Hodge Theory and Moduli*

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Outline

I. Introduction
II. Hodge theory
III. Moduli
IV. I-surfaces and $\overline{\mathcal{M}}_I$

Both Hodge theory and birational geometry/moduli are highly developed subjects in their own right. The theme of this talk will be on the *uses* of Hodge theory to study an interesting geometric question and to illustrate how this works in one particular non-classical example of an algebraic surface.

I.A. Introduction

- classification of algebraic varieties is a central problem in the algebraic geometry (minimal model program)
- It falls into two parts
  - { discrete invariants, Kodaira dimension, Chern numbers
  - { continuous invariants, moduli space $\mathcal{M}$.
- Under the second part a basic issue is

  What singular varieties does one add to construct a canonical completion $\overline{\mathcal{M}}$ of $\mathcal{M}$?

- Basically, given a family $\{X_t\}_{t\in\Delta^*}$ of smooth varieties, how can one determine a *unique* limit X_0?
- A fundamental invariant of a smooth variety X is the Hodge structure $\Phi(X)$ given by linear algebra data on its cohomology $H^*(X)$.
- one knows
  - how $\Phi(X)$ varies in families
  - how to define $\Phi(X)$ when X is singular
  - how to *uniquely* define $\lim \Phi(X_t)$ for $\{X_t\}_{t\in\Delta^*}$, $t \to 0$

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Goal: *Use Hodge theory in combination with standard algebraic geometry to help understand* $\overline{\mathcal{M}}$

- two aspects
  - (A) general theory
  - (B) interesting examples

- under (B) there are
  - the classical case (curves, abelian varieties, K3's, hyperKählers, cubic 4-fold) — space of Hodge structures is a Hermitian symmetric domain
  - some results for Calabi-Yau varieties (especially those motivated by physics)
  - existence of $\overline{\mathcal{M}}$ for X's of general type — not yet any examples of $\partial\mathcal{M}$ (the global structure the singular X's nor the stratification of $\overline{\mathcal{M}}\backslash\mathcal{M}$).
- First non-classical general type surface is the *I-surface* ($p_g(X) = 2,\ q(X) = 0,\ K_X^2 = 1, \dim \mathcal{M}_I = 28$) — informally stated we have the

Main result: *The extended period mapping*

$$\Phi_e : \overline{\mathcal{M}}_I \to \overline{\mathcal{P}}$$

has degree 1 and faithfully captures the boundary structure of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$.

- Analysis of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$ was initiated by FPR — first case beyond $\overline{\mathcal{M}}_g$ ($g \geqq 2$) where the boundary structure of the Kollár-Shepherd-Barron-Alexeev (KSBA) canonical completion $\overline{\mathcal{M}}^{\mathrm{Gor}} \subset \overline{\mathcal{M}}$ is understood.
- Hodge theory (using Lie theory, differential geometry, complex analysis) gives us $\overline{\mathcal{P}} \supset \mathcal{P} = \Phi(\mathcal{M}_I)$ — the result says that Φ extends to Φ_e and the stratification of $\overline{\mathcal{P}}$ determines that of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$ — the non-Gorenstein case is only partially understood.

II. Hodge theory

A. Selected uses of Hodge theory

These include

- topology of algebraic varieties:
  - smooth case (PHS's) — (Hard Lefschetz)
  - singular case (MHS's) — also general case, relative case
  - families of algebraic varieties (LMHS's) — monodromy (local and global)

- geometry of algebraic varieties:
  - {Torelli questions; rationality and stable rationality; character varieties}
  - {algebraic cycles — conjectures of Hodge and Beilinson-Bloch}
  - {direct study of the geometry of algebraic varieties/Riemann Θ-divisor, IVHS}
- moduli of algebraic varieties
  - classical case
  - non-classical case

We will see that geometry, analysis and topology enter here. Not discussed in this talk are other interesting uses of Hodge theory including:

