# Branched harmonic functions and some related developments in differential geometry

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## Part I: Introduction

In complex analysis:

$z^{1/2}$ is a 2-valued, "branched" holomorphic function on **C**.

$\mathrm{Re} z^{1/2}$ is a 2-valued, "branched" harmonic function.
The multi-valued difficulty can be resolved by:

- passing to a cover;
- working with sections of a flat bundle over $\mathbf{C} \setminus \{0\}$.

Let $(M^n, g)$ be a Riemannian manifold and $\Sigma \subset M$ a co-oriented codimension-2 submanifold.

We can consider multivalued functions on $M$, branched over $\Sigma$.

That is, sections $u$ of a flat $\mathbf{R}$-line bundle $L \to M \setminus \Sigma$ with monodromy $-1$ around $\Sigma$.

Local coordinates $(z, t)$:

- $z \in \mathbf{C}$ transverse to $\Sigma$;
- $t \in \mathbf{R}^{n-2}$ along $\Sigma$.

FACT: If $\Delta u = 0$ near $\Sigma$ and $u$ is bounded then $u$ has asymptotics

$$u \sim \mathrm{Re}\left(a(t)z^{1/2} + b(t)z^{3/2}\right) + O(|z|^{5/2}).$$

More invariantly, $a \in \Gamma(\Sigma, N^{-1/2}), b \in \Gamma(\Sigma, N^{-3/2})$ where $N \to \Sigma$ is the normal bundle (regarded as a complex line bundle).

The problem we consider here is to adjust $\Sigma$ so that the leading term *a* vanishes.

**Rotationally invariant model.**
Take $M=\mathbf{C}\times\mathbf{R}$ and $u=r^{-1/2}f(r,t)\cos(\theta/2)$. So $f(r,t)$ is a function on the half-plane $\{r\geq 0\}$. The equation $\Delta u=0$ is the ordinary Laplace equation $(\partial_r^2+\partial_t^2)f=0$.

The conditions that $u$ is bounded and the $r^{1/2}$ term vanishes become the combined **Dirichlet** and **Neumann** boundary conditions $f=0, \partial_r f=0$ on the boundary of the half-plane.

To make things precise, we consider two specific problems, with $M$ compact.

1. ("Poisson equation"). Fix $\rho$ with support away from $\Sigma$. There is a bounded solution of $\Delta u = \rho$, hence harmonic near $\Sigma$. Can we adjust $\Sigma$ to arrange that the $z^{1/2}$ term vanishes?
2. ("Hodge theory"). Let $\tilde{L}$ be a lift of $L$ to an affine bundle with fibre $\mathbf{R}$. There is a section $u$ of $\tilde{L}$ with $\Delta u = 0$. Can we adjust $\Sigma$ to arrange that the $z^{1/2}$ term vanishes?

*Remark on (2)*.
The lifts $\tilde{L}$ of $L$ are classified by elements $\chi$ of a cohomology group $H^1(L)$. If $\tilde{M} \to M$ is the double cover of $M$ branched over $\Sigma$ then $H^1(L)$ can be identified with $H^1(\tilde{M})^-$, the $-1$ eigenspace of the action of the involution on $H^1(\tilde{M})$.

Then the derivative $du$ is a well-defined 1-form on $\tilde{M}$ and when $\Delta u = 0$ it is the harmonic 1-form representing this class in $H^1(\tilde{M})^-$ with respect to the singular Riemannian metric on $\tilde{M}$ lifted from $g$ on $M$.

These problems can be regarded as codimension-2 analogues of *free boundary value problems*. (Where one imposes Dirichlet *and* Neumann boundary conditions, with the boundary as an additional variable.)

We state our main result for the second problem. So for any $\Sigma, g, \chi$ we have terms $a(\Sigma, g, \chi), b(\Sigma, g, \chi)$ in the asymptotic expansion around $\Sigma$.

