Arithmetic Hall algebras

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Hall algebras and dilogarithm identities

$$q = p^r: \text{ power of a prime}, \quad \mathbb{F}_q: \text{ finite field}, \quad Q: \text{ finite quiver}$$

Gives an abelian category $\mathcal{A}$ with *finite* sets of morphisms and extensions.

Definition: *Hall algebra* of $\mathcal{A}$ is a free $\mathbb{Z}$-module spanned by isomorphism classes of objects in $\mathcal{A}$, with associative product given by

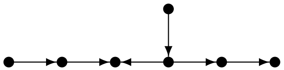
$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = \sum_{\mathcal{E}} c^{\mathcal{E}}_{\mathcal{E}_1,\mathcal{E}_2}[\mathcal{E}]$$

$$c^{\mathcal{E}}_{\mathcal{E}_1,\mathcal{E}_2} := \text{ number of subobjects } \mathcal{F} \subset \mathcal{E} \text{ such that } [\mathcal{F}] = [\mathcal{E}_1],\ [\mathcal{E}/\mathcal{F}] = [\mathcal{E}_2]$$

Example $Q = \bullet$, $\mathcal{A} =$ category of vector spaces. Set of isomorphism classes $= \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. Structure constants are q-binomial coefficients

$$c^{n_1+n_2}_{n_1,n_2} = \# Gr(n_1, n_1+n_2)(\mathbb{F}_q) = \frac{[n_1+n_2]_q!}{[n_1]_q![n_2]_q!}, \quad [n]_q! := \prod_{k=1}^{n}(1 + q + \cdots + q^{k`-1})$$

Why people like Hall algebras?

Theorem (C.Ringel) If Q is Dynkin diagram  then

$$\text{Hall algebra } = U_q \mathfrak{n}_Q^+$$

i.e. quantum deformation of universal enveloping algebra of the upper-triangular part $\mathfrak{n}_Q^+$ of the corresponding semi-simple Lie algebra $\mathfrak{g}_Q$.

Generalization (J.A.Green) For general acyclic quiver, certain *subalgebra* of its Hall algebra is isomorphic to $U_q\, \mathfrak{n}^+$ for the corresponding Kac-Moody algebra.

$\Longrightarrow$ HUGE industry in representation theory.

Non-Dynkin quivers: ∞ many indecomposable representations, Hall algebra is too big.

More recent development: interaction with *Bridgeland stability*, giving multiplicative identities in completed Hall algebras, and then in quantum tori (as certain *quotient algebras* of Hall algebras)

Definition *Central charge* for quiver Q is a collection of complex numbers

$$z_v \in \text{Upper half-plane} := \{z \in \mathbb{C} \,|, \Im z > 0\}, \quad \forall v \in Vertices(Q)$$

For a representation $\mathcal{E} \neq 0$ its *argument* $\arg \mathcal{E} \in (0^o, 180^o)$ (or better $(0, \pi)$) is the argument of non-zero complex number

$$Z(\mathcal{E}) := Z(\overrightarrow{\dim}(\mathcal{E})) = \sum_{v \in Vertices(\mathcal{E})} z_v \cdot \dim \mathcal{E}_v$$

Representation $\mathcal{E} \neq 0$ is *θ-semistable* if

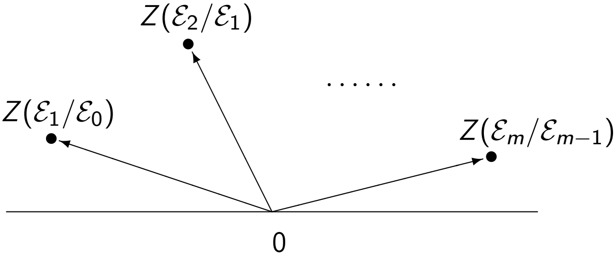
$$\arg \mathcal{E} = \theta \text{ and } \arg \mathcal{F} \leq \theta \quad \forall \mathcal{F} \subsetneq \mathcal{E}, \mathcal{F} \neq 0$$