- physics
  - mirror symmetry
  - $\left\{\begin{array}{l}\text{homological mirror symmetry} \text{ —}\\ \text{Landau-Ginsberg models etc.}\end{array}\right\}$
- Hodge theory and combinatorics

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B. Objects of Hodge theory

These include

- polarized Hodge structures (V, Q, F) (PHS's);
- period domains D and period mappings $\Phi : B \to \Gamma\backslash D$;
- first order variation (V, Q, F, T, δ) of PHS's (IVHS);
- mixed Hodge structures (V, W, F)
- limiting mixed Hodge structures $(V, W(N), F_{\text{lim}})$ (LMHS's);
- IVLMHS.

All of these enter in the result mentioned above.

PHS (V, Q, F) of weight n

- $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$
  Hodge filtration satisfying

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}} \qquad 0 \leqq p \leqq n$$

- setting $V^{p,q} = F^p \cap \overline{F}^q$, this is equivalent to a *Hodge decomposition*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \qquad V^{p,q} = \overline{V}^{q,p}.$$

Given such a decomposition

$$F^p = \bigoplus_{p' \geqq p} V^{p',q}$$

gives a Hodge filtration.

- Hodge-Riemann bilinear relations

$$\nearrow \begin{cases} \text{(HRI)} \\ Q(F^p, F^{n-p+1}) = 0 \end{cases}$$

$$\searrow \begin{cases} \text{(HRII)} \\ i^{p-q}(Q)(F^p, \overline{F}^p) > 0 \end{cases}$$

Notes: One usually defines Hodge structures (V, F) without reference to a Q and HRI, II — only HS's I have seen used in algebraic geometry are polarizable — PHS's form a semi-simple category — in practice there is also usually a lattice $V_{\mathbb{Z}} \subset V$.

Example: The cohomology $H^n(X, \mathbb{Q})$ of a smooth, projective variety is a polarizable Hodge structure of weight n. The class $L \subset H^2(X, \mathbb{Q})$ of an ample line bundle satisfies

$$L^k : H^{n-k}(X, \mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X, \mathbb{Q}) \quad \text{(Hard Lefschetz)}$$

This then completes to the action of an $\mathrm{sl}_2\{L, H, \Lambda\}$ on $H^*(X, \mathbb{Q})$. This is the "tip of the iceberg" for the uses of the Lie theory in Hodge theory.

Note: The reason for using the Hodge filtration rather than the Hodge decomposition is that F varies holomorphically with X.

Period mapping $\Phi : B \to \Gamma\backslash D$: For given (V, Q) and $h^{p,q}$'s

- *period domain* $D = \{(V, Q, F) = \text{PHS}, \dim V^{p,q} = h^{p,q}\}$
- $D = G_{\mathbb{R}}/H$ where $G = \mathrm{Aut}(V, Q)$, $H =$ compact isotropy group of a fixed PHS.

Example: $D = \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2)$

- *period mapping* is given by a complex manifold B and a holomorphic mapping $\Phi \cdot B \to \Gamma\backslash D$ where $\Gamma \subset G_{\mathbb{Z}}$ and

$$\rho : \pi_1(B) \to \Gamma$$

is the induced map on fundamental groups.

MHS: (V, W, F)

- $F^k \subset F^{k-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ Hodge filtration
- $W_0 \subset W_1 \subset \cdots \subset W_\ell = V$ weight filtration
- F induces a HS of weight m on $\mathrm{Gr}_m^W V = W_m / W_{m-1}$

MHS's form an abelian category. A most useful property is that morphisms

$$(V, W, F) \xrightarrow{\psi} (V', W', F')$$

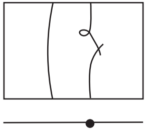
are *strict*; i.e.,

$$\begin{cases} \psi(V) \cap W'_n = \psi(W_n) \\ \psi(V_{\mathbb{C}}) \cap F'^p = \psi(F^p). \end{cases}$$

Example: For X a complete algebraic variety $H^n(X, \mathbb{Q})$ has a functorial MHS (where $k = \ell = n$).