**Theorem** Suppose that $a(\Sigma_0, g_0, \chi_0) = 0$ and that $b(\Sigma_0, g_0, \chi_0)$ is nowhere-vanishing on $\Sigma$. Then for $g, \chi$ sufficiently close to $g_0, \chi_0$ there is a unique solution $\Sigma$ close to $\Sigma_0$ such that $a(\Sigma, g, \chi) = 0$.

Here "close" refers to the $C^\infty$ topology.

In short, the equation $a(\Sigma, g, \chi) = 0$ defines $\Sigma$ implicitly as a function of $g, \chi$, for small variations.

**Part II: Motivation**

Why should we be interested in these questions?

It would not be surprising if similar questions arise in other areas of mathematics, but here we discuss some related topical developments in *differential geometry*, many connected to *special holonomy*. What we outline is in part conjectural or work in progress.

1. **The "classical case"**: $\dim M = 2$.

Here $\Sigma \subset M$ is a finite set and $\tilde{M} \to M$ is a double branched cover of compact Riemann surfaces. If $\Delta u = 0$ the square $(du)^{\otimes 2}$ can be regarded as a quadratic differential on $M$ and the condition that $a = 0$ is the condition that this has zeros rather than poles at $\Sigma$. Our result reduces to the well-known statement that the quadratic differentials on $M$ are locally parametrised by the periods of their square roots, which are 1-forms on $\tilde{M}$.

## 2. **Connections with special Lagrangian geometry**

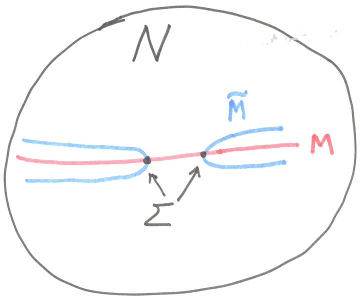
Let $N$ be a complex $n$-dimensional Calabi-Yau manifold with holomorphic $n$-form $\Theta$ and Kähler form $\omega$. Recall that a Lagrangian submanifold $M^n \subset N$ is a submanifold such that $\omega|_M = 0$ and $M$ is *special Lagrangian* if also $\mathrm{Re}(\Theta)|_M$ vanishes. Special Lagrangian submanifolds are *volume minimising*.

One phenomenon of interest for minimal submanifolds is that a sequence of such can converge to a limit with *multiplicity*.

Suppose that $M \subset N$ is special Lagrangian. The deformations of $M$ among Lagrangian submanifolds are locally described as the graphs of derivatives $df$ of functions $f$ on $M$. The linearisation of the special Lagrangian condition, for small variations, is the Laplace equation $\Delta f = 0$.

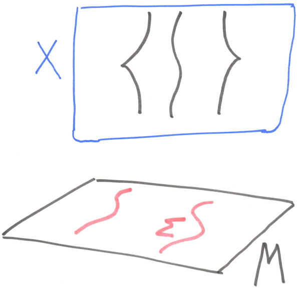
If $u$ is a branched harmonic function with $a = 0$, as above, then the derivative $du$ is $O(|z|^{1/2})$ which fits with the standard model of a double cover $\tilde{M} \subset N$.

It is reasonable to expect that these give deformations of $M$ to 1-parameter families of special Lagrangians converging to $M$, taken with multiplicity 2.



### 3. **Collapsing limits and discriminant sets**

In general, one can consider a manifold $X$ with some special structure fibred over a base $M$, with singular fibres over a "discriminant set" $\Sigma \subset M$.



A well-known case is the Strominger-Yau-Zaslow picture of a Calabi-Yau manifold $X$ with a special Lagrangian fibration.

Here we consider another case when $\dim M = 3$, the manifold $X$ is a 7-dimensional manifold with a $G_2$-structure and the smooth fibres are co-associative submanifolds, diffeomorphic to a K3 surface.