Every representation $\mathcal{E}$ has canonical *Harder-Narasimhan filtration*

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = \mathcal{E}, \quad m \geq 0$$

$$\arg \mathcal{E}_1/\mathcal{E}_0 > \arg \mathcal{E}_2/\mathcal{E}_1 > \cdots > \arg \mathcal{E}_m/\mathcal{E}_{m-1}$$

$$\mathcal{E}_1/\mathcal{E}_0,\ \mathcal{E}_2/\mathcal{E}_1, \ldots \text{ are all semistable}$$



Existence and uniquenes of Harder-Narasimhan filtration $\Longleftrightarrow$ **Product formula**:

$$A_Q^{Hall} = \overset{\curvearrowright}{\prod_\theta} A_{Q,Z,\theta}^{Hall}$$

where

- $A^{Hall} := 1 + \cdots = \sum_{[\mathcal{E}]} [\mathcal{E}]$, the formal sum of all objects,
- $A_{Q,Z,\theta}^{Hall} = 1 + \sum_{\theta-\mathrm{semistable}\,[\mathcal{E}]} [\mathcal{E}]$
- the product is in the clockwise order, the l.h.s. does not depend on the choice of central charge. The product formula defines *uniquely* the r.h.s. from A_Q^{Hall} and Z.

All this is quite abstract, we need to map Hall algebra to something more manageable.

Euler form : $\chi : \mathbb{Z}^I \otimes \mathbb{Z}^I \to \mathbb{Z}, \qquad \chi(d', d'') := \sum_{\substack{\text{vertices} \\ v}} d'_v d''_v - \sum_{\substack{\text{arrows} \\ v \to u}} d'_v d''_u$

Meaning: if d', d'' are dimension vectors of $\mathcal{E}', \mathcal{E}''$ then

$$\chi(d', d'') = \dim Hom(\mathcal{E}', \mathcal{E}'') - \dim Ext(\mathcal{E}', \mathcal{E}'')$$

Define *quantum torus* (associated with quiver Q) as associative algebra over $\mathbb{Q}$ with linear basis $\{\mathbf{e}_d\}$ where $d \in \mathbb{Z}^I_{\geq 0}$ (the set of all possible dimension vectors), with multiplication given by

$$\mathbf{e}_{d'} \cdot \mathbf{e}_{d''} = q^{-\chi(d'', d')} \mathbf{e}_{d'+d''}, \qquad \mathbf{e}_0 = 1$$

Now we have a Homomorphism:

$$\text{Hall algebra} \to \text{Quantum Torus} : \qquad [\mathcal{E}] \mapsto \frac{\mathbf{e}_{\overrightarrow{\dim \mathcal{E}}}}{\# Aut(\mathcal{E})}$$

Apply this homomorphism the product formula:

$$A_Q := \text{ Image of } \sum_{[\mathcal{E}]} [\mathcal{E}] = \sum_{d \in \mathbb{Z}^I_{\geq 0}} \left(\sum_{[\mathcal{E}]: \overrightarrow{\dim} \mathcal{E} = d} \frac{1}{\#Aut(\mathcal{E})} \right) \cdot \mathbf{e}_d =$$

$$= \sum_{d \in \mathbb{Z}^I_{\geq 0}} \frac{q^{\sum_{v \to u} d_v d_u}}{\prod_v \#GL(d_v, \mathbb{F}_q)} \mathbf{e}_d = \sum_{d \in \mathbb{Z}^I_{\geq 0}} \frac{q^{\sum_{v \to u} d_v d_u}}{\prod_v q^{\frac{d_v(d_v-1)}{2}} (q-1)^{d_v} [d_v]_q!} \mathbf{e}_d$$

Define for any angle $\theta \in (0, \pi)$ generating series for θ-semistable representations (plus trivial one):

$$A_{Q,Z,\theta} := \text{ Image of } A_Q^{Hall} = 1 + \sum_{[E]: \theta-\mathrm{semistable}} \frac{1}{\#Aut(\mathcal{E})} \cdot \mathbf{e}_{\overrightarrow{\dim} \mathcal{E}}$$

Product Formula in quantum torus: $A_Q = \overset{\curvearrowright}{\prod_\theta} A_{Q,Z,\theta}$

The product is in the clockwise order, the l.h.s. does not depend on the choice of central charge. This decomposition defines *uniquely* the r.h.s. from A_Q and Z.