Example: $\mathcal{X} \xrightarrow{\pi} B$ is a family of smooth projective varieties $X_b = \pi^{-1}(b)$ and $\rho : \pi_1(B, b_0) \to \mathrm{Aut}(H^n(X_{b_0}, \mathbb{Q}))$ is the monodromy representation. Then $\Phi(b) =$ PHS on $H^n(X_b, \mathbb{Q})$.

Special case: $B = \Delta^* = \{t \cdot 0 < |t| < 1\}$ and we have



- ρ (generator) $= T \in \mathrm{Aut}\, H^m(X_{b_0}, \mathbb{Q})$
- $T = T_{ss} T_u$ where
  - $T_{ss}^m = \mathrm{Id}$
  - $T_u = e^N$ where $N^{m+1} = 0$

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LMHS: $(V, W(N), F_{\lim})$ is a MHS where

- $N \in \mathrm{End}_Q(V)$ and $N^{m+1} = 0$ gives unique monodromy weight filtration

$$W_0(N) \subset W_1(N) \subset \cdots \subset W_{2m}(N)$$

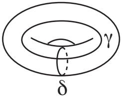
  satisfying

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : \mathrm{Gr}^{W(N)}_{m+k}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{m-k}(V); \end{cases}$$

- $N : F^p_{\lim} \to F^{p-1}_{\lim}$.

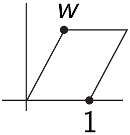
Example: Above example where $B = \Delta^*$ — here $\Gamma = \{T\}$.

Classic Example: X is a compact Riemann surface of genus 1

- 

topological picture

- $y^2 = x^3 + a(t)x^2 + b(t)x + c(t)$, algebraic picture
$\omega = dx/y$

- 

$\mathbb{C}/\mathbb{Z}\cdot w + \mathbb{Z}$ analytic picture

- $w = \int_\gamma \omega / \int_\delta \omega$.

- The space of LMHS's $(V, Q, W(N), F_{\lim})$ has a symmetry group
  - G acts on conjugacy classes of N's;
  - $G_{\mathbb{C}}$ acts transitively on $\check{D} = \{(V, F) : Q(F^p, F^{n-p+1}) = 0\}$;
  - $F_{\lim} \in \check{D}$.

  Thus one may imagine using Lie-theoretic methods to attach to the space $\Gamma\backslash D$ of Γ- equivalence classes of PHS's a set of equivalence classes of LMHS's — then informally stated one has the result

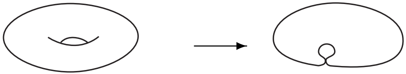
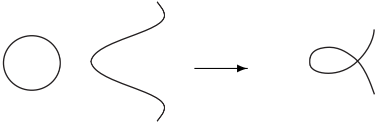
  > *the images $\mathcal{P} \subset \Gamma\backslash D$ of global period mappings have natural completions $\overline{\mathcal{P}}$.*

The proof that $\overline{\mathcal{P}}$ has the structure of a projective variety is a work in progress.

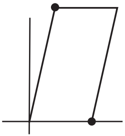
Example: The moduli space of elliptic curves

$$\begin{array}{c}\text{algebro-geometric object } \mathcal{M}_1 \xrightarrow{j} \mathbb{C} \\ \big\downarrow \\ \text{Hodge theoretic object } \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\end{array}$$

completes by adding ∞ corresponding to the LMHS associated to

- 
- 

$$y^2 = x(x-t)(x-1) \longrightarrow y^2 = x^2(x-1)$$



$w \to i\infty$

$$\mathbb{C}/\mathbb{Z}w + \mathbb{Z} \to \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$$

IVHS: (V, F, T, δ) where (V, F) is a HS and

$$\begin{cases} \delta : T \to \overset{p}{\oplus} \operatorname{Hom}\left(\operatorname{Gr}_F^p V_{\mathbb{C}}, \operatorname{Gr}_F^{p-1} V_{\mathbb{C}}\right) \\ [\delta, \delta] = 0. \end{cases}$$

Example: Φ_* for a period mapping.

