arrow[d, "\text{Mirror}", "\cong"'] & K_{amb}(Y) \subset K(Y) \arrow[d, "\cong"'] \\
R_n(q, z) \arrow[r] & H^{n-1}_{res}(\check{Y}) \subset H^{n-1}(\check{Y}; \mathbb{Z})
\end{tikzcd}$$

Isomorphisms preserving pairings, flat conn. & monodromy.

Under our assumption there are no non-polynomial deformations of $Z$

$$K_{amb}(Y) < K(Y) \quad \& \quad H^{n-1}_{res}(\check{Z}) \subset H^{n-1}(\check{Z};\mathbb{Z})$$

are of finite index (the same index)

<u>Cor</u>: In this case the monodromy of $H^{n-1}(\check{Z};\mathbb{Z})$ about the limit bdry point corresponding to the large radius limit of $Y$ satisfies $\mathbb{Q}$-version of MORRISON's Conjecture.

Furthermore, under these hypotheses for CY 3-folds MORRISON's Conj. over $\mathbb{Z}$ holds $\Longleftrightarrow$ the isomorphism extends to $K(Y) \xrightarrow{\cong} H^{n-1}(\check{Z};\mathbb{Z})$.

**Addendum:** There is a Hodge filtration on $H^{2*}(Y)$ defined by $F^p = \bigoplus_{2i \le 2(n-1-p)} H^{2i}(Y; \mathbb{C})$

The monodromy wt filtration in

$$W_{2k} = \bigoplus_{2i \geq 2(n-1-k)} H^{2i}(Y; \mathbb{C})$$

These filtrations agree with those of the limiting MHS for the family $\mathfrak{Z}$ at the corresponding bdry pt.