Example Dynkin quiver for sl_2 : $\overset{1}{\bullet}$

The generating series is the quantum exponent:

$$\sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}}}{\# GL(n, \mathbb{F}_q)} \mathbf{e}_1^n = \sum_{n\geq 0} \frac{(\mathbf{e}_1/(q-1))^n}{[n]_q!} =: \mathbb{E}_q(\mathbf{e}_1)$$

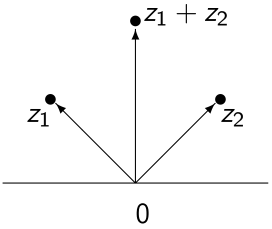
Example: Dynkin quiver for sl_3 : $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$

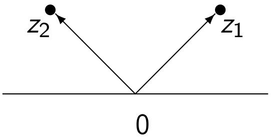
Indecomposable representations $\mathcal{R}_{01}, \mathcal{R}_{11}, \mathcal{R}_{10}$:

$$0 \longrightarrow \mathbb{F}_q \qquad \mathbb{F}_q \xrightarrow{id} \mathbb{F}_q \qquad \mathbb{F}_q \longrightarrow 0$$

Short exact sequence: $0 \to \mathcal{R}_{01} \to \mathcal{R}_{11} \to \mathcal{R}_{10} \to 0.$

Case 1 : $\arg z_1 \geq \arg z_2$: all three $\mathcal{R}_{01}, \mathcal{R}_{11}, \mathcal{R}_{10}$ are semistable,
Case 2 : $\arg z_1 < \arg z_2$: only $\mathcal{R}_{01}$ and $\mathcal{R}_{10}$ are semistable.





Example: Dynkin quiver for sl_3: $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ (continuation)

We get two product decompositions for A_Q, hence an identity

$$\mathbb{E}_q(\mathbf{e}_1) \cdot \mathbb{E}_q(\mathbf{e}_1\mathbf{e}_2) \cdot \mathbb{E}_q(\mathbf{e}_2) = \mathbb{E}_q(\mathbf{e}_2) \cdot \mathbb{E}_q(\mathbf{e}_1), \qquad \mathbf{e}_2 \cdot \mathbf{e}_1 = q\, \mathbf{e}_1 \cdot \mathbf{e}_2$$

This is called quantum dilogarithm identity, as in the formal limit $q \to 1$ it converges to the classical 5-term identity for the dilogarithm

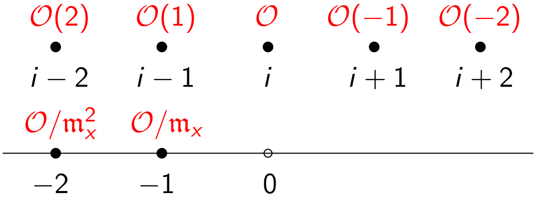
$$Li_2(x) := \sum_{n \geq 1} \frac{x^n}{n^2}$$

From quivers to curves

Representations of quivers over a finite field form an abelian category of *cohomological dimension 1*, similar to the category of coherent sheaves on a smooth compact algebraic curve $\mathcal{C}$ over $\mathbb{F}_q$.
The central charge in this case is a homomorphism of abelian groups

$$Z : K_0(Coh\,\mathcal{C}) \to \mathbb{C}, \quad Z([\mathcal{E}]) := -\deg \mathcal{E} + \sqrt{-1} \cdot \mathrm{rk}\,\mathcal{E}$$

Semistable objects are 1) torsion sheaves, 2) usual semistable vector bundles.
Example: charges of some semistable objects for $\mathcal{C} = \mathbb{P}^1$:



M.Kapranov, O.Schiffmann and E.Vasserot studied Hall algebras for $Coh(\mathcal{C})$, the structure is controlled by cuspidal automorphic forms.