Example: For $\mathcal{M}_g, g \geqq 3$, the IVHS is equivalent to the quadrics through the canonical curve

$$C \to \mathbb{P}^{g-1} = \mathbb{P}H^0(\Omega^1_C)^*.$$

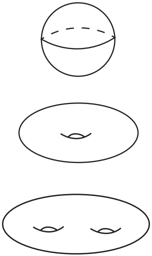
Example (work in progress): The equation of a smooth I-surface can be reconstructed from Φ_*.

III. Moduli

- Basic discrete invariant of an algebraic variety is its *Kodaira dimension* κ

Curves: $g = h^0(\Omega^1_C) = \dim H^0(\Omega^1_C)$

$$\begin{cases} g = 0 & \kappa = -\infty \\ g = 1 & \kappa = 0 \\ g \geqq 2 & \kappa = 1 \end{cases}$$



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Theorem (Deligne-Mumford): *For curves of general type there exists a moduli space* $\mathcal{M}_g$ *with an essentially smooth projective completion* $\overline{\mathcal{M}}_g$

- $\dim \mathcal{M}_g = 3g - 3$;
- *one knows what the boundary curves look like both locally (singularities) and globally.*

Surfaces:

$$\begin{cases} \kappa = -\infty & \text{rational} \\ \kappa = 0 & \text{abelian varieties, K3's} \\ \kappa = 1 & \text{elliptic surfaces} \\ \kappa = 2 & \text{general type.} \end{cases}$$

- For general type surfaces the basic discrete invariants are
  - $c_1^2 = K_X^2$, $c_2 = \chi_{\text{top}}(X)$ — both are positive
  - Noether's inequality for $p_g(X) = h^0(\Omega_X^2)$

$$p_g(X) \leqq \left(\frac{1}{2}\right) c_1^2 + 2;$$

extremal surfaces are of particular interest.

Theorem (Kollár-Shepherd-Barron-Alexeev = KSBA): *For surfaces of general type with given c_1^2, c_2 there exists a moduli space $\mathcal{M}$ with a canonical projective completion $\overline{\mathcal{M}}$.*

- expected dimension $\mathcal{M} = \frac{1}{12}(c_2 - 14c_1^2) = 28$ for I-surfaces
- differences between surface and curve cases
  (a) one knows what the boundary surfaces look like locally, but does not know this globally — only one partial example (FPR);
  (b) $\partial\mathcal{M}$ is (highly) singular.

Main points: *Hodge theory can sometimes help us to understand (a) and (b).*

- Guiding question: Are all of the Hodge-theoretically possible degenerations realized algebro-geometrically?

Theorem (work in progress): *We have*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Phi} & \mathcal{P} \subset \Gamma\backslash D \\ \cap & & \cap \\ \overline{\mathcal{M}} & \xrightarrow{\Phi_e} & \overline{\mathcal{P}}. \end{array}$$

where $\overline{\mathcal{P}}$ is constructed Hodge theoretically.

The stratification of $\overline{\mathcal{P}}$ has two aspects:

(i) using the conjugacy class of N;

(ii) within each D_i corresponding to a particular stratum we have the Mumford-Tate sub-domains.

Example: When the weight $n = 1$ we have $N^2 = 0$ and the only invariant is the rank of N — then (i) gives a schematic (here $\dim V = 2g$)

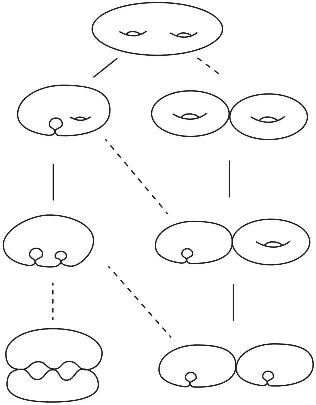
$$I_0 — I_1 — \cdots — I_g .$$

Within each I_k we have for (ii) the Mumford-Tate sub-domains correspond to reducible PHS's that are direct sums.