For a certain natural singularity model ("Kovalev-Lefschetz fibrations") the discriminant set $\Sigma$ is a codimension 2-submanifold. When the fibres are very small (near the *collapsed limit*) there is an adiabatic approximation to the geometry involving a branched section of a flat affine bundle solving a nonlinear variant of the Laplace equation and with $O(|z|^{3/2})$ behaviour transverse to $\Sigma$.

4. **Gauge theory**

This refers to pioneering work of Taubes (from c. 2012), and further developments by Walpuski, Haydys, Doan, Takahashi, Zhang ....

One considers equations for a pair $(A, \phi)$ where $A$ is a connection on a bundle over a Riemannian manifold, usually of dimension 3 or 4, and $\phi$ is an auxiliary field. The issue is the convergence of sequences of solutions. Well-known older results:

- If $\phi$ is absent and the equation is the instanton equation for the connection $A$ over a 4-manifold then there is convergence modulo "Uhlenbeck bubbling" over a finite set in the 4-manifold.
- For the *Seiberg-Witten equations*, when $A$ is a $U(1)$ connection and $\phi$ is a spinor field, the special features of the equation give convergence in a strong sense.

Taubes *et al* discover a new phenomenon in which, for many equations (Vafa-Witten, Kapustin-Witten, ...), there is convergence away from a codimension 2-set and the limit is singular, with a multivalued, branching, behaviour of the kind we discussed above.
Many of these equations arise as dimension reductions of "instanton" equations over manifolds of special holonomy and the results are conjecturally related to singular solutions of these equations in higher dimensions.

**Part III: Proof**

Recall that the assertion is that the equation $a(\Sigma, g, \chi) = 0$ defines the submanifold $\Sigma$ implicitly as a function of the data $g, \chi$.

We ignore the complication that the space $\Gamma(N^{-1/2})$ in which $a(\Sigma, g, \chi)$ lives depends on $\Sigma$.

If we had a suitable Banach space set-up we could invoke the standard implicit function theorem: if $a$ were a smooth map between Banach spaces with the partial derivative $\frac{\delta a}{\delta \sigma}$ surjective then the statement would hold.

The variation $\delta\Sigma$ is given by a normal vector field $v \in \Gamma(N)$. So the partial derivative is a map

$$\mathcal{L}_\Sigma : \Gamma(N) \to \Gamma(N^{-1/2}).$$

When $a = 0$ it is not hard to see that

$$\mathcal{L}_\Sigma(v) = \frac{3}{2} b.v,$$

where the right hand side is the algebraic contraction

$$N^{-3/2} \otimes N \to N^{-1/2}.$$

This is essentially the formula

$$\frac{d}{dz} z^{3/2} = \frac{3}{2} z^{1/2}.$$

If $b$ is nowhere-vanishing then this operator $\mathcal{L}_\Sigma$ is certainly invertible.

But when $a \neq 0$ there is an extra term:

$$\mathcal{L}_\Sigma(v) = \frac{3}{2} b.v + P(a.v)$$

where $P : \Gamma(N^{1/2}) \to \Gamma(N^{-1/2})$ is a *pseudodifferential operator of order +1* over $\Sigma$.

However small $a$ is, the extra term dominates on high frequencies.

This shows that we *cannot* fit our problem into a Banach space set-up.

Fortunately, there is a version of the *Nash-Moser* theory (Nash-Moser-Zehnder-Hamilton) which exactly meets the case.

The pseudodifferential operator $P$ is derived from the Green's function $G$ for the Laplace operator on sections of $L$. This has asymptotics

$$G((z_1,t_1)(z_2,t_2)) \sim \mathrm{Re}\left(\Gamma(t_1,t_2) z_1^{1/2} z_2^{-1/2}\right).$$

The operator $P$ is given by a regularised version of the (divergent) integral

$$(Pf)(t_1) = \int_\Sigma \Gamma(t_1,t_2) f(t_2) dt_2.$$

As in:

$$\lim_{\epsilon\to 0}\left(\int_{|t|>\epsilon}\frac{f(t)}{t^2} - 2\epsilon^{-1}f(0)\right)$$