From curves over finite field to number fields

Category $Coh(\mathcal{C})$ is similar to proto-abelian category of Arakelov coherent sheaves for a number field case. In the case of $\widehat{Spec\,}\mathbb{Z}$ we get pairs

$$(\Gamma, h)$$

where Γ is a finitely generated abelian group (=usual coherent sheaf on $Spec\,\mathbb{Z}$) and h is a positive quadratic form on real vector space $\Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$ (=extension to the archimedean infinity).
Γ is without torsion $\iff$ an Euclidean lattice. Central charge is given by

$$Z(\Gamma, h) := -\log \#\Gamma_{tors} + \log\big(vol(\Gamma_{\mathbb{R}}/\Gamma)\big) + \sqrt{-1} \cdot \mathrm{r}k\,\Gamma$$

Possible values of central charge of semistable objects are

$$\{-\log 2, -\log 3, \dots\} \sqcup \{x + \sqrt{-1} \cdot y \in \mathbb{C} \,|\, x \in \mathbb{R}, y \in \mathbb{Z}_{>0}\}$$

(Oriented) semstable euclidean lattices of covolume $=1$ and of rank $n \geq 1$ are those for which covolume if any sublattice is ≥ 1, they form a closed real-algebraic subset of $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R})$. The case of covolume $\neq 1$ reduces to the case of covolume 1 by the action of 1-parameter group of automorphisms of the category of Arakelov coherent sheaves:

$$(\Gamma, h) \mapsto (\Gamma, e^t \cdot h), \quad t \in \mathbb{R} \simeq Pic(\widehat{Spec\,\mathbb{Z}})$$

Analog of countings of representations and of semistable representations: calculations of volumes of $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R})$ and its semistable part:

$$V(n) = \frac{\mathrm{vol}\,(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R}))}{2\mathrm{vol}(SO(n,\mathbb{R}))} = \frac{\zeta(2)\zeta(3)\ldots\zeta(n)}{\mathrm{vol}(S^0)\ldots\mathrm{vol}(S^{n-1})} \geq V^{ss}(n) > 0$$

Relation between $(V(n))_{n\geq 1}$ and $(V^{ss}(n))_{n\geq 1}$

One can write a recursive formula which in principle allows to write numbers $(V^{ss}(n))_{n\geq 1}$ as complicated polynomial expressions in explicitly known volumes $(V(n))_{n\geq 1}$ (L.Weng).

Alternatively, one can use quantum torus with continuous grading by $\mathbb{Z}\times\mathbb{R}$, and after some manipulations obtain a *functional* relation. Introduce generating series:

$$F(t) := \sum_{n\geq 1} V(n)t^n \in t\,\mathbb{R}_{>0}[[t]], \quad F^{ss}(t) := \sum_{n\geq 1} V^{ss}(n)t^n \in t\,\mathbb{R}_{>0}[[t]]$$

Theorem $\exists!$ series $\Phi(x,y) \in y\,\mathbb{R}[[x,y]]$ such that

- $\frac{x\partial}{\partial x}\left(\frac{x\partial}{\partial x} + \frac{y\partial}{\partial y}\right)\Phi + \frac{x\partial}{\partial x}\left(F^{ss}(x)\Phi\right) = 0$
- $\Phi(t,t) = F^{ss}(t)$
- $\Phi(0,t) = F(t)$

From functions to constructible sheaves

Q: finite quiver, **k**: any field (not necessarily finite).
For any dimension vector $d \in \mathbb{Z}^I_{\geq 0}$ (I=set of vertices) the stack of representations of dimension d is a smooth Artin stack $\mathcal{M}_d$ over **k**, of virtual dimension $= -\chi(d,d)$.
For any d', d'' consider stack $\mathcal{M}_{d',d''}$ parametrizing short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0, \qquad \overrightarrow{\dim}\, \mathcal{E}' = d,\ \overrightarrow{\dim}\, \mathcal{E}'' = d''$$

We have **universal diagram**

$$\begin{array}{ccc} & \mathcal{M}_{d',d''} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{M}_{d'} \times \mathcal{M}_{d''} & & \mathcal{M}_{d'+d''} \end{array}$$

If **k** is finite field, Hall algebra product is $(\pi_2)_* \circ \pi_1^*$ for functions on **k**-points.
For general fields: functor $(\pi_2)_! \circ \pi_1^* = (\pi_2)_* \circ \pi_1^*$ on l-adic constructible sheaves.