Example: When $g = 2$ the Hodge theoretic stratification of $\overline{\mathcal{P}}$ gives for (i)

$$I_0 - I_1 - I_2$$

and using (ii) we get the following stratification of $\overline{\mathcal{M}}_2$:



For use below we remark that a general curve of genus $g = 2$ has the affine equation

$$y^2 = \prod_{i=1}^{6}(x - a_i).$$

In the weighted projective space $\mathbb{P}(1, 1, 3)$ with coordinates (x_0, x_1, y) the equation is

$$y^2 = \prod_{i=1}^{6}(x_1 - a_i x_0) = F_6(x_0, x_1).$$

Example of how Hodge theory is used

- $\mathcal{X}^* \to \Delta^*$ is *any* family of $g = 1$ curves $\{X_t\}_{t \subset \Delta^*}$.
- Monodromy $T : H_1(X_{t_0}, \mathbb{Z}) \to H_1(X_{t_0}, \mathbb{Z})$ — in terms of a standard basis δ, γ

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- $\Phi(t) = \int_\gamma \omega_t / \int_\delta \omega_t$ is the multi-valued period mapping —

$$\Phi(e^{2\pi i} t) = T\Phi(t).$$

- This gives

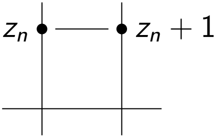
$$\begin{array}{ccccc} z & \in & \mathbb{H} & \xrightarrow{w} & \mathbb{H} \\ \downarrow & & \downarrow & & \downarrow \\ e^{2\pi i z} & = & t \in \Delta^* & \longrightarrow & \{T\}\backslash\mathbb{H} \end{array}$$

$$w(z+1) = Tw(z).$$

Lemma 1: *Eigenvalues* μ *of* T *are roots of unity* $(T_{ss}^k = \mathrm{Id})$*.*

Lemma 2: $\Phi(t) = m\frac{\log t}{2\pi i} + h(t)$ *where* $h(t)$ *holomorphic in* Δ*.*

Proof of Lemma 1: For $t_n = e^{2\pi i z_n} \to 0$, using the $\mathrm{SL}_2(\mathbb{R})$-invariant Poincaré metric $ds_{\mathbb{H}}^2 = \frac{dz d\bar{z}}{(\operatorname{Im} z)^2}$



$$d(z_n, z_n + 1) = \frac{1}{\operatorname{Im} z_n} \to 0$$

Schwarz lemma gives that w is distance decreasing in the Poincaré metric

$$d(w(z_n), Tw(z_n)) \to 0.$$

For $w = i \in \mathbb{H}$ and $w_n = A_n w$, $A_n \in \mathrm{SL}_2(\mathbb{R})$ and invariance of $ds^2_{\mathbb{H}}$ gives

$$\begin{gathered} d(w, A_n^{-1} T A_n w) \to 0 \\ \Downarrow \\ A_n^{-1} T A_n \to \text{ isotropy group } \mathrm{SO}(2) \text{ of } w \\ \Downarrow \\ |\mu| = 1 \\ \Downarrow \\ \mu^m = 1. \end{gathered}$$

Proof of Lemma 2: For a choice of δ, γ we will have $T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ — thus

$$\Phi(t) = \frac{m \log t}{2\pi i} + h(t)$$

where $h(t)$ is single valued. Another use of the Schwarz lemma gives that h is bounded. □

Above argument used

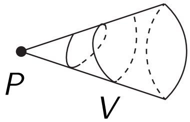
- differential geometry (negative curvature of $ds^2_{\mathbb{H}}$)
- Lie theory ($\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2)$)
- complex analysis.