Cohomological Hall algebra

Trivial observation (MK+Y.Soibelman, 2012): in the universal diagram

$$\begin{array}{ccc} & \mathcal{M}_{d',d''} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{M}_{d'} \times \mathcal{M}_{d''} & & \mathcal{M}_{d'+d''} \end{array}$$

map π_1 is smooth and π_2 is proper $\implies$ a map $(\pi_2)_* \circ \pi_1^*$ on Borel-Moore homology.

Definition (M.K.+ Y.Soibelman,$\sim$ 2012) *Cohomological Hall algebra* $\mathcal{H} = \mathcal{H}_Q$ for Q is $\mathbb{Z}^I \geq 0$-graded associative algebra whose graded component in degree d is given by

$$\mathcal{H}_d := H_\bullet^{BM}(\mathcal{M}_d) \overset{\text{Poincaré}}{=} H^\bullet(\mathcal{M}_d)$$

with the product given by $(\pi_2)^* \circ \pi_1^*$ as above.

Cohomological grading is not preserved by the product.

Cohomological Hall algebra categorifies the universal generating series A_Q in quantum torus. The relation to the *usual* Hall algebra is quite unclear!

Example: $Q = \bullet$, base field $= \mathbb{C}$, cohomology theory=Betti cohomology. As a vector space $\mathcal{H}$ is $\oplus_{n\geq 0} H^\bullet(BU(n), \mathbb{Q}) = \oplus_{n\geq 0}\, \mathbb{Q}[c_1, \ldots, c_n]$. As an abstract algebra COHA is isomorphic to the exterior algebra $\wedge^\bullet(\mathbb{Q}^\infty)$ with infinitely many generators.

Example: $Q = \bullet \longrightarrow \bullet$: COHA is not a familiar object. Quantum dilogarithm identity

$$\mathbb{E}_q(\mathbf{e}_1) \cdot \mathbb{E}_q(\mathbf{e}_1\mathbf{e}_2) \cdot \mathbb{E}_q(\mathbf{e}_2) = \mathbb{E}_q(\mathbf{e}_2) \cdot \mathbb{E}_q(\mathbf{e}_1), \qquad \mathbf{e}_2 \cdot \mathbf{e}_1 = q\, \mathbf{e}_1 \cdot \mathbf{e}_2$$

translates to a bizarre property of $\mathcal{H}$: it contains 3 subalgebras $\mathcal{H}_{(1)}, \mathcal{H}_{(2)}, \mathcal{H}_{(12)}$ (each is isomorphic to $\wedge^\bullet(\mathbb{Q}^\infty)$) such that both multiplication maps

$$\mathcal{H}_{(1)} \otimes \mathcal{H}_{(12)} \otimes \mathcal{H}_{(2)} \to \mathcal{H} \qquad \mathcal{H}_{(2)} \otimes \mathcal{H}_{(1)} \to \mathcal{H}$$

are isomorphisms.
Generalization to quivers with potentials (categorified Donaldson-Thomas invariants), relations to Yangians for symmetric quivers, ...

Arithmetic analog of COHA

Replace stacks $\mathcal{M}_d$ by moduli stacks of Arakelov bundles, i.e. by orbifolds

$$\mathbb{X}_n := GL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/O(n,\mathbb{R}), \qquad n \geq 0$$

In the universal diagram

$$\mathbb{X}_{d'} \times \mathbb{X}_{d''} \xleftarrow{\ \pi_1\ } \mathbb{X}_{d',d''} \xrightarrow{\ \pi_2\ } \mathbb{X}_{d'+d''}$$

now π_1 is proper and π_2 is smooth (opposite to what we have for quiver representations!) $\rightsquigarrow$ we get a *coproduct* $(\pi_1)_* \circ \pi_2^*$ on

$$\oplus_{n\geq 0} H_\bullet^{BM}(\mathbb{X}_n,\mathbb{Q}) \simeq \oplus_{n\geq 0} H^\bullet(GL(n,\mathbb{Z}), \mathit{orientation})$$

Symbols (with Y.Tschinkel and V.Pestun): $\oplus_{n\geq 0} H_n^{BM}(\mathbb{X}_n, \text{certain local systems})$.
Conjecture: this sector of (a version of) Arithmetic Hall coalgebra is freely cogenerated by cuspidal cohomological automorphic forms for $GL(1)$, $GL(2)$, $GL(3)$ only.