IV. I-surfaces and $\overline{\mathcal{M}}_I$

- In $\mathbb{P}(1,1,2,5)$ with coordinates $[x_0, x_1, y, z]$ the equation of the I-surface X is

$$z^2 = F_{10}(x_0, x_1, y) = G_5(x_0, x_1)y + H_{10}(x_0, x_1).$$

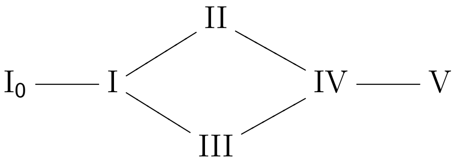
It may be pictured as a 2:1 covering of



$= \mathbb{P}(1,1,2)$

which has branch curve $P + V$ where V is a quintic. Over a ruling of the quadric cone we obtain a covering of $\mathbb{P}^1$ branched over six points; i.e., a pencil of genus 2 curves. Thus I-surfaces are an analogue of $g = 2$ curves.

- The diagram (i) for this case is



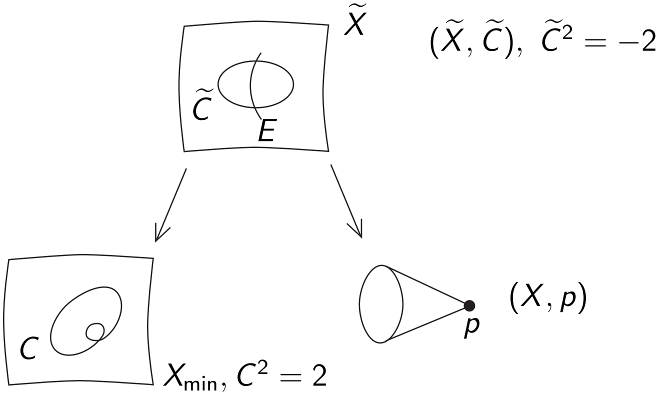
  Note that it is non-linear. It is transitive, but this fails when the weight $n \geqq 3$.
- Within each of these there is the further stratification by Mumford-Tate domains — here the stratification is by the conjugacy class of the semi-simple part of monodromy — for normal Gorenstein I-surfaces the resulting classification is

stratum	dimension	minimal resolution $\widetilde{X}$
I_0	28	canonical singularities
I_2	20	blow up of a K3-surface
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})$=2
$\mathrm{III}_{2,2}$	12	rational surface
$\mathrm{III}_{1,2}$	11	rational surface
$\mathrm{III}_{1,1,R}$	10	rational surface
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})$=0
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})$=0

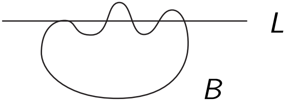
- Difference from $\overline{\mathcal{M}}_g$ is that $\partial\mathcal{M}_I$ is singular — extension data in the LMHS provides a guide as to how to desingularize $\overline{\mathcal{M}}_I$.

Example: $\mathcal{M}_{I_2}$

- Picture of $X \in \mathcal{M}_{I_2}$:



- $X_{\min}$ = degree 2 K3 surface;
- $X_{\min} \to \mathbb{P}^2$ branched over sextic B



$C \to L$

- # moduli $(X_{\min}, C) = 19 + 1 = 20 = \dim \mathcal{M}_{I_2}$;
- $\widetilde{C} \subset \mathbb{P}^2$ is cubic and extension data in the LMHS arising from $H^2(\widetilde{X}, \widetilde{C})$ gives seven points on $\widetilde{C}$;
- blowing these up gives a del Pezzo surface Y — then
- $\widetilde{C} \cup_{\widetilde{C}} Y$ gives the $20 + 7 = 27$ dimensional blowup of $\mathcal{M}_{I_2}$ and provides a desingularization of $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$ along $\mathcal{M}_{I_2}$.

Conclusion: *The structure of* $\overline{\mathcal{P}}$ *may be analyzed using Lie theory — using the extended period mapping it provides a faithful guide to the structure of* $\overline{\mathcal{M}}_I^{\mathrm{Gor}}$.